An Introduction to Kleene Logics, Class Three

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Semilattice direct systems (1)

Definition 1

A *semilattice direct system (of $\mathcal{L}$-algebras)* is a triple

$$\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\} \rangle ,$$

where:

- $\mathbf{I} = \langle I, \vee \rangle$ is a semilattice with induced order $\leq_{\mathbf{I}}$;
- $\{\mathbf{A}_i\}_{i \in I}$ is a family of $\mathcal{L}$-algebras with pairwise disjoint universes;
- for every $i, j \in I$ such that $i \leq_{\mathbf{I}} j$, $p_{ij} \in Hom(\mathbf{A}_i, \mathbf{A}_j)$. Moreover, $p_{ii} = \Delta^{\mathbf{A_i}}$ for every $i \in I$, and if $i \leq_{\mathbf{I}} j \leq_{\mathbf{I}} k$, then $p_{ik} = p_{jk} \circ p_{ij}$.

Semilattice direct systems (2)

A semilattice direct system

$$\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\} \rangle \,,$$

of $\mathcal{L}$-algebras can be viewed as a covariant functor F from $\mathbf{I}$ (viewed as a category) to the algebraic category of $\mathcal{L}$-algebras. The condition that $p_{ii} = \Delta^{\mathbf{A_i}}$ for every $i \in I$ expresses the fact that F preserves identity morphisms, while the condition that if $i \leq_{\mathbf{I}} j \leq_{\mathbf{I}} k$, then $p_{ik} = p_{jk} \circ p_{ij}$ expresses the fact that F preserves composition of morphisms.

Płonka sums (1)

Definition 2

If $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\}\rangle$ is a semilattice direct system of $\mathcal{L}$-algebras, the *Płonka sum* over $\mathbb{A}$ is the $\mathcal{L}$-algebra $\mathrm{Pl}\left(\mathbb{A}\right)$ such that:

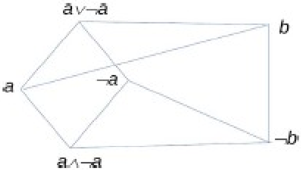
- its universe is the union $\bigcup_{i \in I} A_i$;
- for every n-ary basic operation f (with $n \geqslant 1$) in $\mathcal{L}$, and $a_1, \ldots, a_n \in \bigcup_{i \in I} A_i$,

$$f^{\mathrm{Pl}(\mathbb{A})}(a_1, \ldots, a_n) = f^{\mathbf{A}_j}(p_{i_1 j}(a_1), \ldots, p_{i_n j}(a_n)),$$

  where $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$.

The algebras in $\{\mathbf{A}_i\}_{i \in I}$ are the *fibres* of $\mathrm{Pl}\left(\mathbb{A}\right)$.

Płonka sums (2)



Regular varieties (1)

The problem as to whether all algebras in a variety are representable as Płonka sums has been successfully addressed by Płonka (1967, 1968, 1969).

Definition 3

An identity $\varphi \approx \psi$ is said to be *regular* if $Var(\varphi) = Var(\psi)$.

Theorem 4

If $\mathbb{A}$ is a semilattice direct system of algebras containing at least two algebras, then all regular identities satisfied in all algebras of $\mathbb{A}$ are satisfied in the algebra $\mathrm{Pl}(\mathbb{A})$, whereas any other identity is not satisfied in $\mathrm{Pl}(\mathbb{A})$.

Regular varieties (2)

Definition 5

A variety $\mathcal{V}$ of $\mathcal{L}$-algebras is:

- *regular*, if it satisfies only regular identities;
- *strongly irregular* if there is an $\mathcal{L}$-formula $\varphi(x, y)$ such that $\mathcal{V} \models \varphi(x, y) \approx x$.

Definition 6

If $\mathcal{V}$ is a variety of $\mathcal{L}$-algebras, its *regularisation* $R(\mathcal{V})$ is the variety that satisfies all and only the regular identities holding in $\mathcal{V}$.

Theorem 7

Let $\mathcal{V}$ be a strongly irregular variety of language $\mathcal{L}$, and let $\mathbf{A}$ *be an $\mathcal{L}$-algebra. Then* $\mathbf{A} \in R(\mathcal{V})$ *iff* $\mathbf{A}$ *is decomposable as a Płonka sum over a semilattice direct system of algebras in $\mathcal{V}$.*

Płonka sums of matrices (1)

Definition 8

An *I-direct system* of $\mathcal{L}$-matrices is an ordered pair $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i \in I} \rangle$ such that:

1. $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\} \rangle$ is a semilattice direct system of $\mathcal{L}$-algebras;
2. for every $i \in I$, $F_i \subseteq A_i$;
3. for every $i, j \in I$ such that $i \leq_{\mathbf{I}} j$, $p_{ij}[F_i] \subseteq F_j$.

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Płonka sums of matrices (2)

Definition 9

An *r-direct system* of $\mathcal{L}$-matrices is an ordered pair $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i \in I} \rangle$ such that:

1. $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\} \rangle$ is a semilattice direct system of $\mathcal{L}$-algebras;
2. for every $i \in I$, $F_i \subseteq A_i$;
3. $I^+ := \{i \in I : F_i \neq \emptyset\}$ is the universe of a subsemilattice of $\mathbf{I}$;
4. for every $i, j \in I$ such that $i \leq_{\mathbf{I}} j$, if $F_j \neq \emptyset$, then $p_{ij}^{-1}[F_j] = F_i$.

Płonka sums of matrices (3)

Definition 10

If $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i \in I} \rangle$ is either a l-direct or a r-direct system of matrices, the *Płonka sum* over $\mathbb{M}$ is the matrix $\mathrm{Pl}\,(\mathbb{M}) = \langle \mathrm{Pl}\,(\mathbb{A}), \bigcup_{i \in I} F_i \rangle$.

Whenever $\mathbb{K}$ is a class of matrices, we denote by $\mathcal{P}_{\text{ł}}^{l}(\mathbb{K})$ the class of Płonka sums over the class of all l-direct systems of matrices in $\mathbb{K}$. Similarly for $\mathcal{P}_{\text{ł}}^{r}(\mathbb{K})$ w.r.t. r-direct systems of matrices.

Generalised involutive bisemilattices

Definition 11

The variety $\mathcal{GIB}$ of *generalised involutive bisemilattices* is defined as $R(\mathcal{BA})$; in other words, it is the variety that satisfies exactly the regular identities satisfied in all Boolean algebras.

Since $\mathcal{BA}$ is a strongly irregular variety – just let $\varphi(x, y)$ be $x \wedge (x \vee y)$ – Theorem 7 immediately yields:

Theorem 12

If $\mathbf{A}$ *is an* $\mathcal{L}_0$*-algebra,* $\mathbf{A} \in \mathcal{GIB}$ *iff* $\mathbf{A}$ *is decomposable as a Płonka sum over a semilattice direct system of Boolean algebras.*

Example: Boolean algebras

Boolean algebras can be seen as instances of generalised involutive bisemilattices. More precisely, $\mathbf{A} \in \mathcal{GIB}$ is a Boolean algebra iff one of the following equivalent conditions hold:

1. $\mathbf{A}$ satisfies the absorption identity $x \wedge (x \vee y) \approx x$;
2. $\langle A, \wedge, \vee \rangle$ is a lattice;
3. $\mathbf{A}$ satisfies the identity $x \vee \neg x \approx y \vee \neg y$;
4. its Płonka sum representation has a single fibre.

Example: semilattices

Also semilattices can be seen as instances of generalised involutive bisemilattices. More precisely, **A** $\in \mathcal{GIB}$ is a semilattice iff one of the following equivalent conditions hold:

1. **A** satisfies the identity $x \wedge y \approx x \vee y$;
2. **A** satisfies the identity $x \approx \neg x$;
3. its Płonka sum representation consists entirely of trivial fibres.

Example: **WK**

Another notable example of generalised involutive bisemilattice is the algebra **WK**, which is isomorphic to the unique Płonka sum of Boolean algebras over the 2-element semilattice whose lower fibre is $\mathbf{B}_2$ and whose upper fibre is the trivial Boolean algebra.

Semilattice orderings

If $\mathbf{A} \in \mathcal{GIB}$, we can define two semilattice orderings on **A**, namely

$$x \leq_{\wedge} y \text{ iff } x \wedge y = x$$

and

$$x \leq_{\vee} y \text{ iff } x \vee y = y.$$

These orderings coincide if and only if **A** is a Boolean algebra, and are dual to each other if and only if **A** is a semilattice.

Subdirectly irreducibles and subvarieties

Theorem 13

The only nontrivial subdirectly irreducible generalised involutive bisemilattices are **WK***, the two-element semilattice* $\mathbf{S}_2$*, and the two-element Boolean algebra* $\mathbf{B}_2$*, up to isomorphism. Since* $\mathbf{B}_2$ *is a subalgebra of* **WK** *and* $\mathbf{S}_2$ *is a quotient of such,* $\mathcal{GIB} = \mathrm{HSP}\left(\mathbf{WK}\right)$*.*

Corollary 14

The only nontrivial proper subvarieties of $\mathcal{GIB}$ *are the disjoint varieties* $\mathcal{BA}$ *of Boolean algebras and* $\mathcal{SL}$ *of semilattices.*

Special classes of elements

Definition 15

Given $\mathbf{A} \in \mathcal{GIB}$, an element $a \in A$ is:

- *positive* if $\neg a \leq_{\vee} a$;
- *negative* if $a \leq_{\wedge} \neg a$;
- a *fixpoint*, if $\neg a = a$.

$P(\mathbf{A})$ ($N(\mathbf{A})$) denotes the sets of positive (negative) elements of $\mathbf{A}$. We have that:

- Fixpoints are both negative and positive.
- Elements of the form $a \vee \neg a$ are positive, and coincide with the top elements of the fibre where a lives.
- Elements of the form $a \wedge \neg a$ are negative, and coincide with the bottom elements of the fibre where a lives.
- Thus, he universe of a fibre $\mathbf{A}_i$ the form $[a \wedge \neg a, a \vee \neg a]$ for an arbitrary $a \in A_i$.

Models of weak Kleene logics (1)

Although there are several similarities with the strong Kleene case, we can also find some important differences that render this investigation interesting in its own right.

- The algebras that matter for weak Kleene logics, generally speaking, do not have a lattice reduct. Therefore, the logical filters on such algebras are not lattice filters.
- Using the semilattice orders $\leq_\wedge, \leq_\vee$ it is possible to express conditions of upwards closure and closure under meets that fulfil the same purpose.
- Via the notions of negative and positive elements, we are also in a position to define something analogous to the negative and positive filters from the strong Kleene setting.
- While $\mathsf{Alg}(\mathrm{K}_3) = \mathsf{Alg}(\mathrm{LP}) = \mathcal{KL}$, neither $\mathsf{Alg}(\mathrm{B}_3)$ nor $\mathsf{Alg}(\mathrm{PWK})$ coincides with $\mathcal{GIB}$. The class of algebra reducts of the Suszko reduced models of both logics is a proper subquasivariety of $\mathcal{GIB}$.

Models of weak Kleene logics (2)

Theorem 16

Let $\mathbf{A} \in \mathcal{GIB}$. $F \subseteq A$ *is a* PWK*-filter of* $\mathbf{A}$ *if and only if:*

1. $P(\mathbf{A}) \subseteq F$;
2. $a \in F, a \leq_{\vee} b \Rightarrow b \in F$;
3. $a, b \in F \Rightarrow a \wedge b \in F$.

Theorem 17

Let $\mathbf{A} \in \mathcal{GIB}$. $F \subset A$ *is a* B_3*-filter of* $\mathbf{A}$ *if and only if:*

1. $N(\mathbf{A}) \cap F = \emptyset$;
2. $a \in F, a \leq_{\wedge} b \Rightarrow b \in F$;
3. $a, b \in F \Rightarrow a \wedge b \in F$.

Models of weak Kleene logics (3)

Proof.

Let F be a B-filter of $\mathbf{A}$. F2 and F3 follow from $x \wedge y \dashv\vdash_{\mathrm{B}} x, y$. Assume that $a \in N(\mathbf{A}) \cap F$, by F2 we would have that $\neg a \in F$, and, since $x, \neg x \vdash_{\mathrm{B}} y$, we would have that $F = A$, against our assumption. Conversely, assume that F satisfies F1-F3 and let $\Gamma \vdash_{\mathrm{B}} \varphi$. W.l.g. suppose that $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. We have that $\Gamma \vdash_{\mathrm{CL}} \varphi$ and either: (a) Γ is a CL-antitheorem or: (b) $Var(\varphi) \subseteq Var(\Gamma)$. In case (a), there is ψ such that $\psi, \neg\psi \in \Gamma$; thus, if $v[\Gamma] \subseteq F$, for a given $v \in Hom\left(\mathbf{Fm}\left(\mathcal{L}_0\right), \mathbf{A}\right)$, by F3 $v(\psi) \wedge \neg v(\psi) \in N(\mathbf{A}) \cap F$, contradicting F1. In case (b), since $\Gamma \vdash_{\mathrm{CL}} \varphi$ and $Var(\varphi) \subseteq Var(\Gamma)$, the identity

$$\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \varphi \approx \gamma_1 \wedge \ldots \wedge \gamma_n$$

is valid in $\mathcal{BA}$ and regular, hence it is valid in $\mathcal{GIB}$. Consider again $v \in Hom\left(\mathbf{Fm}\left(\mathcal{L}_0\right), \mathbf{A}\right)$. Then $v(\gamma_1) \wedge \ldots \wedge v(\gamma_n) \leq_{\wedge} v(\varphi)$, and if in addition $v[\Gamma] \subseteq F$, by F3 $v(\gamma_1) \wedge \ldots \wedge v(\gamma_n) \in F$ and thus by F2 $v(\varphi) \in F$, which suffices for our conclusion. $\boxtimes$

Models of weak Kleene logics (4)

Theorem 18

Let $\mathbf{A} \in \mathcal{GIB}$ *be the Płonka sum over the semilattice direct system of Boolean algebras* $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i\in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\}\rangle$*. For* $F \subseteq A$*, the following are equivalent:*

1. F *is a* PWK*-filter of* $\mathbf{A}$*.*
2. *For every* $i \in I$*, there is a* CL*-filter* F_i *of the Boolean algebra* $\mathbf{A}_i$ *such that* $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i\in I}\rangle$ *is an I-direct system of matrices and* $\langle \mathbf{A}, F\rangle$ *is the Płonka sum over* $\mathbb{M}$*.*

Models of weak Kleene logics (5)

Theorem 19

Let $\mathbf{A} \in \mathcal{GIB}$ *be the Płonka sum over the semilattice direct system of Boolean algebras* $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}, \{p_{ij} : i \leq_{\mathbf{I}} j\} \rangle$. *For* $F \subseteq A$, *the following are equivalent:*

1. F *is a* B_3*-filter of* $\mathbf{A}$.
2. *For every* $i \in I$, *there is a* CL*-filter* F_i *of the Boolean algebra* $\mathbf{A}_i$ *such that* $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i \in I} \rangle$ *is an r-direct system of matrices and* $\langle \mathbf{A}, F \rangle$ *is the Płonka sum over* $\mathbb{M}$.

Models of weak Kleene logics (6)

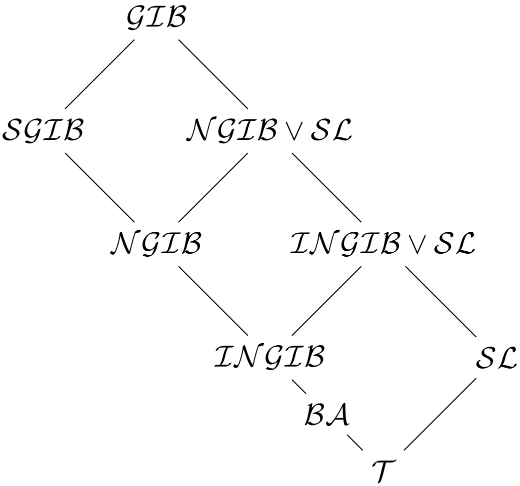
Let $\mathcal{SGIB}$ be the class of all generalised involutive bisemilattices with at most one fixpoint. It is a proper quasivariety.

Theorem 20

The following classes of algebras are pairwise coincident:

1. $Alg(\mathrm{B}_3)$;
2. $Alg(\mathrm{PWK})$;
3. $\mathcal{SGIB}$;
4. $ISP(\mathbf{WK})$.

Models of weak Kleene logics (7)



$\mathcal{GIB}$

$\mathcal{SGIB}$ $\quad$ $\mathcal{NGIB} \vee \mathcal{SL}$

$\mathcal{NGIB}$ $\quad$ $\mathcal{INGIB} \vee \mathcal{SL}$

$\mathcal{INGIB}$ $\quad$ $\mathcal{SL}$

$\mathcal{BA}$

$\mathcal{T}$

Models of weak Kleene logics (8)

Theorem 21

Let $\mathbf{A}$ *be a nontrivial member of* $\mathcal{GIB}$*, and let* $\langle \mathbf{A}, F \rangle$ *be a model of* B_3*, viewed in the Płonka sum representation of Theorem 19. The following are equivalent:*

1. $\langle \mathbf{A}, F \rangle \in \mathrm{Mod}^*(\mathrm{B}_3)$;
2. $I^+ = \{i\}$ *and, either* $\mathbf{A} = \mathbf{A}_i$ *or* $\mathbf{A} = \mathbf{A}_i \oplus \mathbf{1}$*, with* $\langle \mathbf{A}_i, F_i \rangle \in \mathrm{Mod}^*(\mathrm{CL})$.

Weak Kleene logics in the Leibniz hierarchy

Theorem 22

We have that:

1. B_3 *is neither protoalgebraic nor truth-equational.*
2. PWK *is not protoalgebraic, but it is truth-equational.*

Sequent calculus for PWK (1)

$$(Id)\ \frac{}{\phi \Rightarrow \phi}$$

$$(WL)\ \frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta} \qquad (WR)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}$$

$$(\neg L)\ \frac{\Gamma \Rightarrow \Delta, \phi}{\neg\phi, \Gamma \Rightarrow \Delta} \qquad (\neg R)\ \frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi}$$

$$(\neg L^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \neg\phi}{\phi, \Gamma \Rightarrow \Delta} \qquad (\neg R^{\partial})\ \frac{\neg\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}$$

$$(\wedge L)\ \frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\wedge R_1)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}$$

$$(\wedge R_2)\ \frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}$$

$$(\wedge L^{\partial})\ \frac{\phi \wedge \psi, \Gamma \Rightarrow \Delta}{\phi, \psi, \Gamma \Rightarrow \Delta} \qquad (\wedge R_1^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \phi \wedge \psi}{\Gamma \Rightarrow \Delta, \phi, \psi}$$

$$(\wedge R_2^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \phi \wedge \psi}{\phi, \Gamma \Rightarrow \Delta, \psi} \qquad (\wedge R_3^{\partial})\ \frac{\Gamma \Rightarrow \Delta, \phi \wedge \psi}{\psi, \Gamma \Rightarrow \Delta, \phi}$$

Sequent calculus for PWK (2)

$$(\vee L_1)\ \frac{\phi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}$$

$$(\vee L_2)\ \frac{\phi, \Gamma \Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta}{\psi \vee \phi, \Gamma \Rightarrow \Delta} \qquad (\vee R)\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}$$

$$(\vee L_1^\partial)\ \frac{\phi \vee \psi, \Gamma \Rightarrow \Delta}{\phi, \psi, \Gamma \Rightarrow \Delta} \qquad (\vee L_2^\partial)\ \frac{\phi \vee \psi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta, \psi}$$

$$(\vee L_3^\partial)\ \frac{\phi \vee \psi, \Gamma \Rightarrow \Delta}{\psi, \Gamma \Rightarrow \Delta, \phi} \qquad (\vee R^\partial)\ \frac{\Gamma \Rightarrow \Delta, \phi \vee \psi}{\Gamma \Rightarrow \Delta, \phi, \psi}$$

Sequent calculus for PWK (3)

- The weak Kleene conjunction and disjunction are expressible in terms of the strong Kleene connectives as follows:

$$\varphi \wedge^{W} \psi = (\varphi \wedge^{S} \psi) \vee^{S} (\varphi \wedge^{S} \neg\varphi) \vee^{S} ((\psi \wedge^{S} \neg\psi))$$
$$\varphi \vee^{W} \psi = (\varphi \vee^{S} \psi) \wedge^{S} (\varphi \vee^{S} \neg\varphi) \wedge^{S} ((\psi \vee^{S} \neg\psi))$$

- The nonstandard rules for conjunction and disjunction in the PWK calculus are precisely the result of working out the derived rules for these defined connectives in the LP calculus.

Sequent calculus for PWK (4)

Theorem 23

Let $\Gamma \Rightarrow \Delta$ *be a sequent such that* $Var(\Gamma) \cup Var(\Delta) = \{x_1 \ldots x_n\}$*. Then there exists a derivation* $\mathcal{D}$ *of* $\Gamma \Rightarrow \Delta$ *in* $\mathcal{PK}_{Id}^{-*}$ *such that:*

1. *for every branch* $\mathcal{B}_i \in \mathcal{D}$ *there exists a node* $n_i \in \mathcal{B}_i$ *labelled by a sequent of the form* $V_i^1 \Rightarrow V_i^2$*, containing only propositional variables, and such that* $V_i^1 \cup V_i^2 = \{x_1 \ldots x_n\}$*.*
2. *each branch* $\mathcal{B}_i$ *terminates with leaves labelled by* $\Gamma \Rightarrow \Delta$*.*

Sequent calculus for PWK (5)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow x \wedge y, \Sigma}{\Gamma, x \Rightarrow y, \Sigma} \qquad \dfrac{\Gamma \Rightarrow x \wedge y, \Sigma}{\Gamma \Rightarrow x, y, \Sigma}}{\Gamma \Rightarrow x \wedge y, y, \Sigma} \qquad \dfrac{\dfrac{\Gamma \Rightarrow x \wedge y, \Sigma}{\Gamma, y \Rightarrow x, \Sigma} \qquad \dfrac{\Gamma \Rightarrow x \wedge y, \Sigma}{\Gamma \Rightarrow x, y, \Sigma}}{\Gamma \Rightarrow x \wedge y, x, \Sigma}}{\Gamma, \Gamma \Rightarrow x \wedge y, x \wedge y, \Sigma, \Sigma}}{\Gamma \Rightarrow x \wedge y, \Sigma}$$

Sequent calculus for PWK (6)

Lemma 24

The following rule is derivable in $\mathcal{PK}^{-*}_{Id}$ *whenever* $Var(\varphi) \subseteq Var(\Gamma \cup \Delta)$:

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

Theorem 25

The following are equivalent for any $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_0)$:

1. $\{\Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{\mathcal{PK}^{-*}_{Id}} \Rightarrow \varphi$;
2. $\Gamma \vdash_{\mathrm{PWK}} \varphi$.

Generalising the framework (1)

$$\Gamma \vdash_{\mathrm{L}^l} \varphi \iff \text{ there is } \Delta \subseteq \Gamma \text{ s.t. } \mathit{Var}(\Delta) \subseteq \mathit{Var}(\varphi) \text{ and } \Delta \vdash_{\mathrm{L}} \varphi.$$

$$\Gamma \vdash_{\mathrm{L}^r} \varphi \iff \begin{cases} \text{either: } \Gamma \vdash_{\mathrm{L}} \varphi \text{ and } \mathit{Var}(\varphi) \subseteq \mathit{Var}(\Gamma) \\ \text{or: } \Gamma \text{ is an antitheorem of } \mathrm{L}. \end{cases}$$

Generalising the framework (2)

Lemma 26

Let L *be a logic and let* X *be an* l*-direct system of models of* L*. Then* $\mathcal{P}_{\text{ł}}(X)$ *is a model of* L^{l}*.*

Theorem 27

Let L *be a logic. Then* L^{l} *is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}^{l}_{\text{ł}}(\mathsf{Mod}(\mathrm{L})); \quad \mathcal{P}^{l}_{\text{ł}}(\mathsf{Mod}^{*}(\mathrm{L})); \quad \mathcal{P}^{l}_{\text{ł}}(\mathsf{Mod}^{Su}(\mathrm{L})).$$

Theorem 28

Let L *be a logic. If* L *is complete w.r.t. the class of matrices* M*, then* L^{l} *is complete w.r.t. the class* $\{\langle \mathbf{A} \oplus \mathbf{1}, F \cup \{1\}\rangle : \langle \mathbf{A}, F\rangle \in \mathsf{M}\}$*.*

Generalising the framework (3)

Lemma 29

Let L *be a logic with no antitheorems and let* X *be an r-direct system of models of* L. *Then* $\mathcal{P}_{\text{ł}}(X)$ *is a model of* L^r.

Theorem 30

Let L *be a logic without antitheorems such that* $\langle \mathbf{1}, \emptyset \rangle \in \mathit{Mod}(\mathrm{L})$. *Then* L^r *is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}^r_{\text{ł}}(\mathsf{Mod}(\mathrm{L})); \quad \mathcal{P}^r_{\text{ł}}(\mathsf{Mod}^*(\mathrm{L})); \quad \mathcal{P}^r_{\text{ł}}(\mathsf{Mod}^{\mathsf{Su}}(\mathrm{L})).$$

Thank you for your attention!