

An Introduction to Kleene Logics, Class One

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Kleene Logics: the birth (1)

In his *Introduction to Metamathematics*, S.C. Kleene distinguishes between two senses of the propositional connectives when reasoning about partial recursive predicates. If we compute the value of one such predicate P as applied to the argument x , the computation may terminate and return a classical value (0, or 1) if P is defined for x , or else return no value at all. In the latter case, it may be convenient to say that Px has value n (“undefined”). In such a 3-valued setting, how are we to calculate the values of sentential compounds built by means of the connectives of negation, disjunction and conjunction?

Kleene Logics: the birth (2)

Kleene intends to respect two minimal constraints:

- The table of each connective should be exactly the classical one, when all the arguments have classical values.
- The table of each connective should be *regular*, meaning that whenever a column (row) contains a classical value in the row (column) of n , that column (row) should contain *that* classical value everywhere.

Kleene Logics: the birth (3)

Strong tables (**SK**):

\neg	
0	1
<i>n</i>	<i>n</i>
1	0

\wedge	0	<i>n</i>	1
0	0	0	0
<i>n</i>	0	<i>n</i>	<i>n</i>
1	0	<i>n</i>	1

\vee	0	<i>n</i>	1
0	0	<i>n</i>	1
<i>n</i>	<i>n</i>	<i>n</i>	1
1	1	1	1

Weak tables (**WK**):

\neg	
0	1
<i>n</i>	<i>n</i>
1	0

\wedge	0	<i>n</i>	1
0	0	<i>n</i>	0
<i>n</i>	<i>n</i>	<i>n</i>	<i>n</i>
1	0	<i>n</i>	1

\vee	0	<i>n</i>	1
0	0	<i>n</i>	1
<i>n</i>	<i>n</i>	<i>n</i>	<i>n</i>
1	1	<i>n</i>	1

Kleene Logics: the birth (4)

- **SK** + single designated value 1: *strong Kleene logic* (K_3).
- **SK** + designated values 1, n : *Logic of Paradox* (LP: Priest, 1979).
- **WK** + single designated value 1: *Paracomplete weak Kleene logic* (B_3), a fragment of Bochvar's logic (Bochvar, 1938).
- **WK** + designated values 1, n : *Paraconsistent weak Kleene logic* (PWK), a fragment of Halldén's logic (Halldén, 1949).

Kleene Logics: the birth (5)

In other words, given $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_0)$:

- $\Gamma \vdash_{K_3} \varphi$ iff for every \mathcal{L}_0 -valuation to **SK**, if $v[\Gamma] \subseteq \{1\}$, then $v(\varphi) = 1$;
- $\Gamma \vdash_{LP} \varphi$ iff for every \mathcal{L}_0 -valuation to **SK**, if $v[\Gamma] \subseteq \{1, n\}$, then $v(\varphi) \in \{1, n\}$;
- $\Gamma \vdash_{B_3} \varphi$ iff for every \mathcal{L}_0 -valuation to **WK**, if $v[\Gamma] \subseteq \{1\}$, then $v(\varphi) = 1$;
- $\Gamma \vdash_{PWK} \varphi$ iff for every \mathcal{L}_0 -valuation to **WK**, if $v[\Gamma] \subseteq \{1, n\}$, then $v(\varphi) \in \{1, n\}$.

Lemma 1

The law (EM, Excluded Middle) $\vdash \varphi \vee \neg\varphi$ does not hold in K_3 .

Lemma 2

None of the following principles hold in LP:

- *(EAQ, Ex Absurdo Quodlibet) $\varphi, \neg\varphi \vdash \psi$;*
- *(MP, Modus Ponens) $\varphi, \neg\varphi \vee \psi \vdash \psi$.*

Lemma 3

None of the following principles hold in B_3 :

- (EM, Excluded Middle) $\vdash \varphi \vee \neg\varphi$;
- (DA, Addition) $\varphi \vdash \varphi \vee \psi, \psi \vdash \varphi \vee \psi$.

Lemma 4

None of the following principles hold in PWK:

- (EAQ, Ex Absurdo Quodlibet) $\varphi, \neg\varphi \vdash \psi$;
- (CS, Conjunctive Simplification) $\varphi \wedge \psi \vdash \varphi, \varphi \wedge \psi \vdash \psi$;
- (MP, Modus Ponens) $\varphi, \neg\varphi \vee \psi \vdash \psi$.

Definition 5

Let L be a logic of language \mathcal{L} .

- An \mathcal{L} -formula φ is a *theorem* of L if $\vdash_L \varphi$.
- A set Γ of \mathcal{L} -formulas in a single variable is an *antitheorem* of L if for every substitution σ and every \mathcal{L} -formula φ , $\sigma[\Gamma] \vdash_L \varphi$.

Lemma 6

We have that:

- 1 K_3 and B_3 have no theorem, while they have the same antitheorems as CL .
- 2 LP and PWK have no antitheorem, while they have the same theorems as CL .

Proof.

We only prove the claims regarding LP. Suppose ex absurdo that $\sigma[\Gamma] \vdash_{\text{LP}} \varphi$. We lose no generality in assuming that φ is a variable x not occurring in $\sigma[\Gamma]$ (which contains finitely many variables, as Γ contains a single variable). Then we could choose v such that $v(x) = 0$ and $v(y) = n$, for any other variable $y \neq x$. Thus, $v[\sigma[\Gamma]] \subseteq \{n\}$ and $v(x) = 0$. This, however, would contradict our assumption.

For the nontrivial direction of the remaining claim, suppose towards a contradiction that $\vdash_{\text{CL}} \varphi$ but $\not\vdash_{\text{LP}} \varphi$. Then there is v such that $v(\varphi) = 0$. We now define a *classical* valuation w as follows: $w(x) = 0$, if $v(x) = n$, and $w(x) = v(x)$ otherwise. By a routinary induction on the complexity of formulas we have that for any formula ψ , if $v(\psi) \neq n$, then $w(\psi) = v(\psi)$. Therefore $w(\varphi) = 0$, a contradiction. ☒

Adding quantifiers to strong Kleene logics (1)

Let \mathcal{L}_1 be a signature for first-order classical logic, consisting of relation symbols and function symbols of finite (possibly zero) arity. A 3-model for \mathcal{L}_1 is a pair $M = \langle D, I \rangle$ such that:

- $I(P^n) \in \{0, n, 1\}^{D^n}$, for any n -ary relation symbol P^n ;
- $I(x) \in D$, for any variable x ;
- $I(f^n) \in D^{D^n}$, for any n -ary function symbol f^n ;
- for any atomic \mathcal{L} -formula $P^n(t_1, \dots, t_n)$,

$$I(P^n(t_1, \dots, t_n)) = I(P^n)(I(t_1), \dots, I(t_n));$$

- for any \mathcal{L} -formulas φ, ψ , $I(\neg\varphi) = 1 - I(\varphi)$,
 $I(\varphi \wedge \psi) = \min(I(\varphi), I(\psi))$ and $I(\varphi \vee \psi) = \max(I(\varphi), I(\psi))$;
- $I(\forall x\varphi(x)) = \min(I'(\varphi(x)))$, for all x -variants I' of I ;
- $I(\exists x\varphi(x)) = \max(I'(\varphi(x)))$, for all x -variants I' of I .

Adding quantifiers to strong Kleene logics (2)

Given $\Gamma \cup \{\varphi\} \subseteq \text{Sent}(\mathcal{L}_1)$:

- $\Gamma \vdash_{K_3} \varphi$ iff for every 3-model for \mathcal{L}_1 $M = \langle D, I \rangle$, if $I[\Gamma] \subseteq \{1\}$, then $I(\varphi) = 1$;
- $\Gamma \vdash_{LP} \varphi$ iff for every 3-model for \mathcal{L}_1 $M = \langle D, I \rangle$, if $I[\Gamma] \subseteq \{1, n\}$, then $I(\varphi) \in \{1, n\}$.

Applications of strong Kleene logics: the paradoxes (1)

One of the most compelling intuitions about the notion of truth is its *disquotational* character. Namely, it is tempting to assume that the truth predicate obeys the Tarskian equivalence

$$\text{T-Scheme: } T^{\ulcorner} \varphi \urcorner \dashv\vdash \varphi$$

Certain (natural or formal) languages have the expressing resources to formulate such *self-referential sentences* as $\lambda := \neg T^{\ulcorner} \lambda \urcorner$. Applying the T-scheme to λ , we get:

$$\text{L: } T^{\ulcorner} \lambda \urcorner \dashv\vdash \neg T^{\ulcorner} \lambda \urcorner.$$

Applications of strong Kleene logics: the paradoxes (2)

$$\begin{array}{c}
 \begin{array}{c} \vdots \\ T^{\lambda} \vee \neg T^{\lambda} \end{array} \quad \begin{array}{c} [\neg T^{\lambda}] \\ \vdots \\ T^{\lambda} \end{array} \\
 \hline
 T^{\lambda}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c} \vdots \\ T^{\lambda} \vee \neg T^{\lambda} \end{array} \quad \begin{array}{c} [\neg T^{\lambda}] \\ \vdots \\ \neg T^{\lambda} \end{array} \quad \begin{array}{c} [T^{\lambda}] \\ \vdots \\ \neg T^{\lambda} \end{array} \\
 \hline
 \neg T^{\lambda}
 \end{array}
 \\
 \hline
 \Psi
 \end{array}$$

Applications of strong Kleene logics: the paradoxes (3)

Two options:

- 1 *Drop the T-scheme.* In order to keep consistency, the T-scheme must be restricted.
- 2 *Go subclassical (Bar-Hillel, 1973; Kripke, 1975; Martin, 1974; Priest, 1979).* The above derivation is invalid (for different reasons) both in K_3 (which allows for *truth value gaps*) and in LP (which allows for *truth value gluts*).

Against gaps and gluts

Against gaps:

- Strengthened liars.
- The liar does not seem to belong together with other classes of gappy sentences (sentences with category mistakes, sentences whose truth value is under-determined).

Against gluts:

- Are we comfortable with dialetheias?
- Seemingly related paradoxes call for different solutions (the liar, Curry...).

Classical recapture (1)

- CL is simple, well-understood, and has proven successful in many different contexts.
- Many proponents of non-classical logics argue that CL is generally correct, except in certain “anomalous” deductive situation (where we are reasoning from potentially inconsistent assumptions, when self-reference is at issue, etc.)
- Therefore, they aim at showing that CL can be recaptured in some form within their own logic of choice.

Classical recapture (2)

- Can we have our cake and eat it, too? Is it possible to keep a fully disquotational account of the truth predicate without having to forswear CL, and without incurring the penalty of paradoxes?
- Cobreros, Egré, van Rooj, and Ripley (2012, 2015, 2024) think we can – if we're flexible enough in our account of logical consequence. They propose a logic called *Strict-Tolerant Logic* (ST),

There is no need, from an ST-based perspective, ever to criticize (on logical grounds) any classically-valid inference. As such, there is no need for the 'classical recapture' that so exercises many non-classical theorists. If the classical bird is never let out of the cage in the first place, there is no need to recapture it (Ripley, 2013, p. 156).

Mixed logics: Strict-Tolerant and Tolerant-Strict (1)

Given $\Gamma \cup \{\varphi\} \subseteq \text{Sent}(\mathcal{L}_1)$ and a 3-model for \mathcal{L}_1 $M = \langle D, I \rangle$, let us define:

- $\Gamma \vdash_{\text{ST}}^M \varphi$ iff whenever $I[\Gamma] \subseteq \{1\}$, then $I(\varphi) \in \{1, n\}$;
- $\Gamma \vdash_{\text{TS}}^M \varphi$ iff whenever $I[\Gamma] \subseteq \{1, n\}$, then $I(\varphi) \in \{1\}$;
- $\Gamma \vdash_{\text{ST}} \varphi$ iff for every 3-model M for \mathcal{L}_1 , $\Gamma \vdash_{\text{ST}}^M \varphi$;
- $\Gamma \vdash_{\text{TS}} \varphi$ iff for every 3-model M for \mathcal{L}_1 , $\Gamma \vdash_{\text{TS}}^M \varphi$.

Theorem 7

Given $\Gamma \cup \{\varphi\} \subseteq \text{Sent}(\mathcal{L}_1)$:

- 1 $\Gamma \vdash_{\text{ST}} \varphi$ iff $\Gamma \vdash_{\text{CL}} \varphi$;
- 2 *It is never the case that $\Gamma \vdash_{\text{TS}} \varphi$.*

Questions:

- 1 Are they interesting at all?
- 2 Can they handle paradoxes?
- 3 What is their relationship to K_3 and LP?

Mixed logics: Strict-Tolerant and Tolerant-Strict (3)

- ST differs from CL at the *metainferential level*. For example, $\varphi \vdash_{ST}^M \psi$ and $\psi \vdash_{ST}^M \chi$ do not guarantee that $\varphi \vdash_{ST}^M \chi$; also, $\vdash_{ST}^M \varphi \vee \psi$, $\varphi \vdash_{ST}^M \chi$ and $\psi \vdash_{ST}^M \chi$ do not guarantee that $\vdash_{ST}^M \chi$.
- TS is non-empty at the *metainferential level*. For example, $\varphi \vdash_{TS}^M \psi$ and $\psi \vdash_{TS}^M \chi$ do guarantee that $\varphi \vdash_{TS}^M \chi$.

Mixed logics: Strict-Tolerant and Tolerant-Strict (4)

Both ST and TS can support axiomatic truth theories with a fully disquotational truth predicate. In particular, if we buy the idea that ST is CL, we can actually have the best of two worlds.

Mixed logics: Strict-Tolerant and Tolerant-Strict (5)

Given $\Gamma \cup \{\varphi\} \subseteq \text{Sent}(\mathcal{L}_1)$ and a 3-model for \mathcal{L}_1 $M = \langle D, I \rangle$, let us define:

- $\Gamma \vdash_{\text{MinfST}} \varphi$ iff for every 3-model for \mathcal{L}_1 $M = \langle D, I \rangle$, if $\emptyset \vdash_{\text{ST}}^M \gamma$ for all $\gamma \in \Gamma$, then $\emptyset \vdash_{\text{ST}}^M \varphi$.

We have that (cf. Barrio et al., 2015; Dicher and F.P., 2019):

Theorem 8

Let $\Gamma \cup \{\varphi\} \subseteq \text{Sent}(\mathcal{L}_1)$. Then $\Gamma \vdash_{\text{MinfST}} \varphi$ iff $\Gamma \vdash_{\text{LP}} \varphi$.

Logic on the Buenos Aires Plan

- LP coincides with CL at the level of *valid formulas*, but not at the level of *valid inferences*;
- ST coincides with CL at the level of *valid formulas and inferences*, but not at the level of *valid metainferences*;
- It is possible to define a logic TS/ST that coincides with CL up to the level of *valid metainferences*, but not at the level of *valid meta-metainferences*;
- ...

Barrio, Pailos, and Szmuc (2020) argue that a logic is to be identified with an infinite sequence of consequence relations holding between increasingly complex relata: formulae, inferences, metainferences, and so on.

Valuations to **WK** obey a *principle of infectiousness*: if v is a valuation and φ a formula, then $v(\varphi) = n$ if and only if there is a propositional variable x occurring in φ such that $v(x) = n$. By extension, elements like n are called *infectious* as well.

Logics with infectious truth values seem promising if we want to account for reasoning in the presence of possibly *non-significant* sentences.

Interpretation of the third value in B_3

- “meaningless” (Bochvar, 1938);
- “non-significant” (Ferguson, 2014): meaningless, or syntactically ill-formed, or empirically unverifiable, or containing a category mistake...;
- “off-topic” (Beall, 2016; Song et al., 2021).

Interpretation of the third value in PWK (1)

- Interpreting the value n in PWK involves an additional problem. In PWK, n is both infectious and designated – two features that don't seem to sit well with each other.
- Brady and Routley (1973): if the designated values are those we attach to sentences we'd be ready to assert, designating n commits us to “sometimes asserting logical nonsense”
- Halldén (1949): EM and MP together entail that $\vdash\varphi$ (φ is meaningful). By taking n on board as a designated value, we can salvage the classical tautologies while avoiding the commitment to the contention that everything is meaningful.

Interpretation of the third value in PWK (2)

- Ciuni and Carrara (2019): Halldén seems to flout the *Assertion-Designation Harmony* (ADH), which is a popular and widespread assumption among many-valued logicians. Since B_3 does not have the same problem, the two weak Kleene logics are not on an equal footing.
- However, B_3 flouts the dual *Denial-Undesignation Harmony*: we cannot view undesigned values as those that correspond to sentences we'd be ready to deny (Bonzio et al., 2022).

Other logics with infectious values

- S_{FDE} (Deutsch, 1977; Oller, 1999; Fitting, 2006; Ferguson, 2017);
- dS_{FDE} (Szmuc, 2016; Ciuni et al., 2018; Ferguson, 2017);
- HYB_1 and HYB_2 (Szmuc, 2016; Da Ré et al., 2020);
- S_{FDE}^* (Daniels, 1986; Priest, 2010; Ferguson, 2017);
- dS_{FDE}^* (Szmuc, 2016).

Logic and Analyticity

According to a dominant tradition in philosophy, logic is the paradigmatic example of a discipline consisting of *analytic* sentences, whose truth or otherwise can be known simply by an analysis of their meanings, and of the meaning of their constituents. This doctrine has been championed by Rudolf Carnap in his *Logical Syntax of Language* and many other authors. Some dissenting voices:

- The analytic vs synthetic distinction is not legitimate (Quine, 1951);
- Open texture or vagueness can blur the distinction in specific cases (Shapiro, 2014; Burgess, 2004);
- Classical first-order logic is synthetic (Hintikka, 1973);
- The methods of logic are continuous with those of science (Anti-exceptionalism).

Still, a vast majority of discussants agree that a failure to display the marks of analyticity is, for a sentence, a *reductio* of its logical character.

Analyticity and Consequence

From Kant's *Critique of Pure Reason*:

In all judgments, in which the relationship of a subject to a predicate is thought [...], this relationship is possible in two ways. Either the predicate B belongs to the subject A as something which is contained in the concept A (in a hidden way); or B lies entirely apart from that concept A, though it indeed stands in connection with it. In the first case I term the judgment analytical, in the other, synthetic. (B10-11)

In Kant's perspective, analyticity is a property of *judgments*. However, we can extend it to *arguments*. An argument with premisses $\varphi_1, \dots, \varphi_n$ and conclusion ψ is analytic if the truth of ψ can be known simply by an analysis of the meanings of $\varphi_1, \dots, \varphi_n$. The thesis that logic is analytic boils down to the claim that logically valid arguments amount to a species of analytic arguments; thus understood, analyticity is a feature of *consequence*.

Analyticity and the Plurality of Logics

- Does the view that logic is analytic support CL (Classical Logic) against its nonclassical rivals? There are *prima facie* reasons to suspect that it does not.
- The argument with premisses “Snow is white” and “Snow is not white” and with conclusion “Inter will win the Serie A League this year”, an instance of the rule of *Ex Absurdo Quodlibet*:

$$\varphi, \neg\varphi \vdash \psi,$$

is classically valid. It can be plausibly argued, though, that by a sheer analysis of the meanings contained in these particular premisses we can't buttress any conclusion that is about football.

- Whatever notion of analyticity may appeal to those who claim that CL is analytic, it must be somewhat remote from an immediate, literal construal of the term.

Informational Explications

- According to Bar-Hillel and Carnap (1964), the information content of a sentence φ is the set of possible worlds that φ rules out. The information content of a set of sentences Γ is the *union* of the contents of each $\varphi \in \Gamma$.
- All classically valid entailments are analytic in that the information content of their conclusion is included in the content of their set of premisses. In particular, whenever Γ is an inconsistent set of premisses, its content coincides by definition with the set of *all* possible worlds, a circumstance that vindicates the *Ex Absurdo Quodlibet*.
- As an explication of *meaning*, this approach leaves some room for dissatisfaction. For example, we could find it implausible that all tautologies (or, for that matter, all contradictions) have the same meaning and thus co-entail one another.

Semantic Explications (1)

- Intriguing alternatives were proposed at the Philosophy Department at Harvard in the mid- to late 1920s, where C.I. Lewis famously defended an account of entailment in terms of *strict implication*.
- At the time, it was common to regard logics as determined by a set of validities. Logicians expressed their views on consequence, and the attendant analyticity requirements, in terms of connections between the antecedent and the consequent of a formula dominated by an entailment *connective*.
- The informal glosses of entailment provided by Lewis suggest a more radical departure from classical logic than his calculi actually allow. For example, while in his *Survey of Symbolic Logic* Lewis claims that “the ‘implication’ and ‘equivalence’ of ordinary logic are relations of intension or meaning”, he also accepts that a contradiction entails an arbitrary conclusion.
- Typically, his students or younger colleagues would agree with his view of implication in broad strokes, but would translate it into more extreme revisions of CL.

Semantic Explications (2)

- For example, E.J. Nelson (1931) holds that φ entails ψ just in case φ is inconsistent with $\neg\psi$. Although this is nominally the same definition of entailment one finds in Lewis, a different notion of consistency is at stake.
- For Lewis, two sentences are consistent when they are *compossible*: hence φ and ψ might be inconsistent simply because one of them is impossible. According to Nelson, consistency is an inherently relational notion of conflict of meanings. Every sentence (even an impossible one) is consistent with itself and inconsistent with its own negation; moreover, if φ entails ψ , then φ is consistent with ψ .
- Nelson endorses the contra-classical principles $(\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi)$ (*Boethius' law*) and $\neg(\varphi \rightarrow \neg\varphi)$ (*Aristotle's law*). The calculus he presents in his PhD thesis is free from the paradoxes of strict implication.

Semantic Explications (3)

- In order to obtain a consistent logic, Nelson has to forgo much of CL – and not all these sacrifices are exactly palatable. For example, in his *Survey* Lewis had shown that a variant of the *Ex Absurdo Quodlibet* is derivable if we assume Conjunctive Simplification and Antilogism:

$$\begin{aligned} & \varphi \wedge \psi \rightarrow \varphi, \varphi \wedge \psi \rightarrow \psi, \\ & \varphi \wedge \psi \rightarrow \chi \vdash \varphi \wedge \neg\chi \rightarrow \neg\psi. \end{aligned}$$

- Nelson chooses to give up Conjunctive Simplification (an approach he would later disavow). And it wouldn't be long before some commentators noticed that this is at odds with an explication of logical entailment as analytic.
- Körner (1946-47), for example, remarks that if there is an inference that the inference from $\varphi \wedge \psi$ to φ is a paragon of analyticity: how are we to deny that the meaning of a conjunction contains the meanings of its conjuncts?

Syntactic Explications

- William T. Parry obtains his PhD (Harvard, 1931) with a thesis on *analytic implication*. He claims that valid entailments should satisfy a *Proscriptive Principle (PP)*:

No formula is valid that has [analytic implication] as a main relation and has a variable [...] which occurs in the consequent but not in the antecedent (pp. 170-171).

- Parry blames either Addition:

$$\varphi \rightarrow \varphi \vee \psi, \psi \rightarrow \varphi \vee \psi.$$

or Antilogism as the stumbling blocks in Lewis's alleged "proofs" of the paradoxes of strict implication.

- Critics of Parry, like Anderson and Belnap (1975), have found the PP questionable as an analysis of meaning inclusion.
- According to Ferguson (2017), such critics are attacking a strawman. Parry does not have in mind the *semantic* notion of meaning containment, but an irreducibly *syntactic* concept.

From Analyticity to (Non-)Significance

- Analytic containment logics and infectious logics seem totally unrelated. In the former case, the sentences of concern to the logician are just the *significant* sentences, but consequence is understood in terms of meaning inclusion. In the latter, the presence of non-significant sentences in reasoning is no longer ruled out, but consequence is more traditionally interpreted as truth (or non-falsity) preservation.
- However, there is an unexpected, but robust, connection between these approaches.

Theorem 9

[Urquhart, 1986] For any $\Gamma \cup \{\varphi\} \subseteq \text{Fm}(\mathcal{L}_0)$, the following are equivalent:

- 1 $\Gamma \vdash_{\text{B}_3} \varphi$;
- 2 $\Gamma \vdash_{\text{CL}} \varphi$ and either: (a) Γ is an antitheorem of CL, or: (b) $\text{Var}(\varphi) \subseteq \text{Var}(\Gamma)$.

Theorem 10

[Ciuni-Carrara, 2016] For any $\Gamma \cup \{\varphi\} \subseteq \text{Fm}(\mathcal{L}_0)$, the following are equivalent:

- 1 $\Gamma \vdash_{\text{PWK}} \varphi$;
- 2 There exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\text{CL}} \varphi$ and $\text{Var}(\Delta) \subseteq \text{Var}(\varphi)$.

Are Weak Kleene Logics Good Models of Analytic Consequence?

- As much as it is tempting to use B_3 and PWK to model the concept of analytic consequence and its dual in light of Theorems 9 and 10, an objection is to be expected.
- Let us exemplify with B_3 . The intuition is clear: if φ classically follows from Γ , B_3 only accepts this entailment if φ is analytically contained in Γ , i.e., if $Var(\varphi) \subseteq Var(\Gamma)$. Theorem 9, however, provides for an exception: if Γ is an antitheorem of CL (hence also of B_3), the variable inclusion proviso need not apply.
- For arbitrary variables x, y , we have that $x, \neg x \vdash_{B_3} y$. Yet, recall that the validity of EAQ in CL was one of the reasons for deeming CL inadequate as an explication of analytic containment. Trading B_3 in for CL does not seem to have improved the situation.
- Shouldn't we get rid of these exceptions altogether?

Pure Variable Inclusion Companions of Classical Logic (1)

$$\Gamma \vdash_{\text{CL}^{pr}} \varphi \iff \Gamma \vdash_{\text{CL}} \varphi \text{ and } \text{Var}(\varphi) \subseteq \text{Var}(\Gamma);$$

$$\Gamma \vdash_{\text{CL}^{pl}} \varphi \iff \left\{ \begin{array}{l} \text{there exists } \Delta \subseteq \Gamma, \Delta \neq \emptyset \text{ such that} \\ \Delta \vdash_{\text{CL}} \varphi \text{ and } \text{Var}(\Delta) \subseteq \text{Var}(\varphi). \end{array} \right.$$

Theorem 11

We have that:

- 1 CL^{pr} has no single characteristic matrix.
- 2 For any $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_0)$, $\Gamma \vdash_{CL^{pr}} \varphi$ iff: (a) for every valuation v to **WK**, $v[\Gamma] \subseteq \{1\}$ implies $v(\varphi) = 1$ and (b) for every valuation v to **WK**, $v[\Gamma] \subseteq \{1, 0\}$ implies $v(\varphi) \in \{1, 0\}$.

Lemma 12

CL^{PI} has a 5-element characteristic matrix $\langle \mathbf{PK}, \{1, n\} \rangle$, where \mathbf{PK} is described by the tables below:

	\neg	\wedge	0	p	n	m	1	\vee	0	p	n	m	1
0	1	0	0	0	n	0	0	0	0	0	n	1	1
p	m	p	0	p	n	p	0	p	0	p	n	m	1
n	n	n	n	n	n	n	n	n	n	n	n	n	n
m	p	m	0	p	n	m	1	m	1	m	n	m	1
1	0	1	0	0	n	1	1	1	1	1	n	1	1

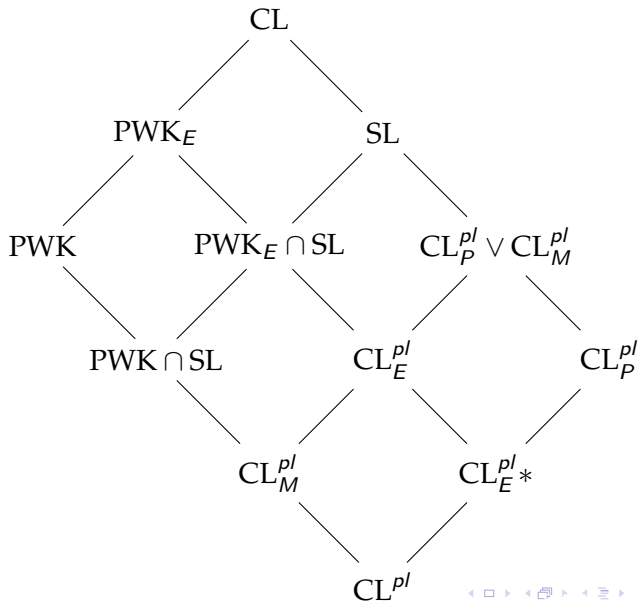
Extensions of Weak Kleene Logics (1)

- The unique logic strictly stronger than B_3 and strictly weaker than CL is *Suszko logic* SL, defined as follows for any $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_0)$:
 $\Gamma \vdash_{SL} \varphi$ iff $\Gamma \vdash_{CL} \varphi$ and $\Gamma \neq \emptyset$.
- The unique logic strictly stronger than PWK and strictly weaker than CL is PWK_E , determined by the matrix $\langle \mathbf{EK}, \{1, \top\} \rangle$, where \mathbf{EK} has the tables

\neg	
0	1
\perp	\top
b	a
a	b
\top	\perp
1	0

\wedge	0	\perp	b	a	\top	1
0	0	0	0	0	0	0
\perp	0	\perp	\perp	\perp	\perp	0
b	0	\perp	b	\perp	b	0
a	0	\perp	\perp	a	a	1
\top	0	\perp	b	a	\top	1
1	0	0	0	1	1	1

Extensions of CL^{pl}



Theorem 13

The following conditions are equivalent for any $\Gamma \cup \{\varphi\} \subseteq Fm(\mathcal{L}_0)$:

- 1 $\Gamma \vdash_{\text{PWKE}} \varphi$;
- 2 $\Gamma \vdash_{\text{L}} \varphi$, where L is the logic obtained by adding (EAQ) to PWK;
- 3 There exists $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\text{CL}} \varphi$ and either (A) $\text{Var}(\Delta) \subseteq \text{Var}(\varphi)$ or (B) Δ is a CL-antitheorem.

- We have just seen that PWK_E admits a linguistic characterisation in terms of a variable inclusion requirement. Exceptions must be provided for, but after all neither PWK nor B_3 is exempt from such dispensations.
- However, its defining matrix contains no infectious value.
- Hence, the relationship between infectiousness and variable inclusion may not be as straightforward as one might think. Indeed, we will see that *logics of variable inclusion* arise in the presence of certain conditions that admit of a convenient semantic description, and of which infectiousness is only a limit case.
- To clarify all this, however, we need a detour through universal algebra.

Thank you for your attention!