How Thick is a Boundary Layer?

1 Boundary Layer

A simple relation can be obtained with the following idea. Take the Navier-Stokes equations:

$$\rho\frac{\partial\vec{V}}{\partial t}+\rho\vec{V}\cdot\vec{\nabla}\vec{V}=-\vec{\nabla}p+\mu\nabla^2\vec{V} \tag{1}$$

Let's imagine a flat plate moving left with a velocity U. How far does the vorticity (or the information) travel in a direction perpendicular to the flat plate?

$$\delta\propto(\nu t)^{1/2} \tag{2}$$

The flat plate in the same time would have moved at a distance

$$x=Ut \tag{3}$$

So if I take a point at a distance x from the leading edge of the flat plate, the boundary layer will have a thickness (this can be obtained by simplifying the time from the 2 previous equations):

$$\delta\propto\left(\frac{\nu x}{U}\right)^{1/2} \tag{4}$$

or, in non-dimensional form:

$$\frac{\delta}{x}\propto\left(\frac{\nu}{xU}\right)^{1/2}=\frac{1}{\mathrm{Re}_x^{1/2}} \tag{5}$$

This means that a boundary layer has a thickness that is inversely proportional to the Reynolds number to the power of 1/2.

$$\mathrm{Re}=10^6 \quad\Rightarrow\quad \frac{\delta}{x}\sim10^{-3} \tag{6}$$

$$\text{so for a 10 cm blade}\quad\Rightarrow\quad\delta\sim10^{-4}\text{ m} \tag{7}$$

2 Boundary Layer Measures

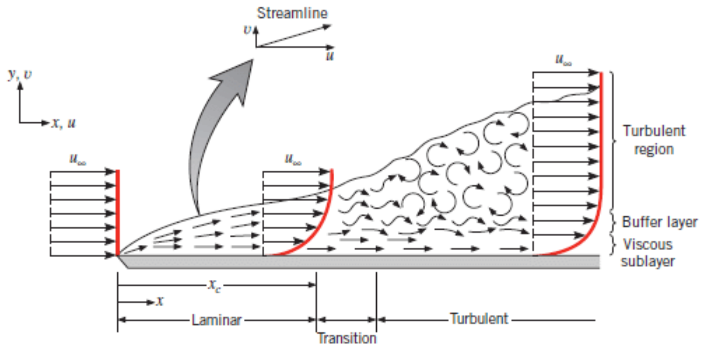
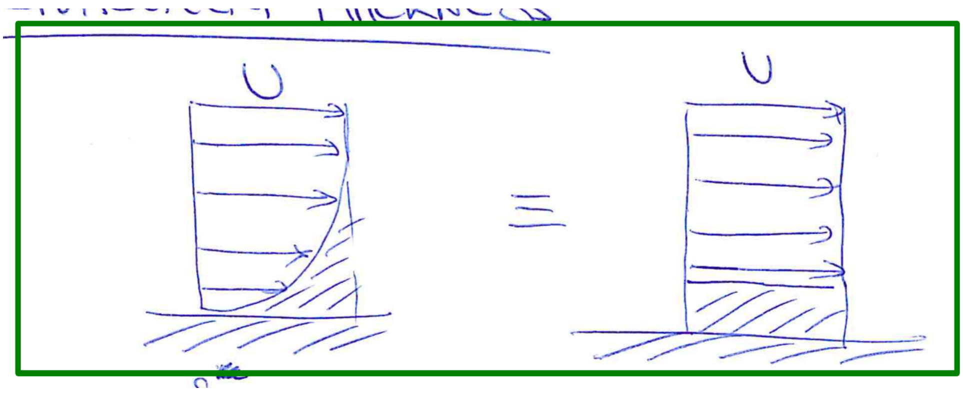


Figure 1: A typical boundary layer

Conventionally, the thickness of the boundary layer δ is the distance where the velocity reaches 99% of the velocity at ∞ (i.e. undisturbed)

Other useful measures measures in a boundary layer are:

- the **DISPLACEMENT THICKNESS** δ^*, which can be defined by equating the momentum deficit in the boundary layer to an uniform flow:



$$\int_0^\delta \rho(U-u)dy = \rho U \delta^* \tag{8}$$

$$\Rightarrow \delta^* = \int_0^\delta \left(1-\frac{u}{U}\right) dy \tag{9}$$

- the **MOMENTUM THICKNESS** θ:

$$\int_0^\delta \rho u(U-u)dy = \rho U^2 \theta \tag{10}$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right) dy \tag{11}$$

- the **ENERGY THICKNESS** δ_e:

$$\int_0^\delta \rho u(U^2-u^2)dy = \rho U^3 \delta_E \tag{12}$$

$$\delta_E = \int_0^\delta \frac{u}{U}\left(1 - \left(\frac{u}{U}\right)^2\right) dy \tag{13}$$

- the **SHAPE FACTOR** H:

  The shape factor is defined as:

  $$H = \frac{\delta^*}{\theta} > 1 \tag{14}$$

  By definition, the shape factor H is always greater than 1. Its value of H is a good indicator of the type of boundary layer (i.e. $H \sim 2.6$ in the laminar regime, $H \sim 1.4$ in the turbulent regime), on a flat plate.

  The value of H is also a good indicator of the pressure gradient. The larger the adverse pressure gradient, the larger H, and separation approximately occurs for:

  $$H \simeq 3.5 \quad \leftarrow \text{ LAMINAR B.L} \tag{15}$$

  $$H \simeq 2.4 \quad \leftarrow \text{ TURBULENT B.L.} \tag{16}$$

3 Boundary Layer - Order of Magnitude Argument

We consider the continuity and momentum equations for a steady incompressible flow:

$$\begin{aligned} &\text{CONT} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &\text{X-MOM} \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ &\text{Y-MOM} \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned} \tag{17}$$

and we analyse the orders of magntude of the different terms.
In the continuity equation:

$$u = O(U_\infty) \quad x = O(L) \quad y = O(\delta) \tag{18}$$

$$\begin{aligned} &\frac{\partial u}{\partial x} = O\left(\frac{U_\infty}{L}\right) \quad \frac{\partial v}{\partial y} = O\left(\frac{U_\infty}{L}\right) \\ &\Rightarrow v = O\left(\frac{U_\infty \delta}{L}\right) \end{aligned} \tag{19}$$

In the momentum equations the different terms are:

$$\begin{aligned} &\text{X-MOM:} \quad \underbrace{u\frac{\partial u}{\partial x}}_{O\left(\frac{U_\infty^2}{L}\right)} + \underbrace{v\frac{\partial u}{\partial y}}_{O\left(\frac{U_\infty^2}{L}\right)} = - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{O\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right)} + \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{O\left(\frac{\nu U_\infty}{L^2}\right)} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{O\left(\frac{\nu U_\infty}{\delta^2}\right)} \\ &\text{Y-MOM:} \quad \underbrace{u\frac{\partial v}{\partial x}}_{O\left(\frac{U_\infty^2 \delta}{L^2}\right)} + \underbrace{v\frac{\partial v}{\partial y}}_{O\left(\frac{U_\infty^2 \delta}{L^2}\right)} = - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial y}}_{O\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right)} + \underbrace{\nu\frac{\partial^2 v}{\partial x^2}}_{O\left(\frac{\nu U_\infty \delta}{L^3}\right)} + \underbrace{\nu\frac{\partial^2 v}{\partial y^2}}_{O\left(\frac{\nu U_\infty}{L\delta}\right)} \end{aligned} \tag{20}$$

If we divide by U_∞^2/L:

$$\begin{aligned} &\text{X-MOM:} \quad \underbrace{\frac{L}{U_\infty^2}u\frac{\partial u}{\partial x}}_{O(1)} + \underbrace{\frac{L}{U_\infty^2}v\frac{\partial u}{\partial y}}_{O(1)} = - \underbrace{\frac{L}{\rho U_\infty^2}\frac{\partial p}{\partial x}}_{?} + \underbrace{\frac{\nu L}{U_\infty^2}\frac{\partial^2 u}{\partial x^2}}_{O\left(\frac{\nu}{L U_\infty}\right)} + \underbrace{\frac{\nu L}{U_\infty^2}\frac{\partial^2 u}{\partial y^2}}_{O\left(\frac{\nu L}{\delta^2 U_\infty}\right)} \\ &\text{Y-MOM:} \quad \underbrace{\frac{L}{U_\infty^2}u\frac{\partial v}{\partial x}}_{O\left(\frac{\delta}{L}\right)} + \underbrace{\frac{L}{U_\infty^2}v\frac{\partial v}{\partial y}}_{O\left(\frac{\delta}{L}\right)} = - \underbrace{\frac{L}{\rho U_\infty^2}\frac{\partial p}{\partial y}}_{?} + \underbrace{\frac{\nu L}{U_\infty^2}\frac{\partial^2 v}{\partial x^2}}_{O\left(\frac{\nu\delta}{L^2 u_\infty}\right)} + \underbrace{\frac{\nu L}{U_\infty^2}\frac{\partial^2 v}{\partial y^2}}_{O\left(\frac{\nu}{\delta U_\infty}\right)} \end{aligned} \tag{21}$$

In a boundary layer ($where \delta << L$), the terms $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 v}{\partial x^2}$ can be neglected. In addition, we consider conditions with $Re = \frac{U_\infty L}{\nu} \to \infty$.

$$\begin{aligned}
&\text{X-MOM: } \underbrace{\frac{L}{U_\infty^2} u \frac{\partial u}{\partial x}}_{O(1)} + \underbrace{\frac{L}{U_\infty^2} v \frac{\partial u}{\partial y}}_{O(1)} = \underbrace{- \frac{L}{\rho U_\infty^2} \frac{\partial p}{\partial x}}_{?} + \underbrace{\frac{\nu L}{U_\infty^2} \frac{\partial^2 u}{\partial y^2}}_{O\left(\frac{\nu L}{\delta^2 U_\infty}\right)} \\
&\text{Y-MOM: } \underbrace{\frac{L}{U_\infty^2} u \frac{\partial v}{\partial x}}_{O\left(\frac{\delta}{L}\right)} + \underbrace{\frac{L}{U_\infty^2} v \frac{\partial v}{\partial y}}_{O\left(\frac{\delta}{L}\right)} = \underbrace{- \frac{L}{\rho U_\infty^2} \frac{\partial p}{\partial y}}_{?} + \underbrace{\frac{\nu L}{U_\infty^2} \frac{\partial^2 v}{\partial y^2}}_{O\left(\frac{\nu}{\delta U_\infty}\right)}
\end{aligned} \tag{22}$$

By looking at the x-momentum equation, we can consider 3 cases:

CASE A

There is an equilibrium between pressure and viscous forces and pressure forces. This would require the viscous terms to dominate over the inertial terms, i.e.

$$\frac{\mu L}{\delta^2 U_\infty} = \frac{1}{Re} \left(\frac{L}{\delta} \right)^2 \gg 1 \tag{23}$$

which means

$$Re \ll \left(\frac{L}{\delta} \right)^2 \tag{24}$$

This would mean that the inertial forces are negligible, which is impossible because it is against the original hypothesis $Re \to \infty$.

CASE B

There is an equilibrium between pressure forces and inertial forces. This is OK, but it basically means that we are not inside the BL (i.e. we are solving Euler equations).

$$\frac{1}{Re} \left(\frac{L}{\delta} \right)^2 \ll 1 \tag{25}$$

CASE C

In this case inertia, pressure and viscous forces are all important. Even in high Reynolds number flow, there is a region where viscous forces cannot be neglected.

$$\frac{1}{Re} \left(\frac{L}{\delta} \right)^2 = O(1) \tag{26}$$

$$\frac{\delta}{L} \not\propto \frac{1}{\mathrm{Re}} \tag{27}$$

Rearranging the previous equation:

$$\left(\frac{\delta}{L} \right) = O\left(\frac{1}{\sqrt{\mathrm{Re}}} \right) \tag{28}$$

Putting all together:

$$
\begin{aligned}
&\text{X-MOM: } \underbrace{\frac{L}{U_\infty^2}u\frac{\partial u}{\partial x}}_{O(1)}+\underbrace{\frac{L}{U_\infty^2}v\frac{\partial u}{\partial y}}_{O(1)}=-\underbrace{\frac{L}{\rho U_\infty^2}\frac{\partial p}{\partial x}}_{?}+\underbrace{\frac{\nu L}{U_\infty^2}\frac{\partial^2 u}{\partial y^2}}_{O(1)}\\
&\text{Y-MOM: } \underbrace{\frac{L}{U_\infty^2}u\frac{\partial v}{\partial x}}_{O(\frac{\delta}{L})}+\underbrace{\frac{L}{U_\infty^2}v\frac{\partial v}{\partial y}}_{O(\frac{\delta}{L})}=-\underbrace{\frac{L}{\rho U_\infty^2}\frac{\partial p}{\partial y}}_{?}+\underbrace{\frac{\nu L}{U_\infty^2}\frac{\partial^2 v}{\partial y^2}}_{O(\frac{\delta}{L})}
\end{aligned}
\tag{29}
$$

Coming now to the components of the pressure gradient:

$$
\begin{aligned}
\left(\frac{\partial(p/\rho U_\infty^2)}{\partial(x/L)}\right) &= O\left(1\right)\\
\left(\frac{\partial(p/\rho U_\infty^2)}{\partial(y/\delta)}\right) &= O\left(\left(\frac{\delta}{L}\right)^2\right)
\end{aligned}
\tag{30}
$$

This means that we can approximate the N-S equations in the boundary layer as:

$$
\begin{aligned}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}\\
\frac{\partial p}{\partial y} &= 0
\end{aligned}
\tag{31}
$$

These are the **PRANDTL BOUNDARY LAYER EQUATIONS** and are valid for $Re \gg 1$.

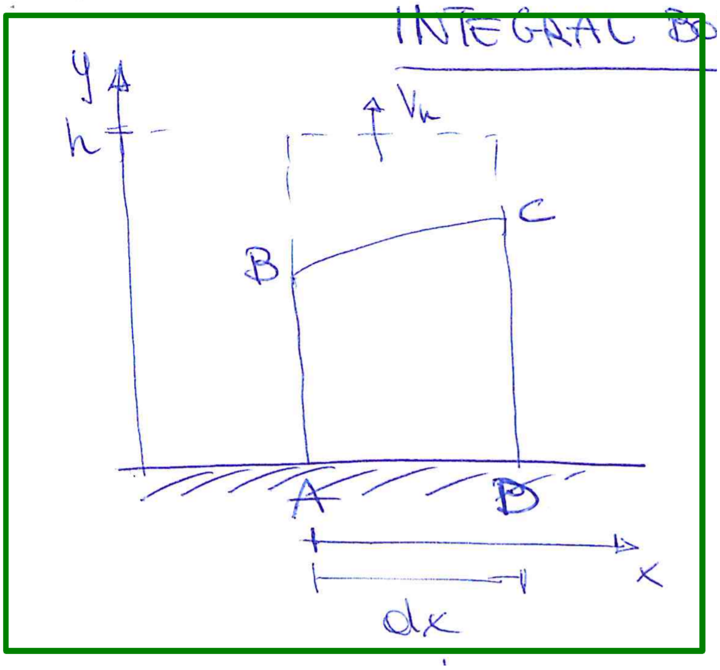
A few considerations:

- In a boundary layer $\frac{\partial^2 u}{\partial x^2}$ is negligible. The term $\frac{\partial^2 u}{\partial y^2}$ is significantly more important
- The pressure is not a function of y, but $p = p(x)$. This means that one could solve the flow outside the boundary layer, and then impose the pressure gradient $\frac{\partial p}{\partial x}$ to the boundary layer flow. If the flow can be assumed incompressible, outside the boundary layer

$$
P+\frac{1}{2}\rho U^2 = 0 \Rightarrow \frac{\partial p}{\partial x} = -\rho U\frac{dU}{dx}
\tag{32}
$$

- The equations are now parabolic and not elliptic, as the $\frac{\partial^2}{\partial x^2}$ terms have been dropped.

4 Integral Boundary Layer



CONTINUITY:

$$-\cancel{\int_0^h \rho u dy} + \cancel{\int_0^h \rho u dy} + \frac{d}{dx}\left(\int_0^h \rho u dy\right) dx + \rho V_h dx = 0 \tag{33}$$

$$\Rightarrow \rho V_h = -\frac{d}{dx}\left(\int_0^h \rho u dy\right) \tag{34}$$

MOMENTUM:

$$-\cancel{\int_0^h \rho u^2 dy} + \cancel{\int_0^h \rho u^2 dy} + \frac{d}{dx}\left(\int_0^h \rho u^2 dy\right) dx + \rho V_h U dx = -\tau_w dx - \left(\frac{\partial p}{\partial x}\right) h dx \tag{35}$$

Substituting V_h obtained from the continuity equation:

$$\frac{d}{dx}\int_0^h \rho u^2 dy - U d\left(\int_0^h \rho u dy\right) = -\tau_w - \left(\frac{\partial p}{\partial x}\right) h \tag{36}$$

We can write:

$$U\frac{d}{dx}\int_0^h \rho u dy = \frac{d}{dx}\left(U\int_0^h \rho u dy\right) - \left(\int_0^h \rho u dy\right)\frac{dU}{dx} \tag{37}$$

and

$$-\frac{dp}{dx} = \rho U \frac{dU}{dx} \tag{38}$$

Hence:

$$\frac{d}{dx}\int_0^h \rho u^2 dy - \frac{d}{dx}U\int_0^h \rho u dy + \int_0^h \rho u dy \frac{dU}{dx} = -\tau_w + \rho U\frac{dU}{dx}h \tag{39}$$

$$\frac{d}{dx}\int_0^h \rho u(u-U)dy + \frac{dU}{dx}\left(\int_0^h \rho u dy - \rho U\right) = -\tau_w \tag{40}$$

$$-\frac{d}{dx}\cdot\rho U^2\int_0^h \frac{u}{U}\left(1-\frac{u}{U}\right)dy - \frac{dU}{dx}\rho U\int_0^h\left(1-\frac{u}{U}\right)dy = -\tau_w \tag{41}$$

Recalling the definitions of boundary layer displacement thickness δ and momentum thickness θ, the above equation can be rewritten as:

$$\frac{d}{dx}\rho U^2\theta + \rho U\delta^*\frac{dU}{dx} = \tau_w \tag{42}$$

or:

$$\frac{d\theta}{dx} + 2\frac{\theta}{U}\frac{dU}{dx} + \frac{\delta^*}{U}\frac{dU}{dx} = \frac{\tau_w}{\rho U^2} \tag{43}$$

Introducing the shape factor H ($\delta^* = H\theta$):

$$\frac{d\theta}{dx} + \frac{H+2}{U}\theta\frac{dU}{dx} = \frac{\tau_w}{\rho U^2} = \frac{c_f}{2} \tag{44}$$

where $c_f = \tau_w/(\frac{1}{2}\rho U^2)$ is the friction coefficient.

The previous equation is valid for both LAMINAR and TURBULENT flows. When the pressure gradient is zero (i.e. for a flat plate) the equation becomes the VON KARMAN INTEGRAL BOUNDARY LAYER EQUATION:

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2} = \frac{C_f}{2} \tag{45}$$

The “beauty” of the Von KARMAN INTEGRAL B.L. equation is that, provided one can assume a reasonable form of $u(y)$, we can get reasonable estimates for a number of important quantities.

4.1 Example 1 (Laminar Boundary Laeyr

Let’s assume a parabolic boundary layer:

$$u(y) = ay^2 + by + c \tag{46}$$

with boundary conditions:

$$\begin{cases} u(y=0) = 0 \\ u(y=\delta) = U \\ \dfrac{\partial u}{\partial y}(y=\delta) = 0 \end{cases} \tag{47}$$

Substituting

$$\begin{cases} c = 0 \\ a\delta^2 + b\delta = U \quad \Rightarrow \quad a = -\dfrac{U}{\delta^2} \\ 2a\delta + b = 0 \quad \Rightarrow \quad b = \dfrac{2U}{\delta} \end{cases} \tag{48}$$

and therefore:

$$u(y) = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \tag{49}$$

$$\tau_w = \mu \frac{du}{dy}\bigg|_{y=0} = 2\mu \frac{U}{\delta} \tag{50}$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) dy = \tag{51}$$

$$= \frac{y^2}{\delta} - \frac{y^2}{\delta^2} + \frac{2y^3}{3\delta^2}\bigg|_0^\delta = \delta - \frac{\delta}{3} - \frac{4\delta}{3} + \frac{2\delta}{3} = \frac{\delta}{3} \tag{52}$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) dy \tag{53}$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right) dy \tag{54}$$

$$= \delta - \frac{\delta}{3} - \frac{4\delta}{3} + \frac{\delta}{2} + \frac{\delta}{2} - \frac{\delta}{5} = \tag{55}$$

$$= 2\delta - \frac{5\delta}{3} - \frac{\delta}{5} = \frac{30 - 25 - 3}{15} = \frac{2\delta}{15} \tag{56}$$

$$H = \frac{\delta^*}{\theta} = \frac{\delta}{3} \cdot \frac{15}{2\delta} = \frac{5}{2} \tag{57}$$

Substituting into the INTEGRAL B.L. EQUATION

$$\frac{d\theta}{dx} = 2\mu \frac{U}{\delta} \cdot \frac{1}{\rho U^2} = \frac{2\mu}{\rho U \delta} \tag{58}$$

$$\frac{2}{15}\frac{d\delta}{dx} = \frac{2\mu}{\rho U \delta} \quad \Rightarrow \quad \delta d\delta = \frac{15\mu}{\rho U} dx \tag{59}$$

The solution is

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U} x \tag{60}$$

$$\delta = \sqrt{30 \cdot \frac{\mu}{\rho U} x} \tag{61}$$

$$\frac{\delta}{x} = \sqrt{30} \cdot \sqrt{\frac{\mu}{\rho U x}} = \frac{\sqrt{30}}{\mathrm{Re}_x^{1/2}} \tag{62}$$

$$\frac{\delta}{x} \simeq \frac{5.5}{\mathrm{Re}_x^{1/2}} \tag{63}$$

We can then obtain

$$\frac{\delta^*}{x} = \frac{\delta}{3x} \simeq \frac{1.83}{\mathrm{Re}_x^{1/2}} \tag{64}$$

$$\frac{\theta}{x} = \frac{2\delta}{15x} \simeq \frac{0.73}{\mathrm{Re}_x^{1/2}} \tag{65}$$

$$c_f = \frac{2\mu U}{\delta} \cdot \frac{1}{\frac{1}{2}\rho U^2} = \frac{4\mu}{\delta \rho U} = \frac{4\mu}{\delta U x} \cdot \frac{x}{\delta} = \frac{4}{\mathrm{Re}_x^{1/2}} \cdot \frac{1}{5.5} \frac{0.73}{\mathrm{Re}_x^{1/2}} \simeq \tag{66}$$

$$c_D = \int_0^L \frac{0.73}{\sqrt{\nu x}} dx = \frac{0.73}{2} \cdot \sqrt{\frac{2 \cdot 0.73}{\mathrm{Re}_L}} = 1.46 \tag{67}$$

4.2 Turbulent Boundary Layer

Let's assume

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \qquad \tau_w = 0.0225\rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4} \tag{68}$$

INTEGRAL B.L. equation:

$$\frac{d\theta}{dx} = \frac{C_f}{2} = \frac{\tau_w}{\rho U^2} \tag{69}$$

$$\begin{aligned} \delta^* &= \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \left(\frac{y}{\delta}\right)^{1/7}\right) dy = \\ &= y - \frac{7}{8}\left(\frac{y}{\delta}\right)^{8/7} \delta \Big|_0^\delta = \delta\left(1 - \frac{7}{8}\right) = \frac{\delta}{8} \end{aligned} \tag{70}$$

$$\begin{aligned} \theta^* &= \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left(1 - \left(\frac{y}{\delta}\right)^{1/7}\right) dy \\ &= \int_0^\delta \left[\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right] dy = \frac{7}{8}\frac{(y/\delta)^{8/7}}{\delta^{1/7}} - \frac{7}{9}\frac{(y/\delta)^{9/7}}{\delta^{2/7}} \\ &= \left(\frac{7}{8} - \frac{7}{9}\right)(\delta) = \frac{63-56}{72}\delta = \frac{7}{72}\delta \end{aligned} \tag{71}$$

$$C_f = 0.045\left(\frac{\nu}{U\delta}\right)^{1/4} \tag{72}$$

Substituting into the integral BL equation:

$$\frac{7}{72}\frac{d\delta}{dx} = 0.0225 \cdot \left(\frac{\nu U}{\delta}\right)^{1/4} \tag{73}$$

$$\delta^{1/4} d\delta = \frac{72}{7} \cdot 0.0225 \cdot \left(\frac{\nu}{U}\right)^{1/4} dx \tag{74}$$

$$\frac{4}{5}\delta^{5/4} = \frac{72}{7} \cdot 0.0225 \cdot \left(\frac{\nu}{U}\right)^{1/4} x \tag{75}$$

$$\frac{\delta}{x} = \left(\frac{5}{4} \cdot \frac{72}{7} \cdot 0.0225\right)^{4/5} \cdot \left(\frac{\nu}{Ux}\right)^{1/5} = \tag{76}$$

$$\frac{\delta}{x} \simeq \frac{0.37}{\mathrm{Re}_x^{1/5}} \tag{77}$$

The other parameters:

$$\frac{\delta^*}{x} = \frac{0.046}{Re^{1/5}} \tag{78}$$

$$\frac{\theta}{x} = \frac{0.036}{Re^{1/5}} \tag{79}$$

$$c_f = 0.045 \cdot \left(\frac{\nu}{\delta U}\right)^{1/4} = 0.045 \cdot \left(\frac{\nu Re_x^{1/5}}{Ux0.37}\right)^{1/4} = \frac{0.045}{\left(\mathrm{Re}^{4/5}\right)^{1/4}} \cdot \frac{1}{(0.37)^{1/4}} \simeq \frac{0.058}{\mathrm{Re}_x^{1/5}} \tag{80}$$

$$\tau_w = 0.029 \cdot \rho U^2 \frac{1}{\mathrm{Re}_x^{1/5}} \tag{81}$$

5 Boundary Layer Separation

The integral boundary layer equation:

$$\frac{d\theta}{dx} + \frac{H+2}{U}\theta\frac{dU}{dx} = \frac{C_f}{2} \tag{82}$$

is valid for both laminar and turbulent boundary layers, even on complex surfaces. On a flat plate (where $\frac{dp}{dx} = 0$ and hence $\frac{dU}{dx} = 0$), $\frac{d\theta}{dx} = \frac{c_f}{2}$:

We can write an equilibrium of forces at the wall:

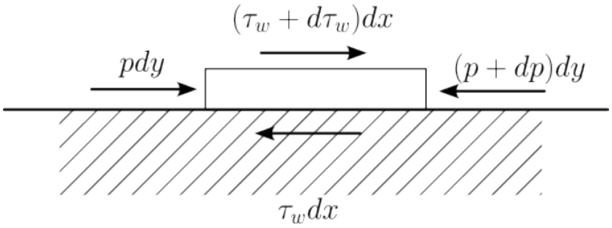


Figure 2: Equilibrium of forces at the wall

$$\frac{dp}{dx} = \frac{d\tau_w}{dy} = \mu\frac{d^2u}{dy^2} \tag{83}$$

$$\mu\left.\frac{d^2u}{dy^2}\right|_{y=0} = -\frac{dp}{dx} \tag{84}$$

The second derivative of the velocity will have the same sign of the pressure gradient. Given that the $\frac{d^2u}{dy^2}$ must be negative at the end of the boundary layer:

- if $\frac{dp}{dx} > 0$, the second derivative must go from positive to negative and therefore there will be an inflection point in the boundary layer.
- for $\frac{dp}{dx} < 0$, there will be NO inflection point within the boundary layer
- $\frac{dp}{dx} = 0$ (flat plate) the inflection point is at the wall.

H (shape factor) is a good indicator for separation: 2.4 for turbulent; 3.5 for laminar B.L.

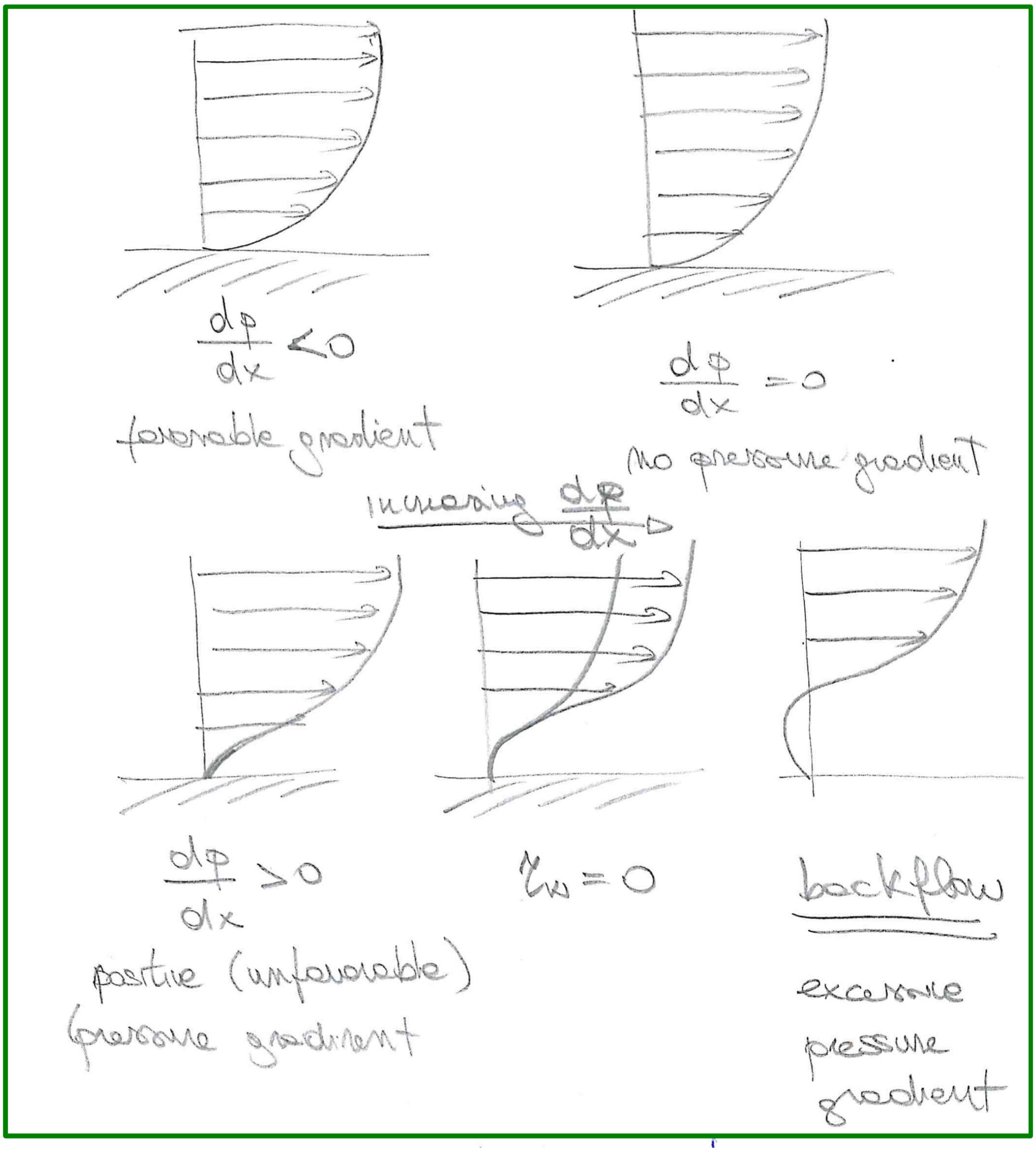


Figure 3: Boundary layer profiles with different pressure gradients

Clearly, in the presence of a pressure gradient, we cannot express the boundary layer with a parabolic shape, as this would lead to a constant curvature.

THWAITES idea was to add a further parameter:

$$m = \frac{\theta^2}{v}\left(\frac{d^2u}{dy^2}\right) \tag{85}$$

m represent a non-dimensional pressure gradient, since

$$\mu\frac{d^2u}{dy^2} = \mu\frac{d\tau_w}{dy} = \frac{dp}{dx} = -\rho U\frac{dU}{dx} \tag{86}$$

Hence

$$m = \frac{\theta^2}{v}\cdot\frac{dp}{dx} = -\frac{\theta^2}{\mu}\cdot\rho U\frac{dU}{dx} \tag{87}$$

He introduced also a non-dimensional wall-shear:

$$\ell = \frac{\theta}{\nu}\frac{du}{dy} \tag{88}$$

ℓ is a non-dimensional wall shear.

He also re-arranged the integral boundary layer equation as follows:

$$\frac{d\theta}{dx} - \frac{H+2}{U}\theta\frac{dU}{dx} = \frac{C_f}{2} \tag{89}$$

$$\frac{d\theta}{dx} - \frac{H+2}{U}\mu m\frac{\theta}{\rho\theta^2} = \frac{\tau_w}{\rho U^2} = \frac{\mu}{\rho U^2}\frac{\ell U}{\theta} \tag{90}$$

$$U\theta\frac{d\theta}{dx} = \nu\left[\ell + m(H+2)\right] \tag{91}$$

or

$$U\frac{d}{dx}(\theta^2) = 2\nu\left[\ell + m(H+2)\right] = \nu F(m) \tag{92}$$

From exaperiments, he approximated the $F(m)$ as

$$F(m) = 0.45 + 6m \tag{93}$$

leading to the following equation:

$$U\frac{d}{dx}(\theta^2) = (0.45 + 6m)\nu \tag{94}$$

$$U\frac{d}{dx}(\theta^2) - 0.45\nu + 6\frac{\theta^2 dU}{dx} = 0 \tag{95}$$

$$\frac{d}{dx}(U^6\theta^2) = 0.45\nu U^5 \tag{96}$$

$$\theta^2(x_1) = \theta_0^2\left(\frac{U(0)}{U(x_1)}\right)^6 + \frac{0.45\nu}{U(x_1)^6}\int_0^{x_1} U^5 dx \tag{97}$$

Using the previous equation, given the distributions of $U(x)$ (or $p(x)$), one can calculate the evolution of the boundary layer. Boundary layer separation is predicted for $m = 0.09$

$$m = -\frac{\theta^2 dU}{\nu dx} = \frac{\theta^2}{\mu U}\cdot\frac{dp}{dx} \tag{98}$$