Relazioni Fondamentali
(Fundamental Relations)

Course Notes

1 Continuity Equation (Mass Flow)

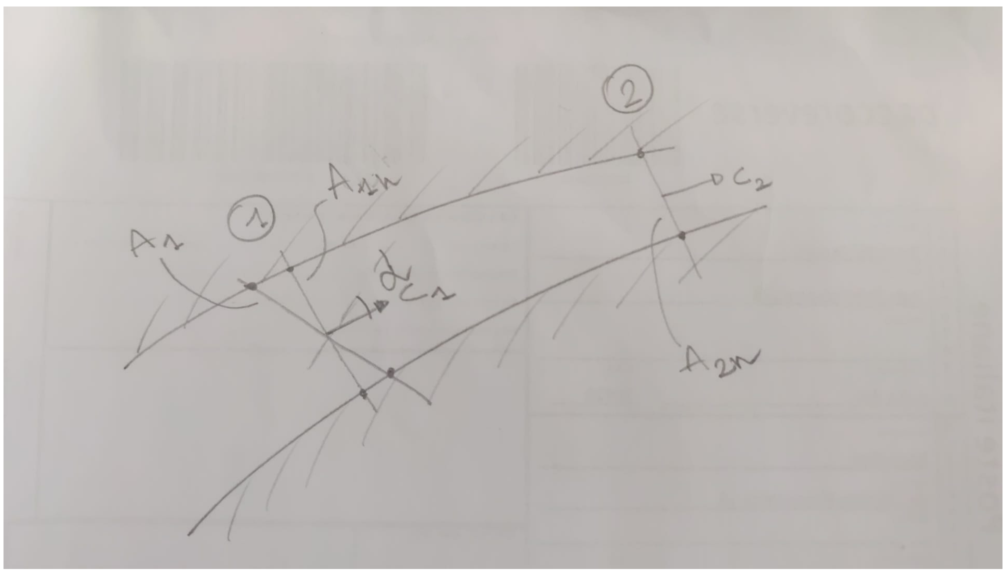

Figure 1: Streamline and control volume element

$$\dot{m} = \rho c A_n = \rho c A \cos\alpha$$

The previous equation assumes a constant velocity c throughout the section. If this is not the case:

$$\dot{m} = \int_{A1} \rho c dA_n = \int_{A1} \rho c \cos(\alpha) dA$$

If there is no accumulation of fluid between stations 1 and 2:

$$\dot{m}_1 = \dot{m}_2 \quad (\rho_1 c_1 A_{n1} = \rho_2 c_2 A_{n2})$$

Otherwise, if a different between $\dot{m}_1$ and $\dot{m}_2$ exists, the variation in mass M in the control volume encolsed between sections 1 and 2 is:

$$\frac{dM}{dt} = \frac{d \int_V \rho dV}{dt} = \dot{m}_1 - \dot{m}_2$$

2 First Law of Thermodynamics

2.1 For a System

$$E_2 - E_1 = Q - W$$

where E is the total energy:

$$E = U + \frac{1}{2}Mc^2 + Mgz$$

For an infinitesimal transformation:

$$dE = \delta Q - \delta W$$

2.2 For a Control Volume

$$\dot{Q} - \dot{W}_x = \dot{m}\left[(h_2 - h_1) + \frac{1}{2}(c_2^2 - c_1^2) + g(z_2 - z_1)\right]$$

$h =$ specific enthalpy, $h = u + \frac{p}{\rho}$
If we introduce the total specific enthalpy $h_0 = h + \frac{c^2}{2} + gz$:

$$\dot{Q} - \dot{W}_x = \dot{m}\left[(h_{02} - h_{01})\right]$$

If the system is adiabatic: $\dot{Q} = 0$
If we have a turbine ($\dot{W} > 0$):

$$\dot{W}_x = \dot{W}_t = \dot{m}(h_{01} - h_{02})$$

If we have a compressor ($\dot{W}_x < 0$):

$$-\dot{W}_x = \dot{W}_c = \dot{m}(h_{02} - h_{01})$$

3 Momentum Equation (Steady Transfer)

It expresses Newton's second law of motion.
For a system:

$$\sum F_x = \frac{d}{dt}(mc_x)$$

For a control volume:

$$\sum F_x = \dot{m}(c_{x2} - c_{x1})$$

3.1 Momentum of Momentum

For a system of mass M:

$$\mathrm{T}_A = m\frac{d}{dt}(rc_\theta)$$

$r \rightarrow$ distance of the mass center from the axis of rotation
$c_\theta \rightarrow$ tangential velocity component
For a control volume:

$$\mathrm{T}_A = \dot{m}(r_2c_{\theta 2} - r_1c_{\theta 1})$$

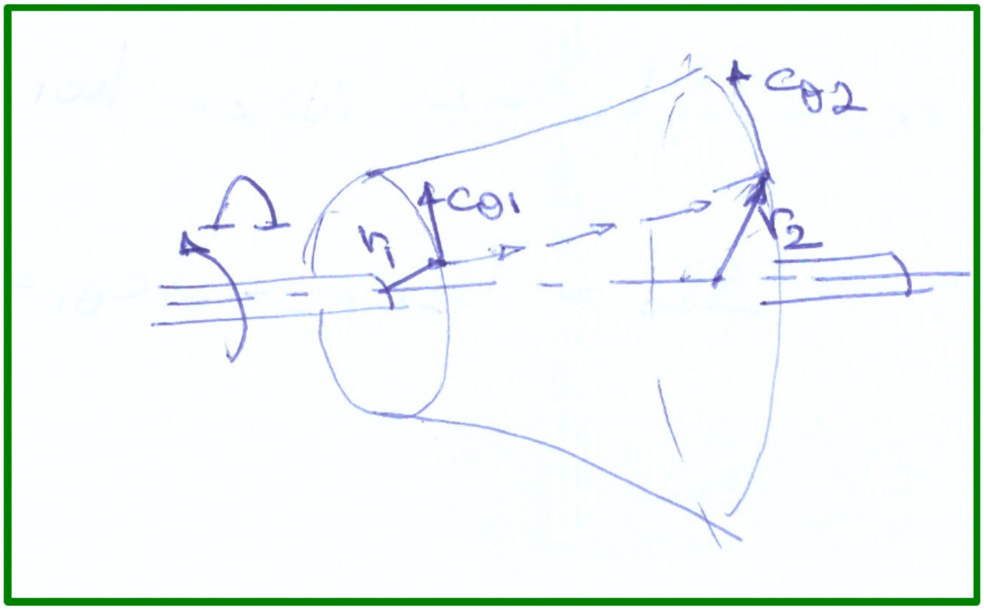

Figure 2: Moment of momentum in a turbomachine

4 Euler Work Equation

For a driven turbomachine (pump or compressor) running at an angular velocity Ω, the rate at which the rotor does work on the fluid is:

$$\dot{W}_c = \mathrm{T}_A\Omega = \dot{m}(r_2c_{\theta 2} - r_1c_{\theta 1})\Omega$$

$$= \dot{m}(U_2c_{\theta 2} - U_1c_{\theta 1})$$

where the blade speed $U = \Omega r$
The specific work is:

$$w_c = \frac{\dot{W}_c}{\dot{m}} = U_2c_{\theta 2} - U_1c_{\theta 1} > 0$$

In a turbine, the fluid does work on the rotor and therefore the sign is reversed:

$$\dot{W}_t = \dot{m}(U_1 c_{\theta 1} - U_2 c_{\theta 2})$$

or in terms of specific work:

$$\Delta w_t = U_1 c_{\theta 1} - U_2 c_{\theta 2}$$

Applying the steady energy equation:

$$-\Delta W_x = \Delta h_0 \implies \Delta W_t = h_{01} - h_{02}$$

COMPRESSOR: $w_c = \Delta h_0 = \Delta(U c_\theta) = U_2 c_{\theta 2} - U_1 c_{\theta 1}$
TURBINE: $w_T = -\Delta h_0 = -\Delta(U c_\theta) = U_1 c_{\theta 1} - U_2 c_{\theta 2}$
These are the general form of the Euler work equation. The assumptions we have made:

- adiabatic flow
- on a streamline

It is applicable to both viscous and inviscid flow, since the torque can be provided both by pressure and friction forces. It is strictly valid only for steady flows but it can be applied also to unsteady flows provided that averaging is done over a long enough time period.

NOTE: For a stationary blade $U = 0$ and therefore $\Delta h_0 = 0$

5 Rothalpy and General Relative Velocities

The Euler work equation ($\Delta h_0 = \Delta(U c_\theta)$) can be also written in the following form:

$$\Delta(h_0 - U c_\theta) = 0$$

We define the rothalpy $\boxed{I = h_0 - U c_\theta}$, that is always constant in a turbomachine (along a streamline, and provided that the flow is adiabatic).

$$I = h_0 - U c_\theta = h + \frac{1}{2}c^2 - U c_\theta$$

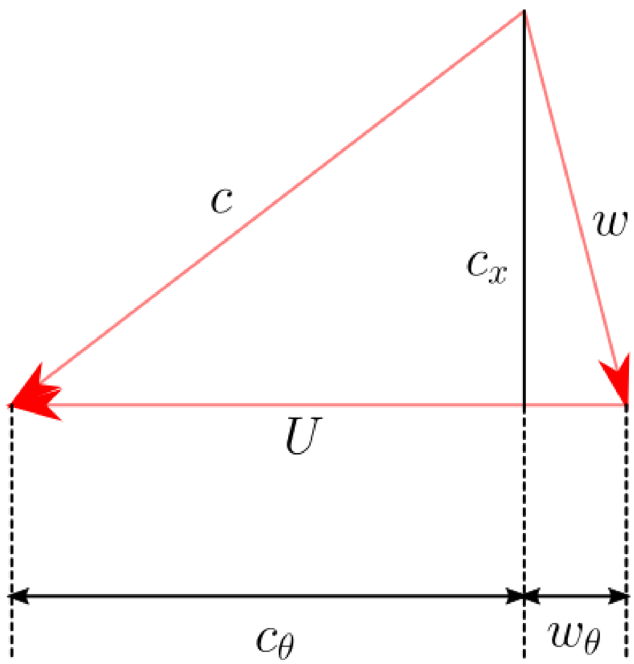

Figure 3: Velocity triangle

We can also write:

$$c^2 = c_x^2 + c_\theta^2 = c_x^2 + (U - w_\theta)^2$$

$$= c_x^2 + w_\theta^2 - 2w_\theta U + U^2$$

$$= w^2 + 2w_\theta U + U^2$$

Therefore:

$$I = h + \frac{1}{2}w^2 + \frac{1}{2}U^2 - w_\theta U - U(U - w_\theta) =$$

$$= h + \frac{1}{2}w^2 - \frac{1}{2}U^2 = h_{0,\mathrm{rel}} - \frac{1}{2}U^2$$

$$\boxed{I = h_{0,\mathrm{rel}} - \frac{1}{2}U^2 = h_0 - Uc_\theta}$$

This third form of the Euler turbomachinery equation shows that provided that U is constant, the relative stagnation enthalpy is constant.

6 Second Law of Thermodynamics

The inequality of Clausius states that for a system passing through a cycle involving heat exchanges:

$$\oint \frac{\delta Q}{T} \leq 0$$

If the process is reversible:

$$\oint \frac{dQ_R}{T} = 0$$

The entropy is defined as:

$$S_2 - S_1 = \int_1^2 \frac{dQ_R}{T}$$

In incremental form:

$$dS = mds = \frac{dQ_R}{T}$$

For a control volume:

$$\frac{\dot{Q}}{T} \leq \dot{m}(s_2 - s_1)$$

Alternatively, we can write:

$$\dot{m}(s_2 - s_1) = \frac{\dot{Q}_R}{T} + \Delta \dot{S}_{\text{IRREV}}$$

For an adiabatic process: $\dot{Q} = 0$, and therefore:

$$\boxed{s_2 \geq s_1}$$

and, if the process is reversible, $s_2 = s_1$
For a System of Mass m Undergoing a Reversible Process:

$$\delta Q = dQ_R = mTds$$

$$\delta W = dW_R = mpdv$$

Substituting into the first law of thermodynamics

$$dU = mdu = mTds - mpdv$$

And therefore

$$du = Tds - pdv$$

Since $h = u + \frac{p}{\rho} = u + pv \implies dh = du + pdv + vdp$

$$dh = Tds + vdp = Tds + \frac{dp}{\rho}$$

Since these equations are written in terms of properties of the system, they can be applied to a system undergoing any process.

Entropy in a turbomachine is a very important property as it is related to the loss of work. It is therefore important to track the sources of entropy.

7 Bernoulli's Equation

Let's consider the energy equation without heat transfer and work:

$$(h_2 - h_1) + \frac{1}{2}(c_2^2 - c_1^2) + g(z_2 - z_1) = 0$$

or $\Delta h_0 = 0$
In differential form:

$$dh + cdc + gdz = 0$$

If the flow is isentropic, i.e., no mixing or friction forces acting on our streamline:

$$dh = Tds + vdp = \frac{dp}{\rho}$$

and therefore:

$$\frac{dp}{\rho} + cdc + gdz = 0$$

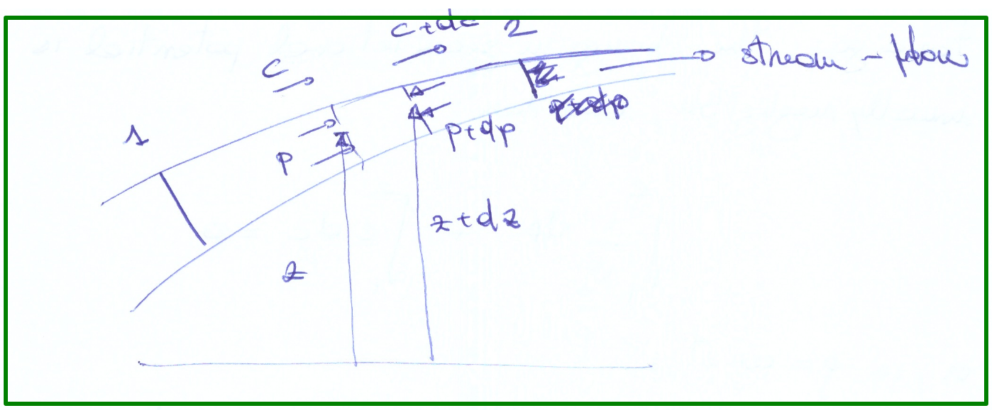

Figure 4: Streamline flow with pressure and velocity variations

By integrating:

$$\int_1^2 \frac{dp}{\rho} + \int_1^2 cdc + \int_1^2 gdz = 0$$

If the Flow is Incompressible ($\rho = \text{const}$)

$$\frac{1}{\rho}(p_2 - p_1) + \frac{1}{2}(c_2^2 - c_1^2) + g(z_2 - z_1) = 0$$

We can introduce the total pressure $p_0 = p + \frac{1}{2}\rho c^2$ (total pressure)

$$\frac{1}{\rho}(p_{02} - p_{01}) + g(z_2 - z_1) = 0$$

In hydraulic machines we often define the total head:

$$H = z + \frac{p_0}{g}$$

$$H_2 - H_1 = 0$$

For a gas, the change in gravitational potential is usually negligible, and then:

$$\int_1^2 \frac{dp}{\rho} + \int_1^2 c dc = 0$$

or, if $\rho = \text{const}$:

$$p_{02} - p_{01} = 0 \implies p_{02} = p_{01}$$

8 Ideal Gases

Ideal gases obey an equation of state

$$\frac{p}{\rho} = RT \qquad R = \frac{R_0}{M}$$

$$R_0 = 8314 \frac{\text{J}}{\text{kmol K}} \qquad M = 28.97 \text{ kg/kmol}$$

$$R_{\text{AIR}} = 287 \frac{\text{J}}{\text{kg K}}$$

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p = \frac{dh}{dT} \implies h = c_p T$$

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v = \frac{du}{dT} \implies u = c_v T$$

8.1 Perfect Gas

A perfect gas is a gas where c_p and c_v are constant:

$$T ds = dh - \frac{dp}{\rho} = c_p dT - \frac{dp}{\rho}$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R_v \ln \frac{p_2}{p_1}$$

9 Compressible Flows

In a compressible flow:

$$M = \frac{c}{a} = \frac{c}{\sqrt{\gamma RT}}$$

For a perfect gas:

$$h_0 = h + \frac{1}{2}c^2 \implies c_p T_0 = c_p T + \frac{1}{2}M^2 \gamma RT$$

$$= c_p T\left(1 + \frac{M^2\gamma R}{2c_p}\right)$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2$$

For an isentropic flow:

$$c_p dT = \frac{dp}{\rho} \implies \frac{dT}{T} = \frac{dp}{\rho T c_p} = \frac{\gamma - 1}{\gamma}\frac{dp}{p}$$

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

The same relationship can be used along a streamline for an isentropic process:

$$\frac{p_{02}}{p_{01}} = \left(\frac{T_{02}}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}}$$

Similarly

$$c_p dT = dh = \frac{dp}{\rho} = R\frac{d(\rho T)}{\rho} = RTd\rho + RdT$$

$$c_v\frac{dT}{T} = R\frac{d\rho}{\rho} \implies \frac{1}{\gamma - 1}\frac{dT}{T} = \frac{d\rho}{\rho}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

Another important Relation

$$\dot{m} = \rho c A n = \rho_0 \cdot \frac{\rho}{\rho_0} \cdot M\sqrt{\gamma R T_0}\sqrt{\frac{T}{T_0}}\frac{A_n}{\sqrt{\frac{T}{T_0}}}$$

$$= \frac{p_0}{\sqrt{RT_0}} \cdot \sqrt{\gamma R T_0}M\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)} A_n =$$

$$\dot{m} = \frac{p_0}{\sqrt{RT_0}}\sqrt{\gamma}M\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)} A_n$$

$$\frac{\dot{m}\sqrt{c_p T_0}}{A_n p_0} = \frac{\gamma}{\sqrt{\gamma - 1}}M\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)} = f(\gamma, M)$$