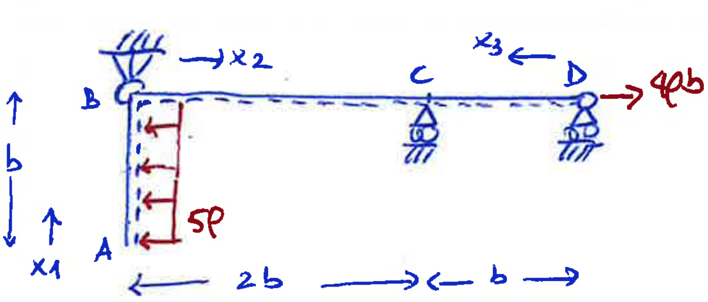
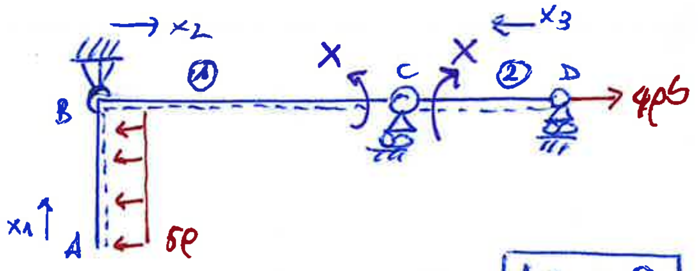
SdC SdA 2026 11 13 P2 TESTO 2

Ex. 1 RISOLVERE MEDIANTE IL PLV LA STRUTTURA IPERSTATICA ASSUMENDO COME INCOGNITA IPERSTATICA IL MOMENTO FLETTENTE SULL'APPOGGIO DI CONTINUITA' M_C
N.B. TENERE CONTO SOLO DELLA DEFORMABILITA' FLESSIONALE.

STRUTTURA REALE — 1 VOLTA IPERSTATICA: 1 CORPO; 3 GDL; 4 GDV

STRUTTURA CONGRUENTE — ISOSTATICA: 2 CORPI; 6 GDL; 6 GDV

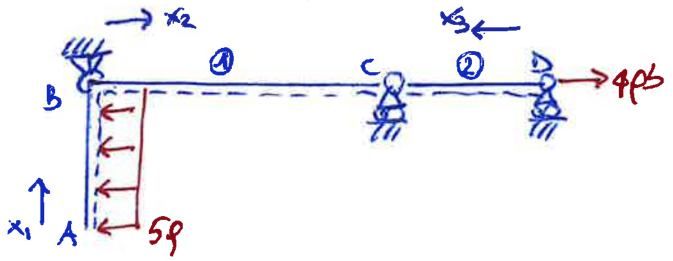




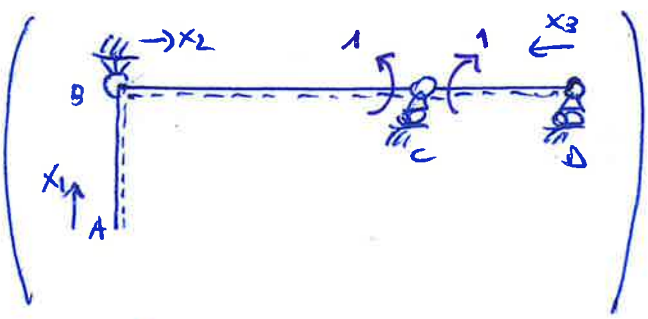
EQ. DI CONGRUENZA: $\boxed{\Delta\varphi_C = 0}$

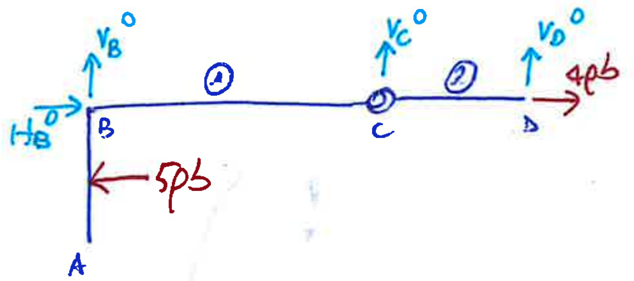
SISTEMA PRINCIPALE SP0

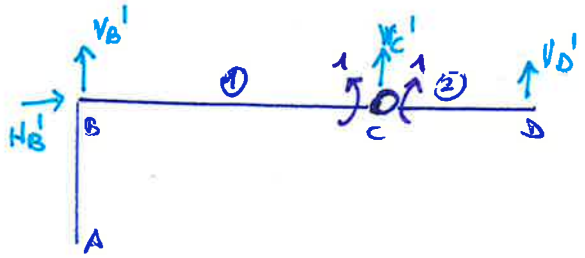
SISTEMA AUSILIARIO 1 SA1



$+ X$







EQ. CARDINALI (1+2)

$\rightarrow R_x = 0$ [1] $H_B^0 - 5pb + 4pb = 0$

$\uparrow R_y = 0$ [2] $V_B^0 + V_C^0 + V_D^0 = 0$

$\circlearrowleft M_{z(B)} = 0$ [3] $V_C^0\, 2b + V_D^0\, 3b - \frac{5}{2} p b^2 = 0$

EQ. AUSILIARIE: $M^{(1)}_{z(C)} = 0$ oppure $M^{(2)}_{z(D)} = 0$

$M^{(2)}_{z(C)} = 0$ [4] $V_D^0 b = 0$ $\quad \underline{V_D^0 = 0}$

[1] $H_B^0 - pb = 0$ $\quad \underline{H_B^0 = pb}$

[3] $V_C^0\, 2b + \cancel{V_D^0\, 3b} - 5/2\, pb^2 = 0$

$V_C^0\, 2b = 5/2\, pb^2$ $\quad \underline{V_C^0 = 5/4\, pb}$

[2] $V_B^0 + V_C^0 + \cancel{V_D^0} = 0$ $\quad \underline{V_B^0 = -V_C^0 = -5/4\, pb}$

EQ. CARDINALI (1+2)

$\rightarrow R_x = 0$ [5] $\underline{H_B^1 = 0}$

$\uparrow R_y = 0$ [6] $V_B^1 + V_C^1 + V_D^1 = 0$

$\circlearrowleft M_{z(B)} = 0$ [7] $V_C^1\, 2b + \cancel{1} - \cancel{1} + V_D^1\, 3b = 0$

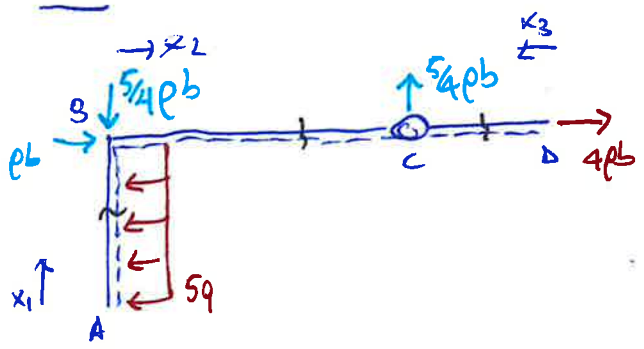
EQ. AUSILIARIE: $M^{(1)}_{z(C)} = 0$ oppure $M^{(2)}_{z(C)} = 0$

$M^{(2)}_{z(C)} = 0$ [8] $V_D^1 b - 1 = 0$ $\quad \underline{V_D^1 = 1/b}$

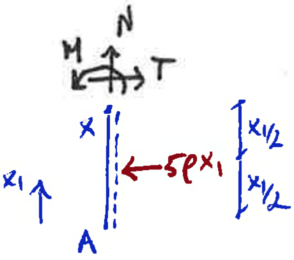
[7] $V_C^1\, 2b = -V_D^1\, 3b = -\frac{1}{\cancel{b}} \cdot 3\cancel{b}$ $\quad \underline{V_C^1 = -\frac{3}{2b}}$

[6] $V_B^1 = -V_C^1 - V_D^1 = +\frac{3}{2b} - \frac{1}{b}$ $\quad \underline{V_B^1 = \frac{1}{2b}}$

SP_0



$A \to B \quad 0 < x_1 < b$

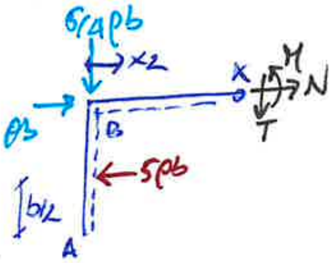


$N^0_{AB} = 0$

$T^0_{AB} = 5\rho b x_1$

$M^0_{AB} = 5/2\, \rho x_1^2$

$B \to C \quad 0 < x_2 < 2b$



$N^0_{BC} = 4\rho b$

$T^0_{BC} = -5/4\, \rho b$

$M^0_{BC} = 5/2\, \rho b^2 - 5/4\, \rho b x_2$

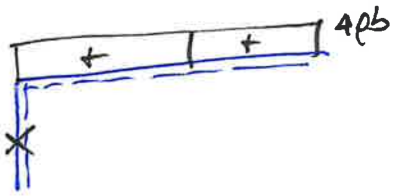
$D \to C \quad 0 < x_3 < b$

$N^0_{DC} = 4\rho b$

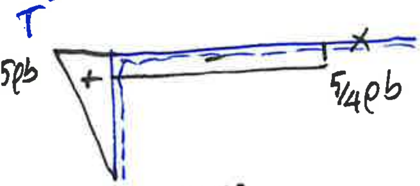
$T^0_{DC} = 0$

$M^0_{DC} = 0$

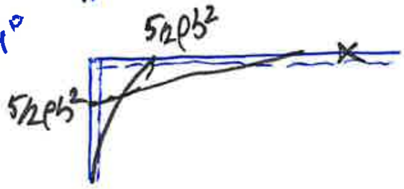
N^0



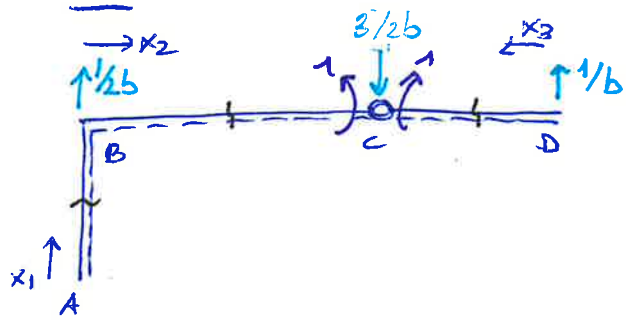
T^0



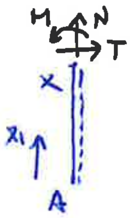
M^0



SX_1



$A \to B \quad 0 < x_1 < b$

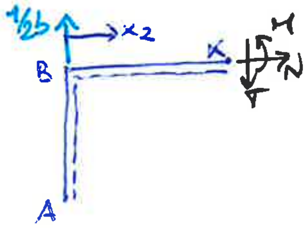


$N'_{AB} = 0$

$T'_{AB} = 0$

$M'_{AB} = 0$

$B \to C \quad 0 < x_2 < 2b$

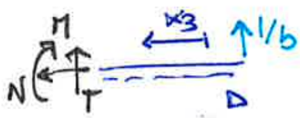


$N'_{BC} = 0$

$T'_{BC} = 1/2b$

$M'_{BC} = 1/2b\, x_2$

$D \to C \quad 0 < x_3 < b$



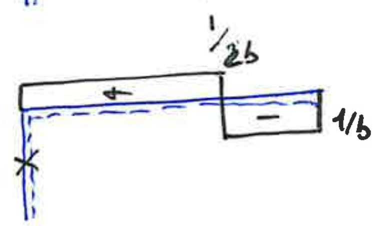
$N'_{DC} = 0$

$T'_{DC} = -1/b$

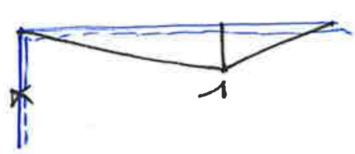
$M'_{DC} = 1/b\, x_3$

N'

T'



M'



PRINCIPIO DEI LAVORI VIRTUALI

$$\boxed{\delta \mathcal{L}_e = \delta \mathcal{L}_i}$$

$\delta \mathcal{L}_e = 1 \cdot (\varphi_c^1 - \varphi_c^2) = 1 \cdot \Delta\varphi_c = 1 \cdot 0 = 0$ N.B. EQ. CONGRUENZA: $\Delta\varphi_c = 0$

$\delta \mathcal{L}_i = \int_{\ell_i} M'(x)\,\chi_x\,dx = 0$ N.B. SI CONSIDERA SOLO LA DEFORMABILITÀ FLESSIONALE

$$\chi_x = \frac{M^0(x) + X M'(x)}{EJ} \rightarrow \delta \mathcal{L}_i = \int_{\ell_i} M'(x)\,\frac{M^0(x)+XM'}{EJ}\,dx = \underbrace{\int_{\ell_i} \frac{M^0(x)M'(x)}{EJ}\,dx}_{\eta_{10}} + X\underbrace{\int_{\ell_i} \frac{M'^2(x)}{EJ}\,dx}_{\eta_{11}}$$

$$\eta_{10} + X\eta_{11} = 0 \rightarrow X = -\frac{\eta_{10}}{\eta_{11}}$$

TRATTO	LUNGHEZZA	M^0	M'	M^0M'	M'^2
AB	b	$5/2\,pb\,x_1^2$	0	0	0
BC	$2b$	$5/2\,pb^2 - 5/4\,pb\,x_2$	$1/2b\;x_2$	$\frac{5}{4}pb\,x_2 - \frac{5}{8}q\,x_2^2$	$\frac{1}{4b^2}x_2^2$
DC	b	0	$1/b\;x_3$	0	$1/b^2\;x_3^2$

$$\eta_{10} = \int_{\ell_i} \frac{M^0(x)M'(x)}{EJ}\,dx = \frac{1}{EJ}\int_0^{2b}\left(\frac{5}{4}qb\,x_2 - \frac{5}{8}q\,x_2^2\right)dx =$$

$$= \frac{1}{EJ}\left[\frac{5}{4}pb\,\frac{x_2^2}{2} - \frac{5}{8}q\,\frac{x_2^3}{3}\right]_0^{2b} =$$

$$= \frac{1}{EJ}\left(\frac{5}{4}pb\,\frac{4b^2}{2} - \frac{5}{8}q\,\frac{8b^3}{3}\right) = \frac{1}{EJ}\left(\frac{5}{2}qb^3 - \frac{5}{3}qb^3\right) = \frac{5pb^3}{6EJ}$$

$$\eta_{11} = \int_{\ell_i} \frac{M'^2(x)}{EJ}\,dx = \frac{1}{EJ}\int_0^{2b}\frac{1}{4b^2}x_2^2\,dx + \frac{1}{EJ}\int_0^{b}\frac{1}{b^2}x_3^2\,dx =$$

$$= \frac{1}{EJ}\left[\frac{1}{4b^2}\frac{x_2^3}{3}\right]_0^{2b} + \frac{1}{EJ}\left[\frac{1}{b^2}\frac{x_3^3}{3}\right]_0^{b} =$$

$$= \frac{1}{EJ}\left(\frac{1}{4b^2}\frac{8b^3}{3}\right) + \frac{1}{EJ}\left(\frac{1}{b^2}\frac{b^3}{3}\right) = \frac{1}{EJ}\left(\frac{2b}{3} + \frac{b}{3}\right) = \frac{b}{EJ}$$

$$X = -\eta_{10}/\eta_{11} = -\frac{5pb^3}{6EJ}\cdot\frac{EJ}{b} = -\frac{5pb^2}{6} \qquad \boxed{X = -5/6\,pb^2}$$

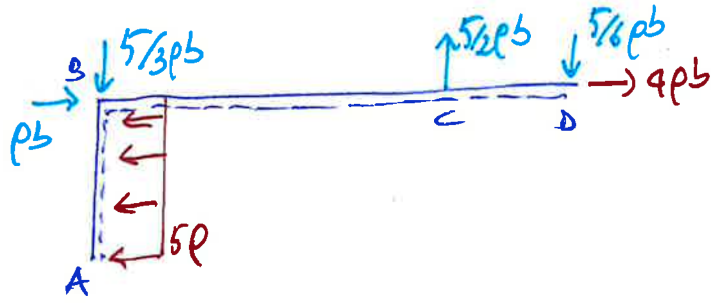
SOVRAPPOSIZIONE DEGLI EFFETTI

$$H_B = H_B^0 + X(H_B') = qb - \cancel{\tfrac{5}{6}pb^2(0)} = pb$$

$$V_B = V_B^0 + X(V_B') = -\tfrac{5}{4}pb - \tfrac{5}{6}pb^2\left(\tfrac{1}{2b}\right) = -\tfrac{5}{3}pb$$

$$V_C = V_C^0 + X(V_C') = \tfrac{5}{4}pb - \tfrac{5}{6}pb^2\left(-\tfrac{3}{2b}\right) = \tfrac{5}{2}pb$$

$$V_D = V_D^0 + X(V_D') = 0 - \tfrac{5}{6}pb^2\left(\tfrac{1}{b}\right) = -\tfrac{5}{6}pb$$



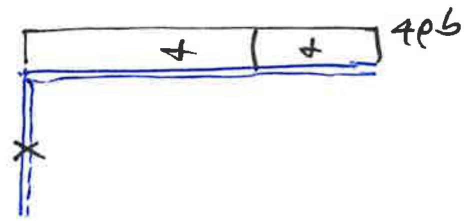
(*)

$$N_{AB} = N^0_{AB} + X N'_{AB} = 0 - \cancel{\tfrac{5}{6}pb^2(0)} = 0 \quad (**)$$

$$N_{BC} = N^0_{BC} + X N'_{BC} = 4pb - \cancel{\tfrac{5}{6}pb^2(0)} = 4qb$$

$$N_{DC} = N^0_{DC} + X N'_{DC} = 4pb - \cancel{\tfrac{5}{6}pb^2(0)} = 4pb$$

N

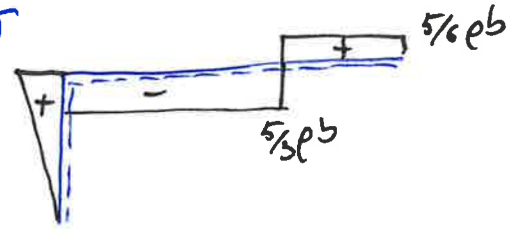


$$T_{AB} = T^0_{AB} + X T'_{AB} = 5pbx_1 - \cancel{\tfrac{5}{6}pb^2(0)} = 5pbx_1 \quad (**)$$

$$T_{BC} = T^0_{BC} + X T'_{BC} = -\tfrac{5}{4}pb - \tfrac{5}{6}pb^2\left(\tfrac{1}{2b}\right) = -\tfrac{5}{3}pb$$

$$T_{DC} = T^0_{DC} + X T'_{DC} = 0 - \tfrac{5}{6}pb^2\left(-\tfrac{1}{b}\right) = \tfrac{5}{6}pb$$

T

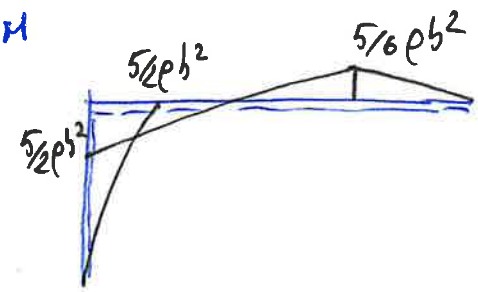


$$M_{AB} = M^0_{AB} + X M'_{AB} = \tfrac{5}{2}px_1^2 - \cancel{\tfrac{5}{6}pb^2(0)} = \tfrac{5}{2}pX_2^1 \quad (**)$$

$$M_{BC} = M^0_{BC} + X M'_{BC} = \left(\tfrac{5}{2}pb^2 - \tfrac{5}{4}pbx_2\right) - \tfrac{5}{6}pb^2\left(\tfrac{1}{2b}x_2\right) =$$
$$= \tfrac{5}{2}pb^2 - \tfrac{5}{3}pbx_2$$

$$M_{DC} = M^0_{DC} + X M'_{DC} = 0 - \tfrac{5}{6}pb^2\left(\tfrac{1}{b}x_3\right) = -\tfrac{5}{6}pbx_3$$

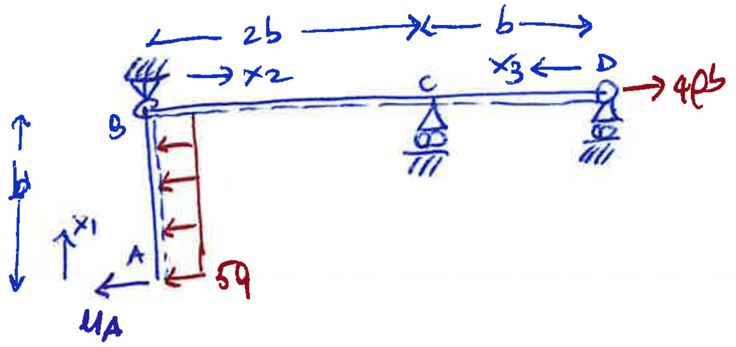
M



(*) N.B. PUÒ ESSERE PIÙ VELOCE CALCOLARE DIRETTAMENTE N, T, M UNA VOLTA NOTE LE REAZIONI VINCOLARI (OTTENUTE DALLA SOVRAPPOSIZIONE DEGLI EFFETTI)

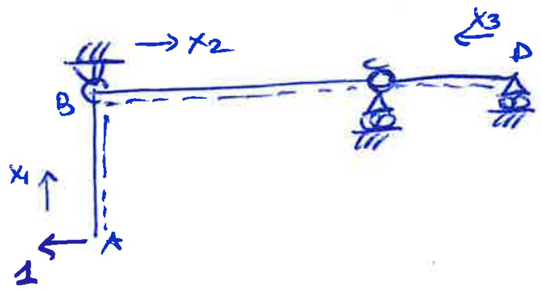
(**) N.B. IL TRATTO AB È UN'APPENDICE ISOSTATICA, I VALORI DELLE AZIONI INTERNE NON DIPENDONO DALLA X

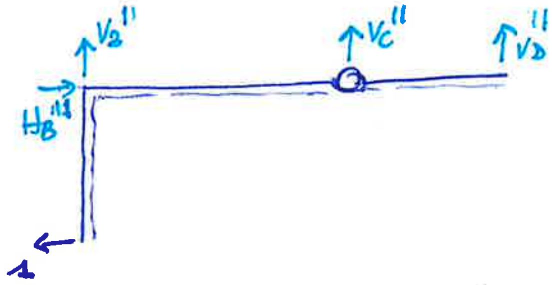
EX 1 CALCOLARE, APPLICANDO IL PLV, LA COMPONENTE DI SPOSTAMENTO ORIZZONTALE DEL PUNTO A, u_A.



$u_A = ?$

SISTEMA AUSILIARIO 2





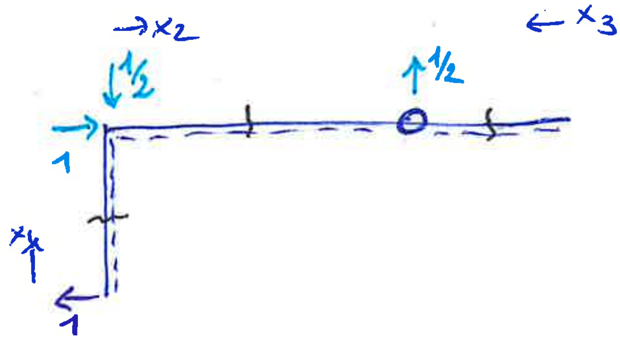
$$\begin{cases} \rightarrow R_x = 0 \quad [1]\ H_B'' - 1 = 0 \\ \uparrow R_y = 0 \quad [2]\ V_B'' + V_C'' + V_D'' = 0 \\ \circlearrowleft M_{t(B)} = 0 \quad [3]\ V_C'' 2b + V_D'' 3b - 1b = 0 \end{cases}$$

$\underline{H_B'' = 1}$

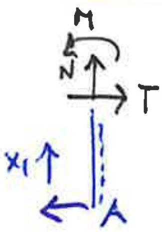
$\circlearrowleft M^{(2)}_{t(C)} = 0 \quad [4]\ V_D'' b = 0 \qquad \underline{V_D = 0}$

$[3]\quad V_C'' 2b - 1b = 0 \qquad V_C'' = \underline{1/2}$

$[4]\quad V_B'' + 1/2 = 0 \qquad V_B'' = \underline{-1/2}$



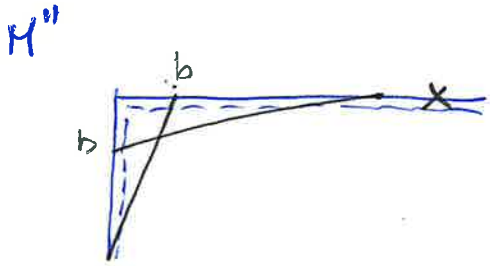
A → B $\quad 0 < x_1 < b$



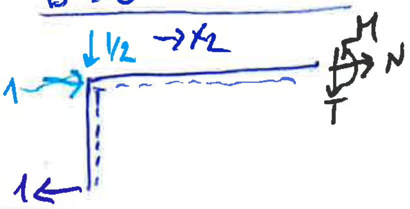
$M''_{AB} = x_1$

D → C $\quad 0 < x_3 < b$

$M''_{DC} = 0$



B → C $\quad 0 < x_1 < 2b$



$M''_{BC} = b - \frac{1}{2} x_2$

PLV $\boxed{\delta\mathcal{L}_e = \delta\mathcal{L}_i}$

(*) N.B. M(x) È IL MOMENTO DELLA STRUTTURA REALE

$$\delta\mathcal{L}_e = 1\cdot u_A$$

$$\delta\mathcal{L}_i = \int_{\ell_i} M''(x)\,\chi_x\,dx = \int_{\ell_i} M''(x)\,\frac{M(x)^{(*)}}{EJ}\,dx$$

TRATTO	LUNGHEZZA	M	M''	$M M''$
AB	b	$\frac{5}{2}p x_1^2$	x_1	$\frac{5}{2}p x_1^3$
BC	$2b$	$\frac{5}{2}pb^2 - \frac{5}{3}pb x_2$	$b - \frac{x_2}{2}$	$\frac{5}{2}pb^3 - \frac{5}{4}pb^2 x_2 - \frac{5}{3}pb^2 x_2 + \frac{5}{6}pb x_2^2$
DC	b	$-\frac{5}{6}pb x_3$	0	0

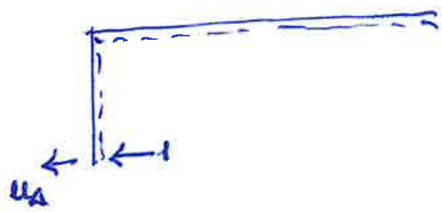
$$\delta\mathcal{L}_i = \frac{1}{EJ}\int_0^b \frac{5}{2}p x_1^3\,dx + \frac{1}{EJ}\int_0^{2b}\left(\frac{5}{2}pb^3 - \frac{5}{4}pb^2 x_2 - \frac{5}{3}pb^2 x_3 + \frac{5}{6}pb\,x_2^2\right)dx =$$

$$= \frac{1}{EJ}\left[\frac{5}{2}p\frac{x_1^4}{4}\right]^b + \frac{1}{EJ}\left[\frac{5}{2}pb^3 x_2 - \frac{5}{4}pb^2\frac{x_2^2}{2} - \frac{5}{3}pb^2\frac{x_2^2}{2} + \frac{5}{6}pb\frac{x_2^3}{3}\right]^{2b} =$$

$$= \frac{1}{EJ}\left(\frac{5pb^4}{8} + 5pb^4 - \frac{20pb^4}{8} - \frac{20pb^4}{6} + \frac{40pb^4}{18}\right) = \frac{1}{EJ}pb^4\left(\frac{45-360-180-240+160}{72}\right) = \frac{145pb^4}{72EJ}$$

$$\delta\mathcal{L}_e = \delta\mathcal{L}_i \rightarrow \boxed{\frac{145pb^4}{72EJ} = u_A}\ ^{(**)}$$

(**) N.B. ABBIAMO IPOTIZZATO:



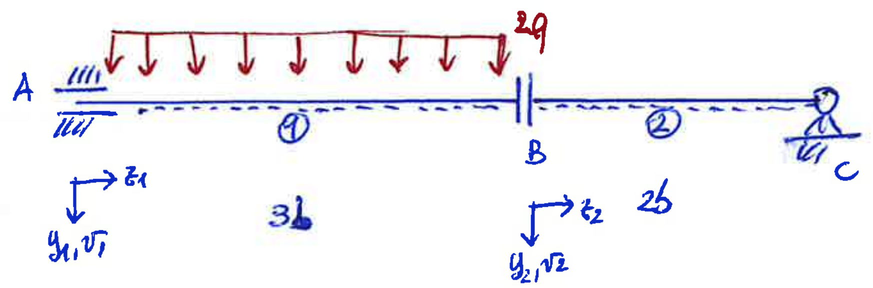
u_A E LA FORZA UNITARIA SONO CONCORDI $\rightarrow$ $\delta\mathcal{L}_i$ POSITIVO

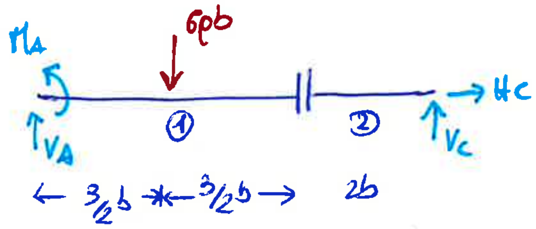
$\delta\mathcal{L}_e$ È ANCHE ESSO POSITIVO, PERTANTO IL VERSO DI u_A È QUELLO IPOTIZZATO, MA NEL NOSTRO SISTEMA DI RIFERIMENTO $\overset{y}{\uparrow}\!\rightarrow x$ È NEGATIVO, PERTANTO

$$u_A = -\frac{145\,pb^4}{72\,EJ}$$

Ex. 2

Per la struttura isostatica determinare reazioni e azioni interne, le condizioni a contorno nei punti A, B e C, e utilizzare l'equazione della linea elastica per determinare la deformata della linea d'asse $v(z)$ e la sua derivata prima $v'(z)$, la rotazione del punto C φ_C e lo spostamento verticale del punto B del tratto AB $v_B^{(1)}$





eq. cardinali (1+2)

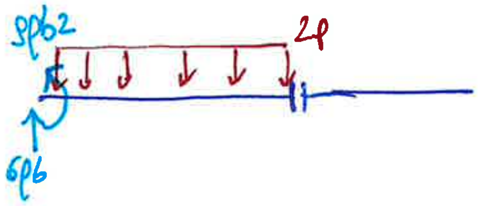
$$\begin{cases} \rightarrow R_x = 0 & [1] \quad \underline{H_C = 0} \\ \uparrow R_y = 0 & [2] \quad V_A + V_C - 6pb = 0 \\ \circlearrowleft M_{z(A)} = 0 & [3] \quad M_A - 6pb\frac{3}{2}b + V_C 5b = 0 \end{cases}$$

eq. ausiliarie: $R_y^{(1)} = 0$ oppure $R_y^{(2)} = 0$

$R_y^{(2)} = 0 \quad [4] \quad \underline{V_C = 0}$

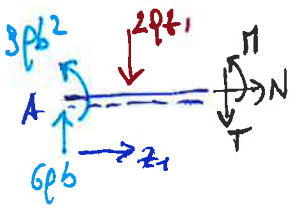
$[2] \quad V_A + \cancel{V_C} - 6pb = 0 \quad \underline{V_A = 6pb}$

$[3] \quad M_A - 9pb^2 + \cancel{V_C 5b} = 0 \quad \underline{M_A = 9pb^2}$



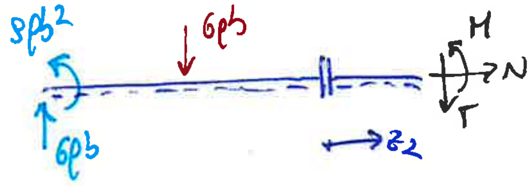
A→B

$0 < z_1 < 3b$



B→C

$0 < z_2 < 2b$



$N_{AB} = 0$

$T_{AB} = 6pb - 2pz_1$

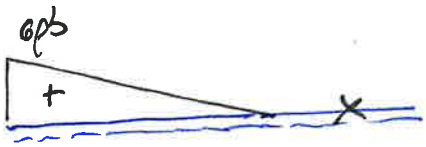
$M_{AB} = -9pb^2 + 6pbz_1 - pz_1^2$

$N_{BC} = 0$

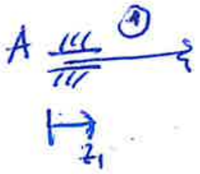
$T_{BC} = 0$

$M_{BC} = 0$

T

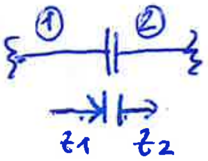


M

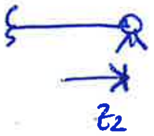
A 

CONDIZIONI A CONTORNO I $v_1(z_1=0)=0$

II $v_1'(z_1=0)=0$

B 

// // III $v_1'(z_1=3b)=v_2'(z_2=0)$

C 

// // IV $v(z_2=2b)=0$

1 $A \rightarrow B$ $0<z_1<3b$

$$v_1''(z_1)=-\frac{M_x}{EJ} \rightarrow \frac{9pb^2}{EJ}-\frac{6pb}{EJ}z_1+\frac{q}{EJ}z_1^2$$

$$v_1'(z_1)=\frac{9pb^2}{EJ}z_1-\frac{6pb}{EJ}\frac{z_1^2}{2}+\frac{q}{EJ}\frac{z_1^3}{3}+A_1$$

$$v_1(z_1)=\frac{9pb^2}{EJ}\frac{z_1^2}{2}-\frac{6pb}{2EJ}\frac{z_1^3}{3}+\frac{q}{3EJ}\frac{z_1^4}{4}+A_1z_1+A_2$$

2 $B \rightarrow C$ $0<z_2<2b$

$$v_2''(z_2)=-\frac{M_x}{EJ} \rightarrow 0$$

$$v_2'(z_2)=B_1$$

$$v_2(z_2)=B_1z_2+B_2$$

4 COSTANTI DI INTEGRAZIONE A_1, A_2, B_1 E B_2 DA DETERMINARE MEDIANTE LE CONDIZIONI A CONTORNO

I $v_1(z_1=0)=0 \rightarrow A_2=0$

II $v_1'(z_1=0)=0 \rightarrow A_1=0$

III $\quad v_1'(z_1=3b) = v_2'(z_2=0)$

$$[\circ] \quad \frac{9pb^2}{EJ}3b - \frac{6pb}{EJ}\frac{(3b)^2}{2} + \frac{9}{EJ}\frac{(3b)^3}{3} + \cancel{A_1} = B_1$$

IV $\quad v_2(z_2=2b)=0 \quad [\circ\circ] \; B_1 2b + B_2 = 0 \qquad B_2 = -B_1 2b$

$$[\circ] \; B_1 = \frac{27pb^3}{EJ} - \frac{54pb^3}{2EJ} + \frac{27pb^3}{3EJ} = \cancel{\frac{27pb^3}{EJ}} - \cancel{\frac{27pb^3}{EJ}} + \frac{9pb^3}{EJ} = \frac{9pb^3}{EJ}$$

$$[\circ\circ] \quad B_2 = -B_1 2b = -\frac{9pb^3}{EJ}\cdot 2b = -\frac{18pb^4}{EJ}$$

$$v_1(z_1) = \frac{9pb^2}{2EJ}z_1^2 - \frac{pb}{EJ}z_1^3 + \frac{9}{12EJ}z_1^4$$

$$v_1'(z_1) = \frac{9pb^2}{EJ}z_1 - \frac{3pb}{EJ}z_1^2 + \frac{9}{3EJ}z_1^3$$

$$v_2(z_2) = \frac{9pb^3}{EJ}z_2 - \frac{18pb^4}{EJ}$$

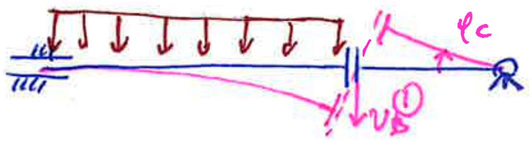
$$v_2'(z_2) = \frac{9pb^3}{EJ}$$

ROTAZIONE IN C, φ_C : $\quad v_2'(z_2=2b) \rightarrow \quad \varphi_C = \dfrac{9pb^3}{EJ} \quad (\circlearrowright)$

SPOSTAMENTO VERTICALE IN B DEL TRATTO AB, $v_B^{(1)}$: $\quad v_1(z_1=3b)$

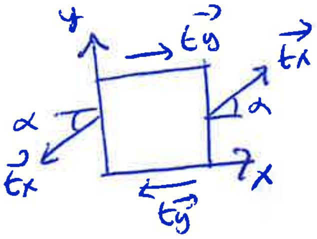
$$v_C = \frac{9pb^2}{2EJ}(3b)^2 - \frac{pb}{EJ}(3b)^3 + \frac{9}{12EJ}(3b)^4 = \frac{81pb^4}{2EJ} - \frac{27pb^4}{EJ} + \frac{\overset{27}{\cancel{81}}pb^4}{\underset{4}{\cancel{12}}EJ} =$$

$$= \frac{(162 - 108 + 27)pb^4}{4EJ} = \frac{81pb^4}{4EJ} \quad (\downarrow)$$



DEFORMATA (QUALITATIVA)

Ex. 3



$|\vec{t_x}| = 20\ MPa$

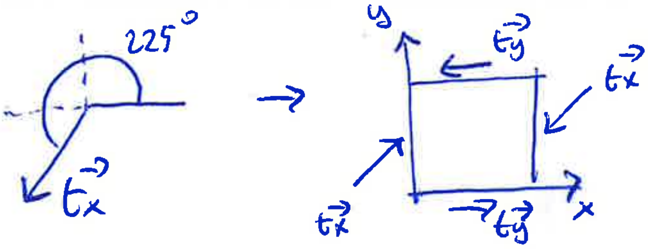
$\alpha = 225°$

DETERMINARE $\sigma_x, \sigma_y, \tau_{xy}$

COSTRUIRE IL CERCHIO DI MOHR

DETERMINARE $\sigma_1, \sigma_2, \tau_{max}$ E L'ANGOLO φ TRA L'ASSE x E LA NORMALE DELLA FACCIA SU CUI AGISCE σ_1

$\alpha = 225° \rightarrow$



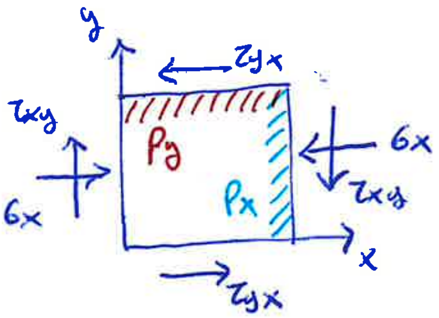
$$\sigma_x = |\vec{t_x}| \cdot \cos\alpha = -20 \cdot \frac{\sqrt{2}}{2} = -14,142\ MPa$$

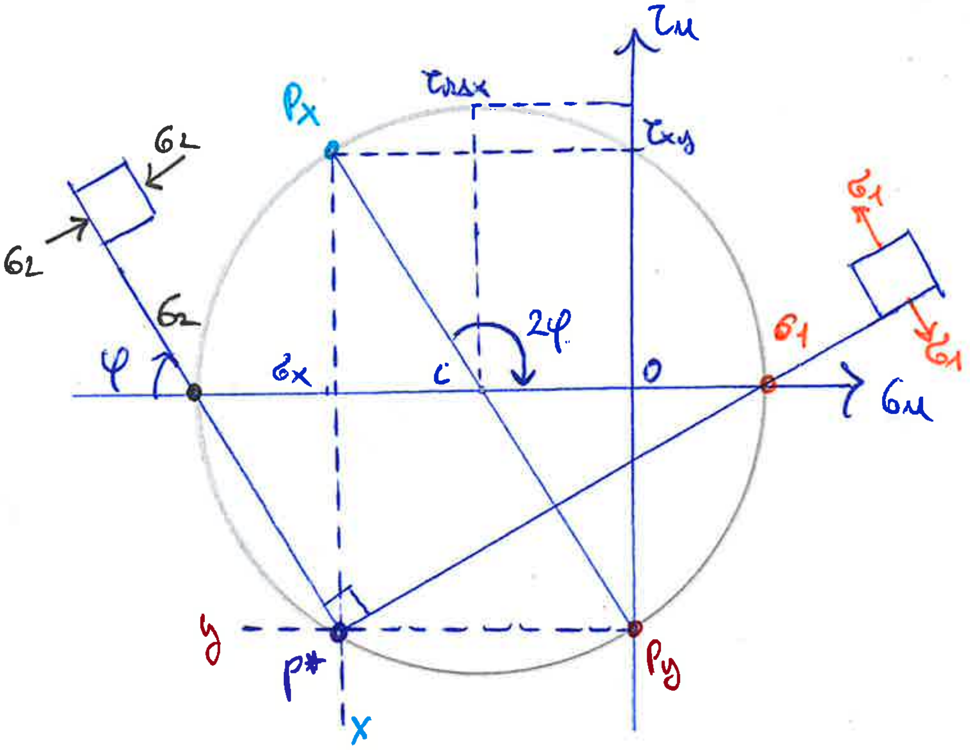
$$\sigma_y = 0,000\ MPa$$

$$\tau_{xy} = |\vec{t_x}| \cdot \sin\alpha = -20\frac{\sqrt{2}}{2} = -14,142\ MPa$$

$$\tau_{xy} = \tau_{yx} = -14,142\ MPa$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} -14,142 & -14,142 & 0 \\ -14,142 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$





$P_x = (-14,142;\ +14,142)$

$P_y = (0,000;\ -14,142)$

N.B.

$$C = \left(\frac{\sigma_x+\sigma_y}{2};0\right) = (-7,071;0)$$

$$R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = \sqrt{(-7,071)^2+14,142^2} = 15,811$$

$\tau_{max} = 15,811\ MPa$

$$\sigma_1 = \sigma_c + R = (-7,071+15,811) = \underline{8,740\ MPa}$$

$$\sigma_2 = \sigma_c - R = (-7,071-15,811) = \underline{-22,882\ MPa}$$