On mathematical models for competitive cycling

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Introduction

The essence of mathematical physics — as opposed to theoretical physics — is a quantitative analogy of physical phenomena without necessarily searching for their fundamental causes. For instance, we argue, in fluid mechanics, that the power required to overcome the resistance of the fluid is linearly proportional to the cube of speed. Models resulting from such formulations are referred to as phenomenological; they must be consistent with laws of physics but are not derived from them. These models allow for an intuitive understanding of physical process but require experimental measurements to support their empirical adequacy, which — notably — is their very purpose. The emphasis of this presentation is not on technical details — which, notably, are available in the two articles listed in references below — but on epistemological issues encountered in such studies.

Mathematical formulations

Velodrome: Let us consider the power required by a cyclist to maintain a steady effort on a velodrome. As discussed by Bos et al. (2023), the corresponding phenomenological model — in terms of dissipating forces — can be written as

$$P_F = \frac{1}{1-\lambda}\Bigg\{ \tag{1a}$$

$$\left(\mathrm{C_{rr}}\underbrace{\overbrace{m\,g}^{F_g}(\sin\theta\tan\vartheta+\cos\theta)}_{N}\cos\theta+\mathrm{C_{sr}}\left|\underbrace{\overbrace{m\,g}^{F_g}\frac{\sin(\theta-\vartheta)}{\cos\vartheta}}_{F_f}\right|\sin\theta\right)v \tag{1b}$$

$$+\ \tfrac{1}{2}\,\mathrm{C_d A}\,\rho\,V^3\Bigg\}, \tag{1c}$$

where m is the combined mass of the cyclist and the bicycle, g is the acceleration due to gravity, θ is the track-inclination angle, ϑ is the bicycle-cyclist lean angle, $\mathrm{C_{rr}}$ is the rolling-resistance coefficient, $\mathrm{C_{sr}}$ is the coefficient of the lateral friction, $\mathrm{C_d A}$ is the air-resistance coefficient, ρ is the air density and λ is the drivetrain-resistance coefficient.

In expression (1), v is the speed of the contact point between the wheels and the track; we refer to it as the black-line speed. The wheels are assumed to roll without slipping, so that v is also the tangential speed of a point on the circumference of a wheel with respect to the axle. V is the centre-of-mass speed. Since lateral friction is a dissipative force, it does negative work, and the work done against it — as well as the power — is positive. For this reason, in expression (1b), we consider the magnitude, $|\;|$.

We assume that the steady effort of a cyclist is tantamount to a constant centre-of-mass speed. Hence, there is no change in kinetic energy. Furthermore, we assume the cyclist to be instantaneously in rotational equilibrium. Hence, there is work done to increase potential energy, ΔU, resulting from a monotonic decrease in the lean angle from ϑ_1 to ϑ_2. The average power, per lap, required for this increase is

$$\overline{P}_U = \frac{1}{1-\lambda}\,\frac{\Delta U}{t_\circlearrowleft}\,, \tag{2}$$

where $t_\circlearrowleft$ is the lap time and

$$\Delta U = 2\,m\,g\,h\,\left(\cos\vartheta_2 - \cos\vartheta_1\right)\,,$$

since the bicycle-cyclist system straightens twice per lap, upon exiting a curve. In total, P is the sum of P_F and $\overline{P}_U$. This model lends itself to examining strategies to obtain desired results in terms of available power.

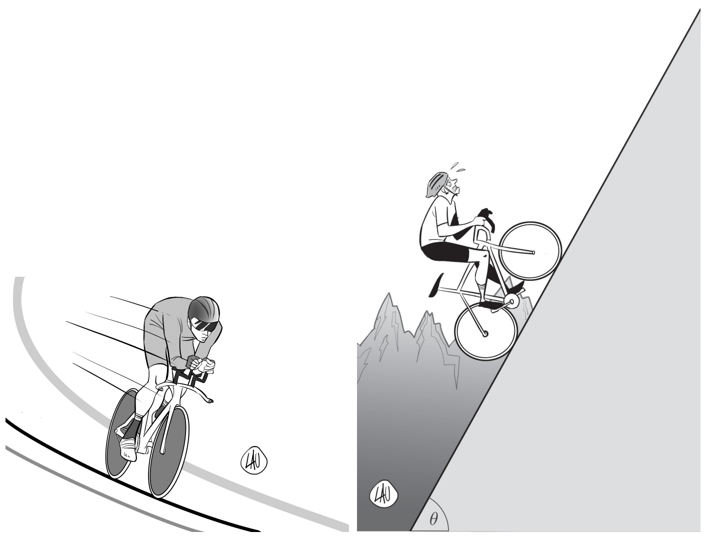

Figure 1: Cycling efforts on a velodrome and an uphill

Uphill: Now, let us consider the power required by a cyclist to climb a hill. As discussed by Bos et al. (2024), the corresponding phenomenological model can be written as

$$P = \frac{\overbrace{m\,\frac{\mathrm{d}V}{\mathrm{d}t}}^{\text{change in speed}} + \overbrace{m\,g\sin\theta}^{\text{change in elevation}} + \overbrace{\mathrm{C_{rr}}\underbrace{m\,g\cos\theta}_{\text{normal force}}}^{\text{rolling resistance}} + \overbrace{\tfrac{1}{2}\,\mathrm{C_d}\mathrm{A}\,\rho\,V^2}^{\text{air resistance}}}{\underbrace{1-\lambda}_{\text{drivetrain efficiency}}}\,V\,, \tag{3}$$

where symbols stand for the same quantities as in expression (1), except for θ, which stands for the slope, as indicated in Figure 1. Model (3) lends itself to optimization to establish strategies to minimize the ascent time, given the constrain of an average speed. Bos et al. (2024) formulate such strategies in the context of a constant instantaneous speed and a constant instantaneous power.

Let us highlight a few differences between the two models. In expression (1), we distinguish between V and v; in expression (3), we do not. Since we assume that the cyclist does not lean, $V \equiv v$, and consequently, $C_{sr} \equiv 0$. Furthermore, in expression (1), there is no term analogous to the first term in the numerator of expression (3), since V is assumed to be constant.

One could argue that such choices are arbitrary. Hence, the need for an experimental support.

Experimental measurements

To illustrate experimental measurements we examine briefly the model stated in expression (3).[1] To do so, we use the hill — referred to as the San Bernardo segment — whose elevation gain is 162 m. As shown in Figure 2, the slope is variable; the average and maximum grades are 4.8 % and 8.4 %, respectively. The time of the ascent performed specifically for our study was 543 s, which is 9 minutes and 3 seconds.

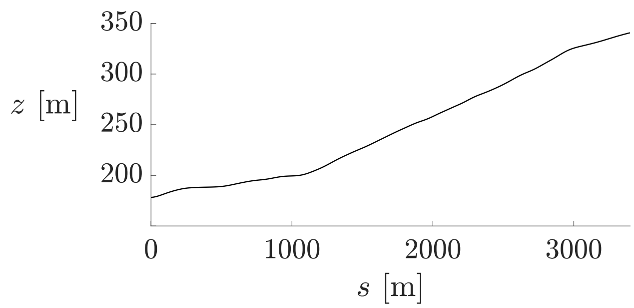

Figure 2: San Bernardo segment, which is a 3rd category climb in the Italian foothills near Ovada; the horizontal axis is with respect to arclength, s, along its trajectory.

The ascent was attempted with constant power to obtain experimental measurements whose data can be analyzed to examine the model in question. As shown in Figure 3, the measured power oscillates about its average of $\overline{P} = 322.0699$, in such a manner that its standard deviation is 33.9902 W. These oscillations are — at least in part — a consequence of cyclocomputer measurements which are sampled only once a second and, hence, occur necessarily at random instants of pedal revolutions.

To examine the model, we consider the following values in expression (3): $m = 78.6\,\mathrm{kg}$, as reported by the rider, $\mathrm{C_dA} = 0.35\,\mathrm{m}^2$, $\mathrm{C_{rr}} = 0.005$, $\lambda = 0.02$, as estimated from standard measurements, $\rho = 1.1464\,\mathrm{kg/m^3}$, as calculated based on altitude and temperature, and $g = 9.81\,\mathrm{m/s^2}$. According to the constant-power strategy — with $P_0 = 322\,\mathrm{W}$ — the ascent time is 541.7489 s, which is 9 minutes and 2 seconds; the ascent times agree to within one second. This specific agreement is

[1]Measurements on a velodrome will be discussed during the presentation in the context of strategizing the two successful Hour-Record attempts in 2022: Daniel Bigham and Filippo Ganna, for whom we formulated and calibrated the model stated in expressions (1) and (2); I was present at the velodrome for both preliminary tests and setting the records.

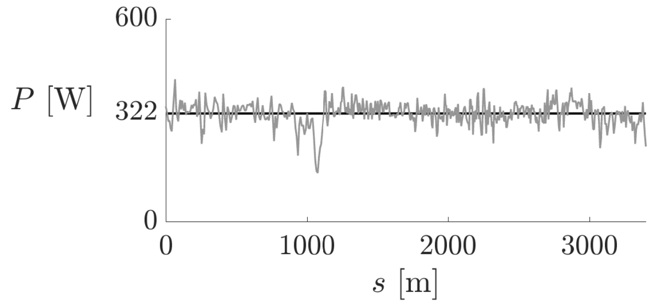

Figure 3: Comparison between average (black) and instantaneous (grey) power

obtained by adjusting — within the range of values expected for the rider in question — the three model parameters, C_dA, C_{rr} and λ. This limited range of adjustments ensures that the measurements can refute the conjectured model, in the spirit of Popper's conjectures and refutations. In other words, an agreement between predictions and measurements supports the empirical adequacy of the model.

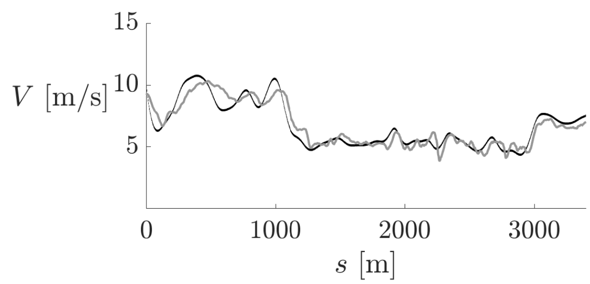

Figure 4: Comparison between modelled (black) and measured (grey) instantaneous ground speeds

Another agreement between the model and measurements is the instantaneous speed, shown in Figure 4. This agreement is symptomatic of the model's pertinence in examining certain subtleties, not only its capacity of predicting average or global quantities. It indicates that the modeling process is analogous with the temporal evolution of phenomena in question.

This initial examination supports the empirical adequacy of the model. More rides are planned in the foreseeable future to be followed by a statistical analysis.

References

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