07 - 11 - 25

## Note (tutoraggio del 31-10):

L'es. 3.2 contiene un errore di trascrizione: le derivate parziali sono state calcolate correttamente

$$f_x = (3x^2-1)(y^2-1) \quad ; \qquad f_y = (x^3-x)(2y) \, ;$$

ma poi $f_y$ è stata copiata male nel sistema $\nabla f = 0$:

$$\begin{cases} (3x^2-1)(y^2-1) = 0 \\ (x^3-1)\cdot 2y = 0 \end{cases}$$

I punti stazionari corretti sono:

$$\begin{cases} (3x^2-1)(y^2-1) = 0 \\ (x^3-x)\cdot 2y = 0 \end{cases} ; \quad \begin{cases} (3x^2-1)(y^2-1) = 0 \\ y \cdot x\,(x^2-1) = 0 \end{cases} ;$$

$$\begin{cases} (3x^2-1)(y^2-1) = 0 \\ y = 0 \end{cases} \vee \begin{cases} (3x^2-1)(y^2-1) = 0 \\ x = 0 \end{cases} \vee \begin{cases} (3x^2-1)(y^2-1) = 0 \\ x^2-1 = 0 \end{cases} ;$$

$$\begin{cases} 3x^2-1 = 0 \\ y = 0 \end{cases} \vee \begin{cases} y^2-1 = 0 \\ x = 0 \end{cases} \vee \begin{cases} y^2 = 1 \\ x^2 = 1 \end{cases} ;$$

$$\begin{cases} x^2 = \frac{1}{3} \\ y = 0 \end{cases} \vee \begin{cases} y^2 = 1 \\ x = 0 \end{cases} \vee \begin{cases} y^2 = 1 \\ x = +1 \end{cases} \vee \begin{cases} y^2 = 1 \\ x = -1 \end{cases}$$

$A = \left(\sqrt{\frac{1}{3}}, 0\right)$ $\qquad C = (0,1)$ $\qquad E = (1,1)$ $\qquad G = (-1,1)$

$B = \left(-\sqrt{\frac{1}{3}}, 0\right)$ $\qquad D = (0,-1)$ $\qquad F = (1,-1)$ $\qquad H = (-1,-1)$

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## Soluzioni esercizi dello scorso tutoraggio:

- DEF: Sia $f$ definita in un dominio $D \subseteq \mathbb{R}^2$. Il punto $P_0 = (x_0, y_0) \in D$ si definisce PUNTO DI SELLA per $f$ se è un punto stazionario $(\nabla f(P_0) = 0)$ e $\forall B_\delta(P_0)$

esistono $P_1, P_2 \in B_\delta(P_0)$ tali che

$$f(P_1) < f(P_0) < f(P_2)$$

- Es. 3.2. b) $\quad g(x,y) = xy\, e^{-(x^2+y^2)}$

$N = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) \rightsquigarrow \mathrm{Hess}_g(N) = \begin{pmatrix} e^{-1}\cdot(-2) & 0 \\ 0 & e^{-1}\cdot(-2) \end{pmatrix}$

$\to \det \mathrm{Hess}_g(N) = 4e^{-2} > 0 \quad$ e $\quad f_{xx}(N) = -2e^{-1} < 0 \to$ MAX. REL

$P = \left(+\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) \rightsquigarrow \mathrm{Hess}_g(P) = \begin{pmatrix} e^{-1}(+2) & 0 \\ 0 & e^{-1}\cdot(+2) \end{pmatrix}$

$\to \det \mathrm{Hess}_g(P) = 4e^{-2} > 0 \quad$ e $\quad f_{xx}(P) = +2e^{-1} > 0 \to$ MIN. REL

$Q = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) \rightsquigarrow \mathrm{Hess}_g(Q) = \begin{pmatrix} e^{-1}(+2) & 0 \\ 0 & e^{-1}\cdot(+2) \end{pmatrix}$

$\to \det \mathrm{Hess}_g(Q) = 4e^{-2} > 0 \quad$ e $\quad f_{xx}(Q) = +2e^{-1} > 0 \to$ MIN. REL

$R = \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) \rightsquigarrow \mathrm{Hess}_g(R) = \begin{pmatrix} e^{-1}\cdot(-2) & 0 \\ 0 & e^{-1}\cdot(-2) \end{pmatrix}$

$\to \det \mathrm{Hess}_g(R) = 4e^{-2} > 0 \quad$ e $\quad f_{xx}(R) = -2e^{-1} < 0 \to$ MAX. REL

- Es. 3.5) $\quad f(x,y) = \frac{\sqrt{3}}{3}x + y$

$f_x = \frac{\sqrt{3}}{3} \quad ; \quad f_y = 1 \quad ; \quad \nabla f = \left(\frac{\sqrt{3}}{3}, 1\right)$ costante.

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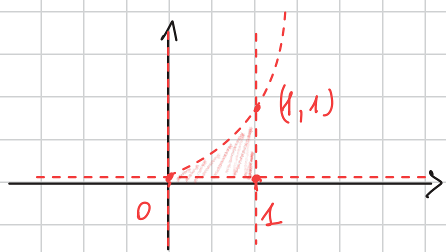
Lista 4 — Integrali Doppi

[Ripasso di geometria e domini normali rispetto ad un asse]

Alcuni esempi:




$A = \{ x \in [0,1], \quad 0 \le y \le x^2 \}$

$0 \le y \le x^2$ é compreso tra $\begin{matrix} y = 0 \\ y = x^2 \end{matrix}$

(dominio normale rispetto all'asse $x$).

$B = \{ e^y \le x \le e \,, \quad 0 \le y \le 1 \}$

(dominio normale rispetto all'asse $y$).

Per integrali doppi su domini normali abbiamo 2 formule di riduzione:

$$\iint_D f(x,y)\,dx\,dy$$

D NORM. RISP. ASSE $x$

$$\int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$$

D NORM. RISP. ASSE $y$

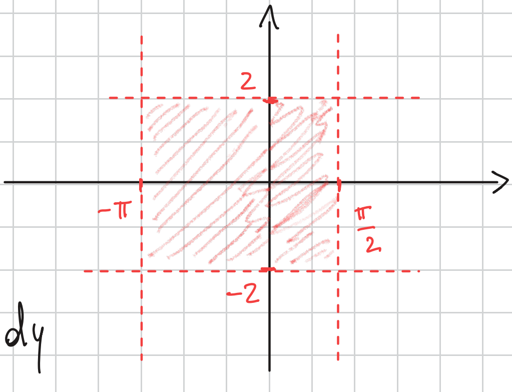
$$\int_c^d dy \int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx$$

Esercizi:

4.1. $$\iint_D \left(y \cdot \cos x - x e^y\right) dx\,dy \,, \qquad D = \underbrace{\left[-\pi, \frac{\pi}{2}\right]}_{\Downarrow \, x} \times \underbrace{\left[-2, 2\right]}_{\Downarrow \, y}$$

D é un dominio normale sia all'asse $x$, sia all'asse $y$.

Vediamolo come norm. risp. asse $x$:




$$\iint_D \left(y \cos x - x e^y\right) dx\,dy = \int_{-\pi}^{\frac{\pi}{2}} dx \int_{-2}^{2} \left(y \cos x - x e^y\right) dy$$

$$= \int_{-\pi}^{\frac{\pi}{2}} dx \left[\frac{y^2}{2} \cdot \cos x - x e^y\right]_{-2}^{2} = \int_{-\pi}^{\frac{\pi}{2}} \left(\frac{4}{2} \cos x - x e^2\right) - \left(\frac{4}{2} \cos x - x e^{-2}\right) dx$$

$$= \int_{-\pi}^{\frac{\pi}{2}} \cancel{2\cos x} - xe^2 - \cancel{2\cos x} + xe^{-2}\, dx = \int_{-\pi}^{\frac{\pi}{2}} (-e^2 + e^{-2})\, x\, dx$$

$$= (e^{-2} - e^2) \int_{-\pi}^{\frac{\pi}{2}} x\, dx = (e^{-2} - e^2) \left[ \frac{x^2}{2} \right]_{-\pi}^{\frac{\pi}{2}} = (e^{-2} - e^2)\left( \frac{\pi^2}{8} - \frac{\pi^2}{2} \right)$$

$$= \left( \frac{1}{e^2} - e^2 \right)\left( \frac{\pi^2 - 4\pi^2}{8} \right) = \frac{1-e^4}{e^2} \cdot \frac{-3\pi^2}{8} = \boxed{-\frac{3\pi^2(1-e^4)}{8e^2}}$$

Vediamolo come norm. risp. asse y:

$$\iint_D (y \cdot \cos x - xe^y)\, dx\, dy = \int_{-2}^{2} dy \int_{-\pi}^{\frac{\pi}{2}} (y \cos x - x e^y)\, dx$$

$$= \int_{-2}^{2} dy \left[ y \cdot \operatorname{sen} x - \frac{x^2}{2} e^y \right]_{-\pi}^{\frac{\pi}{2}}$$

$$= \int_{-2}^{2} \left( y \cdot 1 - \frac{\pi^2}{8} e^y \right) - \left( y \cdot 0 - \frac{\pi^2}{2} e^y \right) dy$$

$$= \int_{-2}^{2} \left( y - \frac{\pi^2}{8} e^y + \frac{\pi^2}{2} e^y \right) dy = \int_{-2}^{2} y - \left( \frac{\pi^2}{8} - \frac{\pi^2}{2} \right) e^y\, dy$$

$$= \left[ \frac{y^2}{2} - \left( \frac{\pi^2}{8} - \frac{\pi^2}{2} \right) e^y \right]_{-2}^{2}$$

$$= \frac{4}{2} - \left( \frac{\pi^2}{8} - \frac{\pi^2}{2} \right) e^2 - \left[ \frac{4}{2} - \left( \frac{\pi^2}{8} - \frac{\pi^2}{2} \right) e^{-2} \right]$$

$$= \cancel{2} - \left( \frac{\pi^2}{8} - \frac{\pi^2}{2} \right) e^2 - \cancel{2} + \left( \frac{\pi^2}{8} - \frac{\pi^2}{2} \right) e^{-2}$$

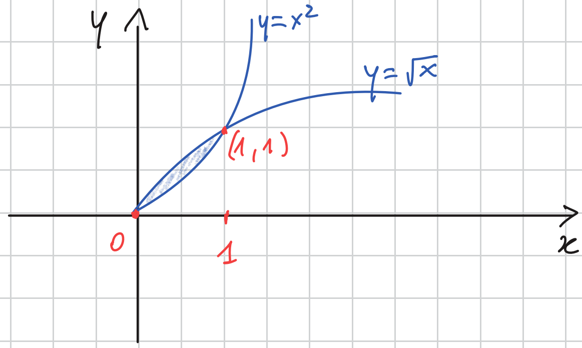
$$= -\left( \frac{\pi^2}{8} - \frac{\pi^2}{2} \right)\left( e^2 - e^{-2} \right) = -\frac{\pi^2 - 4\pi^2}{8} \; \frac{e^4 - 1}{e^2}$$

$$= \boxed{\frac{3\pi^2}{8} \cdot \frac{e^4 - 1}{e^2}}$$

4.2.

a) $\iint_E 4x^3y \, dx\,dy$ , $E = \{0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$

(Normale risp. asse $x$)




$$\int_0^1 dx \int_{x^2}^{\sqrt{x}} 4x^3y\, dy$$

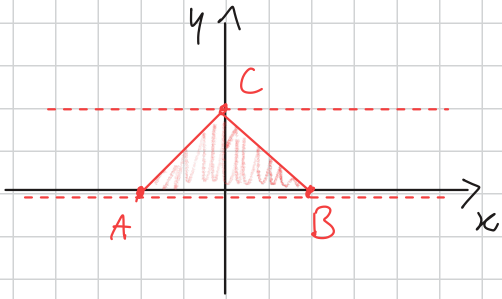
$$= \int_0^1 dx \left[\cancel{4}^{2} x^3 \frac{y^2}{\cancel{2}}\right]_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 (2x^3x - 2x^3x^4)\, dx = \int_0^1 2x^4 - 2x^7\, dx = \left[2\frac{x^5}{5} - 2\frac{x^8}{8}\right]_0^1 =$$

$$= \left(2 \cdot \frac{1}{5} - \cancel{2} \cdot \frac{1}{\cancel{8}_4}\right) - 0 = \frac{2}{5} - \frac{1}{4} = \frac{8-5}{20} = \boxed{\frac{3}{20}}$$

b) $\iint_T xy + e^x \, dx\,dy$ $\quad T = \overset{\Delta}{ABC}$ ,

$A = (-2,0)$, $B = (2,0)$, $C = (0,2)$




Esercizio: scrivere $T$ come dominio normale rispetto all'asse $x$.

Noi lo scriviamo come norm. risp. asse $y$:

$$0 \le y \le 2$$

Gli altri "bordi" hanno eq. delle rette per $AC$ e $BC$:

$$y = mx + q \;\begin{cases} \xrightarrow{B=(2,0)} & 0 = m \cdot 2 + q \\ \xrightarrow{C=(0,2)} & 2 = m \cdot 0 + q \end{cases} \qquad \begin{cases} 2m + q = 0 \\ q = 2 \end{cases} ; \quad \begin{cases} m = -1 \\ q = 2 \end{cases}$$

$\rightsquigarrow$ $y = -x + 2$ retta per $BC$

$x = -y + 2$ " " "

$$y = mx + q \longrightarrow \begin{matrix} \xrightarrow{A=(-2,0)} & 0 = m\cdot(-2)+q \\ \xrightarrow[C=(0,2)]{} & 2 = m\cdot 0 + q \end{matrix} \qquad \begin{cases} -2m+q=0 \\ q=2 \end{cases} ; \quad \begin{cases} m=1 \\ q=2 \end{cases}$$

$$\rightsquigarrow \quad y = x+2 \quad \text{retta per } AC$$
$$x = y-2 \quad \text{" \quad " \quad "}$$

$$\Rightarrow \quad y-2 \le x \le -y+2 \quad \Rightarrow \quad T = \{ 0 \le y \le 2, \ y-2 \le x \le -y+2 \}$$

$$\iint_T xy^2 + e^x \, dx\,dy = \int_0^2 dy \int_{y-2}^{-y+2} (xy^2+e^x)\,dx$$

$$= \int_0^2 dy \left[ \frac{x^2}{2} y^2 + e^x \right]_{y-2}^{-y+2}$$

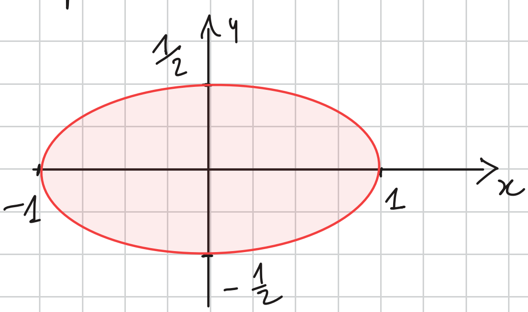
$$= \int_0^2 \left[ \left( \cancel{\frac{(-y+2)^2}{2}} \cdot y^2 + e^{-y+2} \right) - \left( \cancel{\frac{(y-2)^2}{2}} \cdot y^2 + e^{y-2} \right) \right] dy$$

$$\left[ \ (-y+2)^2 = (-y+2)(-y+2) = [-(y-2)][-(y-2)] \right.$$
$$\left. = +(y-2)(y-2) = (y-2)^2 \ \right]$$

$$= \int_0^2 (e^{-y+2} - e^{y-2})\,dy = \left[ -e^{-y+2} - e^{y-2} \right]_0^2$$

$$= \left( -e^{-2+2} - e^{2-2} \right) - \left( -e^2 - e^{-2} \right) = \boxed{-2 + e^2 + e^{-2}}$$

c) $\displaystyle\iint_F dx\,dy \quad (= \text{area}(F))$ $\qquad F = \{ x^2 + 4y^2 \le 1 \}$




Coord. Polari generalizzate:

$$\begin{cases} x = \rho \cos\theta \\ y = \frac{1}{2} \rho \,\text{sen}\,\theta \end{cases} , \qquad \begin{matrix} \rho \in [0,1] \\ \theta \in [0, 2\pi] \end{matrix}$$

$$\rightsquigarrow \quad x^2+4y^2 = \rho^2\cos^2\theta + \cancel{4}\cdot\frac{1}{\cancel{4}}\rho^2 \operatorname{sen}^2\theta = \rho^2\left(\cos^2\theta+\operatorname{sen}^2\theta\right) = \rho^2 \le 1$$

In generale, se $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1$ è l'eq. di un'ellisse, le coord. polari sono:

$$\begin{cases} x = a\cdot\rho\cos\theta \\ y = b\cdot\rho \operatorname{sen}\theta \end{cases}, \qquad \begin{aligned} &\rho\in[0,1] \\ &\theta\in[0,2\pi] \end{aligned}$$

L'integrale doppio con un cambio di coord. si trasforma:

$$\iint_F dx\,dy = \iint_F |\det J|\, d\rho\, d\theta = \iint_F \frac{1}{2}\rho\, d\rho\, d\theta = ⚘$$

dove

$$J = \begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial y}{\partial\rho} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta}\end{pmatrix} = \begin{pmatrix} \cos\theta & \frac{1}{2}\operatorname{sen}\theta \\ -\rho\operatorname{sen}\theta & \frac{1}{2}\rho\cos\theta \end{pmatrix}$$

$$\Rightarrow \quad \det J = \frac{1}{2}\rho\cos^2\theta + \frac{1}{2}\rho\operatorname{sen}^2\theta = \frac{1}{2}\rho$$

Ora svolgo l'integrale doppio ⚘ con le formule di riduzione, notando che $F$ nelle coord. $\rho$ e $\theta$ è normale rispetto a entrambi gli assi:

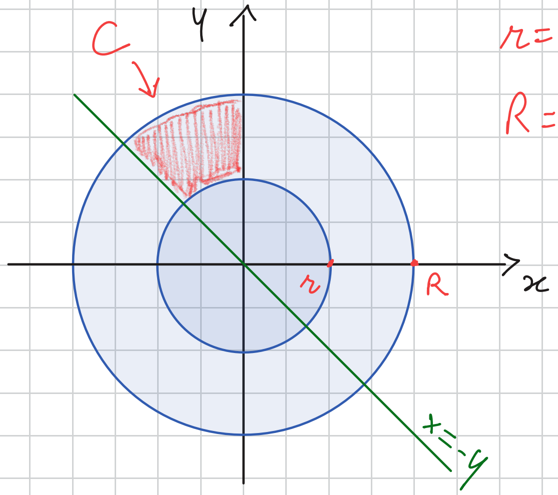
$$\iint_F \frac{1}{2}\rho\, d\rho\, d\theta = \frac{1}{2}\int_0^1 d\rho \int_0^{2\pi}\rho\, d\theta = \frac{1}{2}\int_0^1 d\rho\,\left[\rho\cdot\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\int_0^1 \left[\rho\cdot 2\pi - 0\right] d\rho = \frac{1}{\cancel{2}}\int_0^1 \cancel{2}\pi\rho\, d\rho = \pi\int_0^1 \rho\, d\rho$$

$$= \pi\left[\frac{\rho^2}{2}\right]_0^1 = \boxed{\frac{\pi}{2}}$$

4.3. Esercizio.

4.4. $$\iint_C \frac{x\cos^2\left(\sqrt{x^2+y^2}\right)}{y^2}\,dx\,dy$$




$r = \frac{\pi}{2}$

$R = \pi$

C é la corona circolare tra $r, R$ intersecate con $\{ x > -y,\ x < 0,\ y > 0 \}$

$x = -y$

Parametrizziamo C con coord. polari:

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} , \quad \begin{matrix} \rho \in [r, R] = \left[\frac{\pi}{2}, \pi\right] \\ \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right] \end{matrix}$$

$|\det J|$

$$\Rightarrow \iint_C \frac{x \cos^2\left(\sqrt{x^2+y^2}\right)}{y^2} \, dx\, dy = \iint_C \frac{\rho\cos\theta \cdot \cos^2\left(\sqrt{\rho^2\cos^2\theta + \rho^2 \operatorname{sen}^2\theta}\right)}{(\rho \operatorname{sen}\theta)^2} \cdot \rho \, d\rho \, d\theta$$

$$= \iint_C \frac{\cancel{\rho}\cos\theta \cdot \cos^2\rho}{\cancel{\rho^2}\operatorname{sen}^2\theta} \cancel{\rho} \, d\rho\, d\theta = \iint_C \frac{\cos\theta \cdot \cos^2\rho}{\operatorname{sen}^2\theta} \, d\rho\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} d\rho \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{\cos\theta \cos^2\rho}{\operatorname{sen}^2\theta} \, d\theta = \underbrace{\int_{\frac{\pi}{2}}^{\pi} \cos^2\rho \, d\rho}_{I_1} \cdot \underbrace{\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{\cos\theta}{\operatorname{sen}^2\theta} \, d\theta}_{I_2}$$

$I_1$ : sfruttiamo la formula $\cos^2\rho = \frac{1}{2} - \frac{1}{2}\cos 2\rho$

$$\cos 2\rho = \operatorname{sen}^2\rho - \cos^2\rho = 1 - \cos^2\rho - \cos^2\rho = 1 - 2\cos^2\rho$$

$$\rightarrow 2\cos^2\rho = 1 - \cos 2\rho$$

$$\cos^2\rho = \frac{1}{2} - \frac{1}{2}\cos 2\rho$$

$$\Rightarrow I_1 = \int_{\frac{\pi}{2}}^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2\rho\right) d\rho = \left[\frac{1}{2}\rho - \frac{1}{2}\,\frac{\operatorname{sen} 2\rho}{2}\right]_{\frac{\pi}{2}}^{\pi} = \frac{1}{2}\pi - \frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \pi/4$$

$I_2$ : facciamo una sostituzione: $\operatorname{sen}\theta = t \to \cos\theta\, d\theta = dt$

$$\Rightarrow I_2 = \int \frac{\cos\theta}{\operatorname{sen}^2\theta}\, d\theta = \int \frac{1}{t^2}\, dt = -\frac{1}{t} = -\frac{1}{\operatorname{sen}\theta}$$

$$\Rightarrow \left[-\frac{1}{\operatorname{sen}\theta}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}/2} + \frac{1}{1} = -\frac{2}{\sqrt{2}} + 1 = -\sqrt{2} + 1$$

Risultato finale:

$$I_1 \cdot I_2 = \boxed{\frac{\pi}{4} \cdot \left(-\sqrt{2} + 1\right)}$$