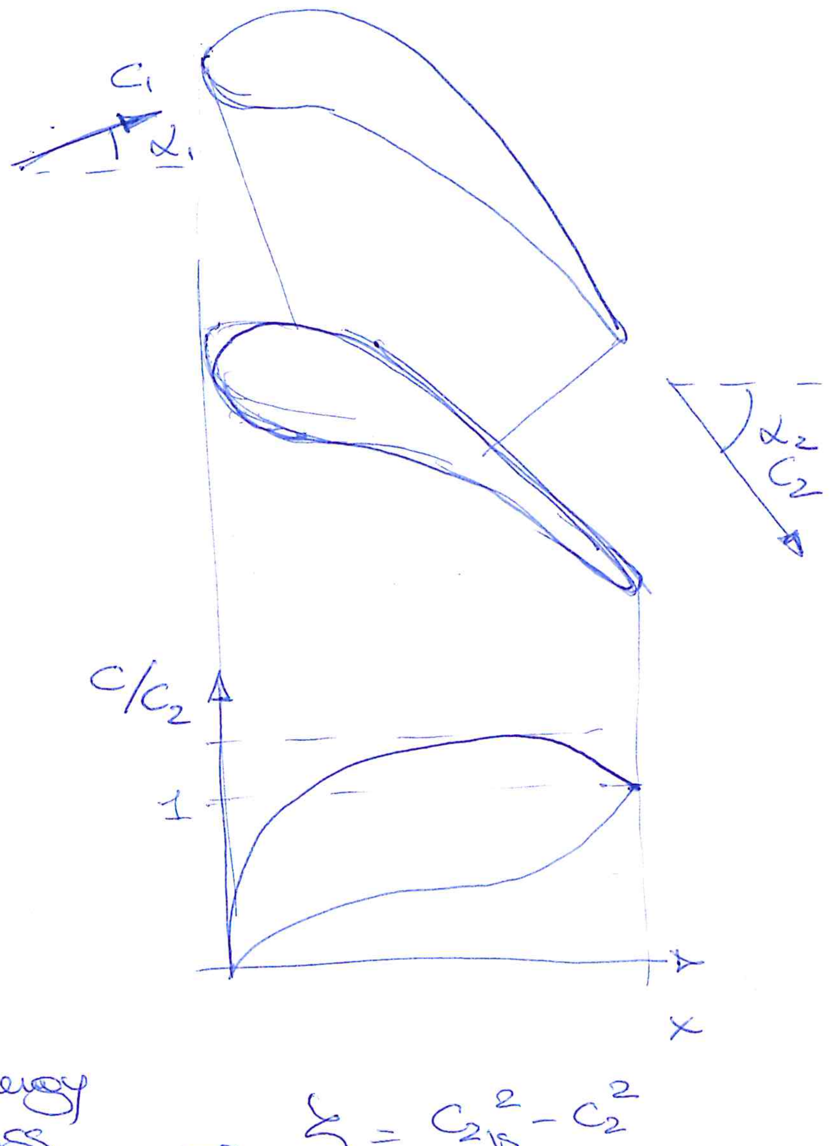
TURBINE PROFILE LOSSES (SODERBERG'S CORRELATION)

The correlations were obtained by Soderberg (1949) from a large number of tests performed on steam turbines and cascades and extended to fit data obtained from small turbines with very low aspect ratio blading.
The data and correlations were then reviewed by Horlock (1960)

Horlock, J. H. (1960) "Losses and efficiency in axial flow turbines", Int. J. of Mechanical Science, 2, 48



energy loss coefficient $\rightarrow \xi = \dfrac{c_{2is}^2 - c_2^2}{c_{2is}^2}$

Soderberg claims that at optimum space-chord ratio, turbine losses could be correlated with space-chord ratio, blade aspect-ratio, thickness-chord ratio, and Reynolds number.

$$\zeta = \zeta\left(\cancel{\frac{s}{c}} \; \frac{1}{\sigma}, AR, \frac{t}{c}, Re\right)$$

For $Re = 10^5$, $AR = 3$, and at optimum load coefficient,

$$\zeta^* = 0.04 + 0.06\left(\frac{\varepsilon}{100}\right)^2$$

$\varepsilon \leftarrow$ fluid deflection in degrees $\varepsilon = \alpha_1 + \alpha_2$
(see previous page)

It is important to note how:

- this expression fits Soderberg's curve for $\frac{t_{max}}{l} = 0.2$ and $\varepsilon \leq 120°$
- the flow deflection is very similar to the blade deflection, as long as the blade incidence is small, given that the deviation in a turbine cascade is generally quite small.

For different AR, the losses can be corrected as follows

NOZZLES $$1+\zeta_1 = (1+\zeta^*)\left(0.993+\frac{0.021}{AR}\right)$$

ROTORS $$1+\zeta_1 = (1+\zeta^*)\left(0.975+\frac{0.075}{AR}\right)$$

$\zeta_1 \Rightarrow$ energy loss coefficient for $Re = 10^5$

Another correction can be made if $Re \neq 10^5$, where Re is the Reynolds number based on exit velocity and hydraulic diameter at the throat section:

$$Re = \frac{\rho_2 c_2 D_h}{\mu}$$

$$D_h = \frac{2 s H \cos\alpha_2}{(s\cos\alpha_2 + H)} = \frac{4A}{P}$$ perimeter

The correction is

$$\zeta_2 = \left(\frac{10^5}{Re}\right)^{1/4} \zeta_1$$

Mach number effect

losses vary with Mach number, mainly because of the presence of shocks, This is not considered in Soderberg's correlations.

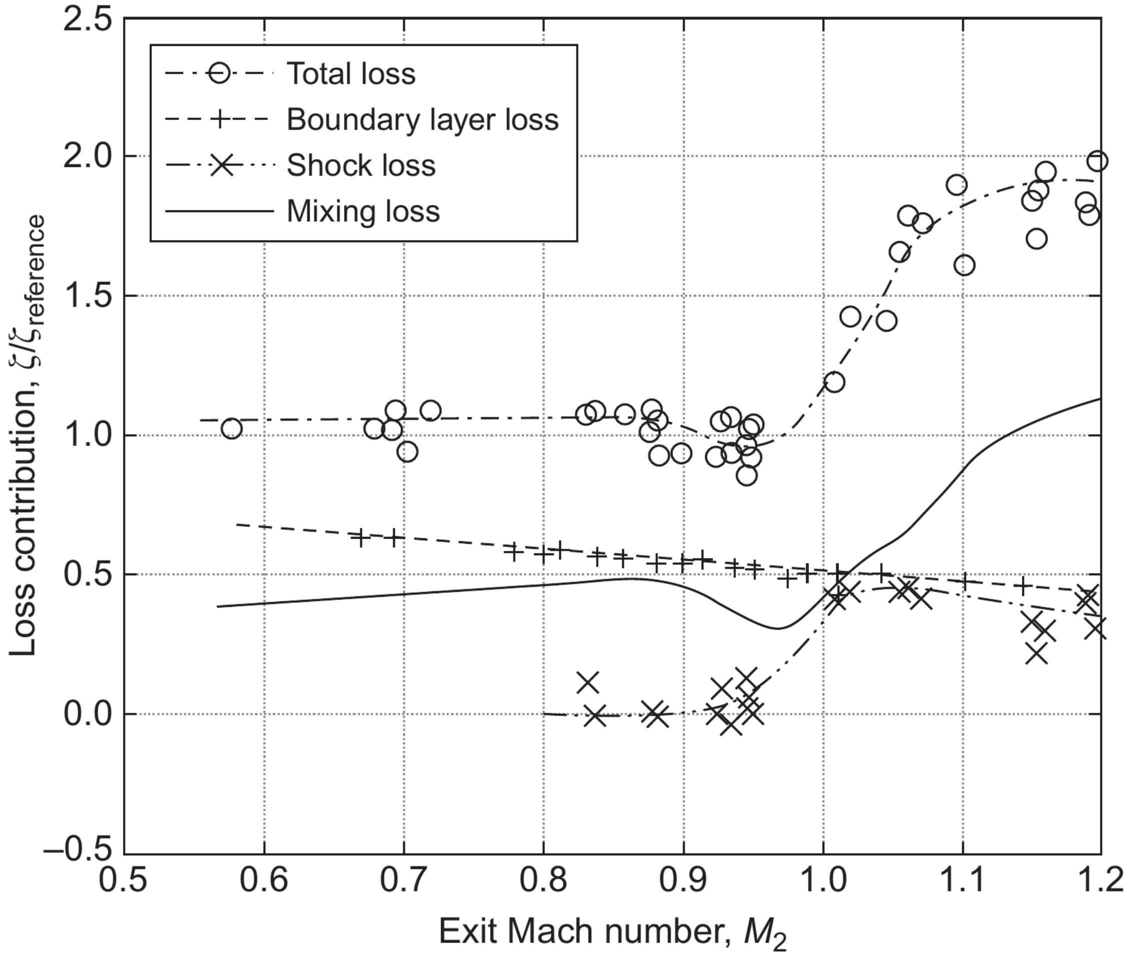
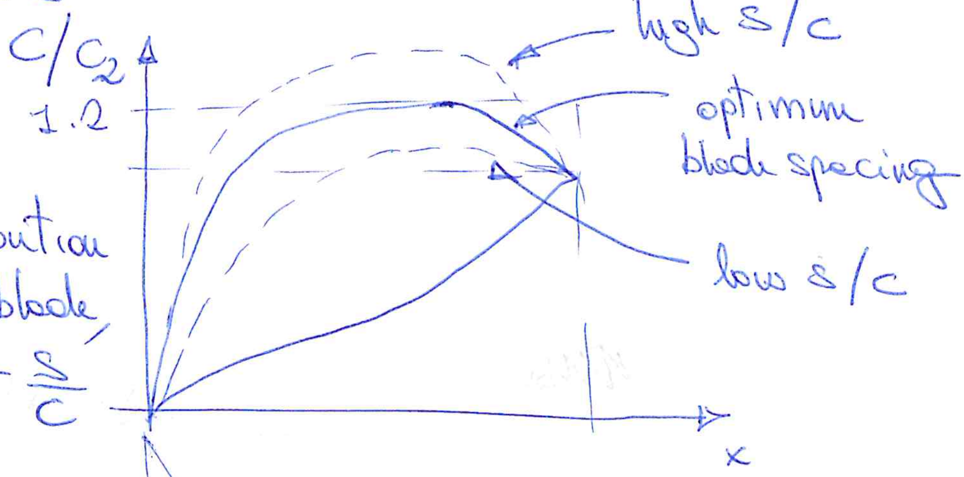


Figure: variation of loss coefficient with Mach number for a turbine cascade at a Reynolds number of 10^6

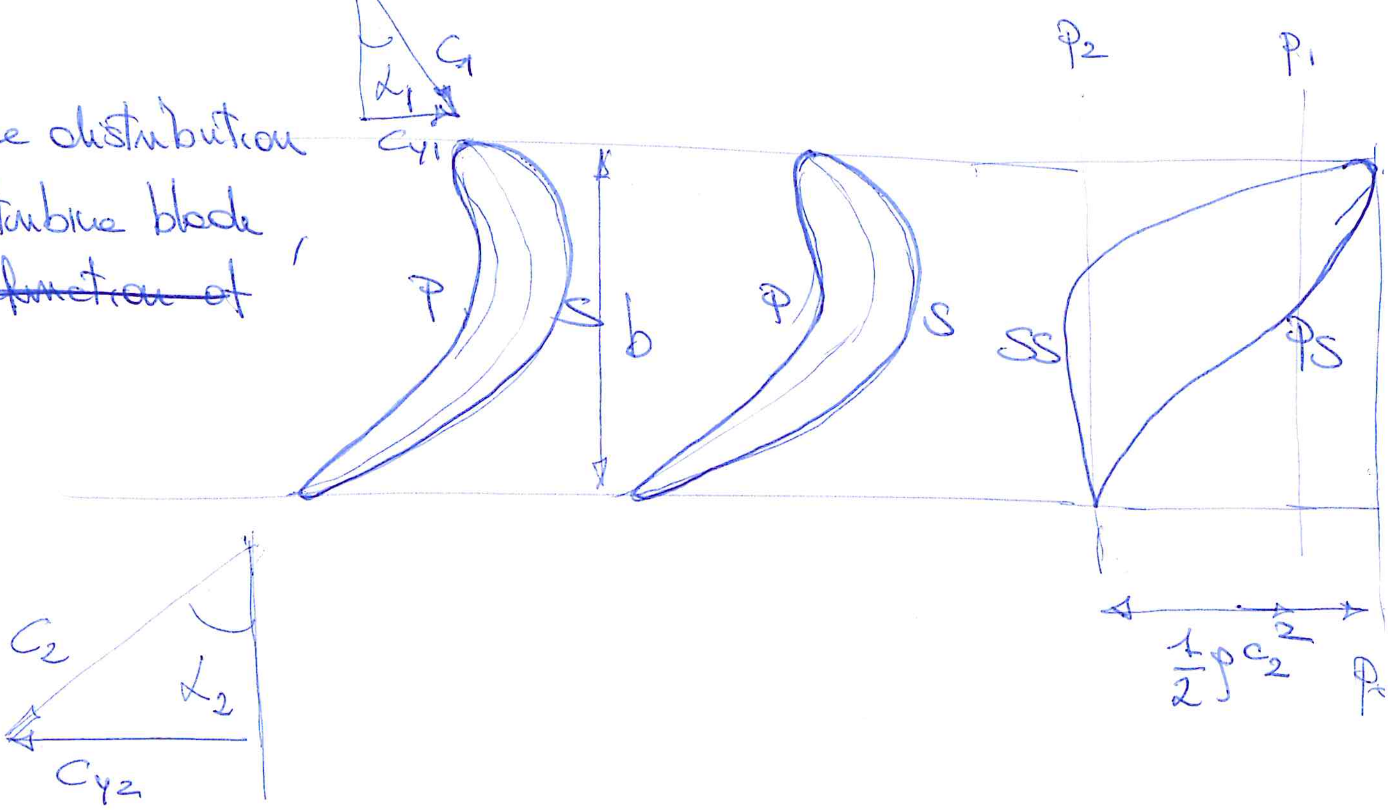
ZWEIFEL CRITERION

For turbine cascades, there is an optimum $\frac{s}{c}$ ratio that gives minimum losses

Variation of velocity distribution on a turbine-blade as a function of $\frac{s}{c}$



pressure distribution on a turbine blade, ~~as a function of~~



The tangential blade load is $Y = \dot{m}\,(c_{y1} + c_{y2})$

An ideal load is defined as follows:

$$Y_{id} = (p_{01} - p_2)\, bH$$

$b \rightarrow$ axial blade chord $\quad H \rightarrow$ blade height

$$Z = \frac{Y}{Y_{id}} = \frac{\dot{m}\,(c_{y1} + c_{y2})}{(p_{01} - p_2)\, bH} =$$

for an incompressible fluid

$$c_y = c_x \tan\alpha \qquad \dot{m} = \rho s c_x H$$

$$Z = \frac{\cancel{\rho}\, \cancel{c_x^2}\, \cancel{H}\, s\,(\tan\alpha_1 + \tan\alpha_2)}{\frac{1}{2}\, \frac{\cancel{\rho}\, \cancel{c_x^2}}{\cos^2\alpha_2}\; b\cancel{H}}$$

$$Z = 2\cos^2\alpha_2\,(\tan\alpha_1 + \tan\alpha_2)\,\frac{s}{b}$$

Zweifel found (from a number of experiments on turbine cascades) that at low Mach numbers, for minimum losses, $Z \simeq 0.8$. This means that

$$0.8 = 2\cos^2\alpha_2\,(\tan\alpha_1 + \tan\alpha_2)\,\frac{s}{b}$$

$$\left(\frac{s}{b}\right)_{OPTIMAL} = \frac{0.4}{\cos^2\alpha_2}\;\frac{1}{\tan\alpha_1 + \tan\alpha_2}$$

$\Rightarrow$ High turning blades (large $\tan\alpha_1 + \tan\alpha_2$) need to have low $\frac{s}{b}$, i.e. closer blades.

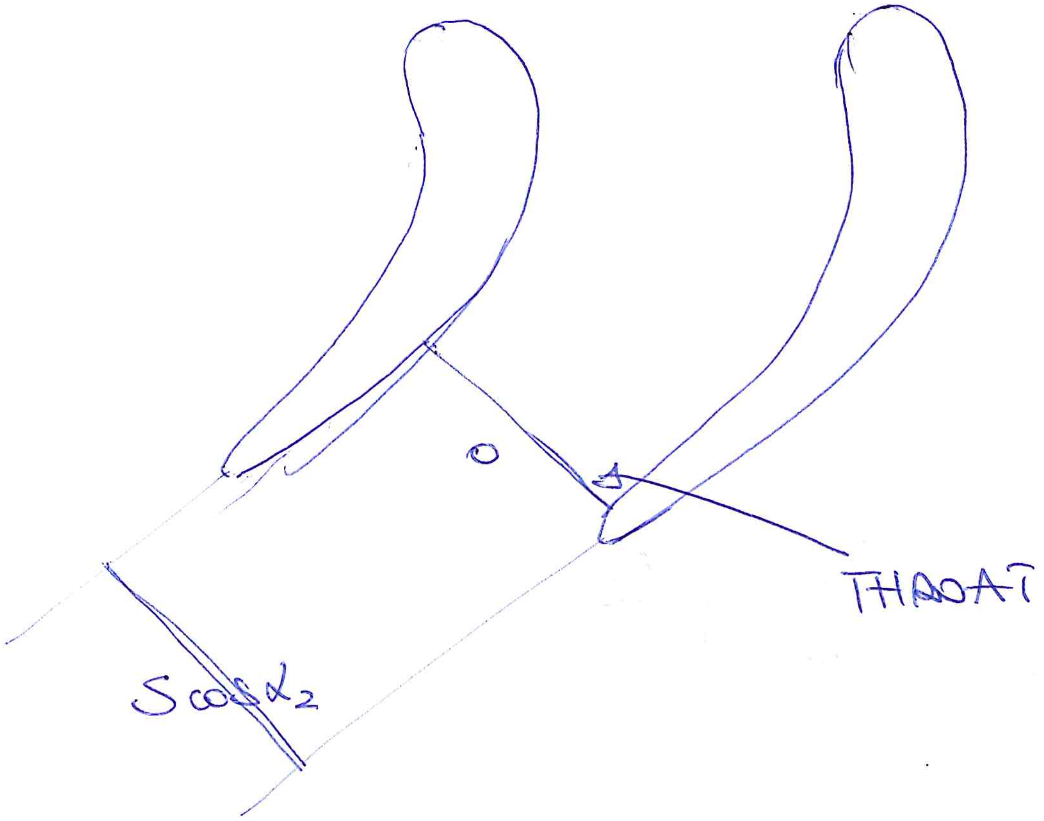
Zweifel's criterion works well for $\alpha_2 \in [60° : 70°]$, less accurate outsize this range. Higher values of Z (ie. less blades) are common in jet engines, where it is important to minimise the weight ($Z > 1$).

For compressible flows we need to use the full relation:

$$Z = \frac{Y}{Y_{id}} = \frac{\dot{m}(c_{y1} + c_{y2})}{(p_{01} - p_2) bH}$$

The optimum Z decreases for high Mach numbers because the difference between p_{01} and p_2 increases, leading to a larger ~~blad~~ ideal blade force.

FLOW EXIT ANGLE



In subsonic blades, the deviation angle is small, because of the small adverse pressure gradient.

The deviation angle is larger in supersonic blades, because of the so-called supersonic deviation.

Let's consider the throat to be choked:

$$\frac{\dot{m}\sqrt{c_p T_0}}{H\, A\, p_0^*} = Q(1)$$

where $Q(1)$ is the non-dim mass-flow function with $M=1$

At exit:

$$\frac{\dot{m}\sqrt{c_p T_0}}{p_{02} H s \cos\alpha_2} = Q(M_2)$$

Therefore:

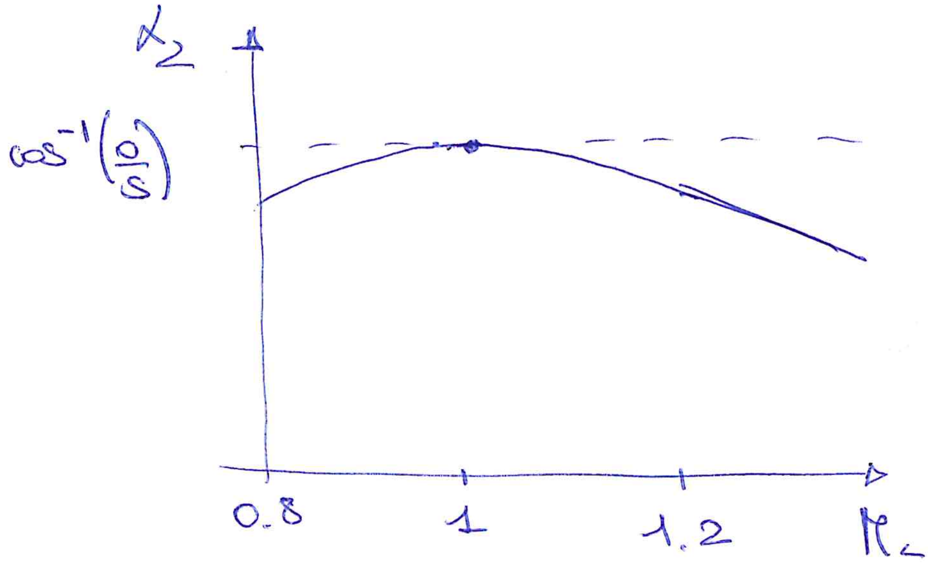
$$Q(1)\cancel{H}\, o\, p_0^* = Q(M_2)\cancel{H}\, s \cos\alpha_2$$

$$\cos\alpha_2 = \left(\frac{p_0^*}{p_{02}}\right) \cdot \frac{o}{s} \cdot \frac{Q(1)}{Q(M_2)}$$

If $M_2 > 1$ (the flow is expanding after the throat, due to a lower static pressure)

$$\frac{Q(M_2)}{Q(1)} < 1 \qquad \left(\frac{p_0^*}{p_{02}} \text{ negligible}\right)$$

Hence $\cos\alpha_2$ increases, and therefore α_2 decreases



The maximum exit speed can be calculated imposing that

$$M_{2,x,lim} = 1$$

$$M_{2,lim} = \frac{M_{2,x,lim}}{\cos\alpha_2} = \frac{1}{\cos\alpha_2}$$

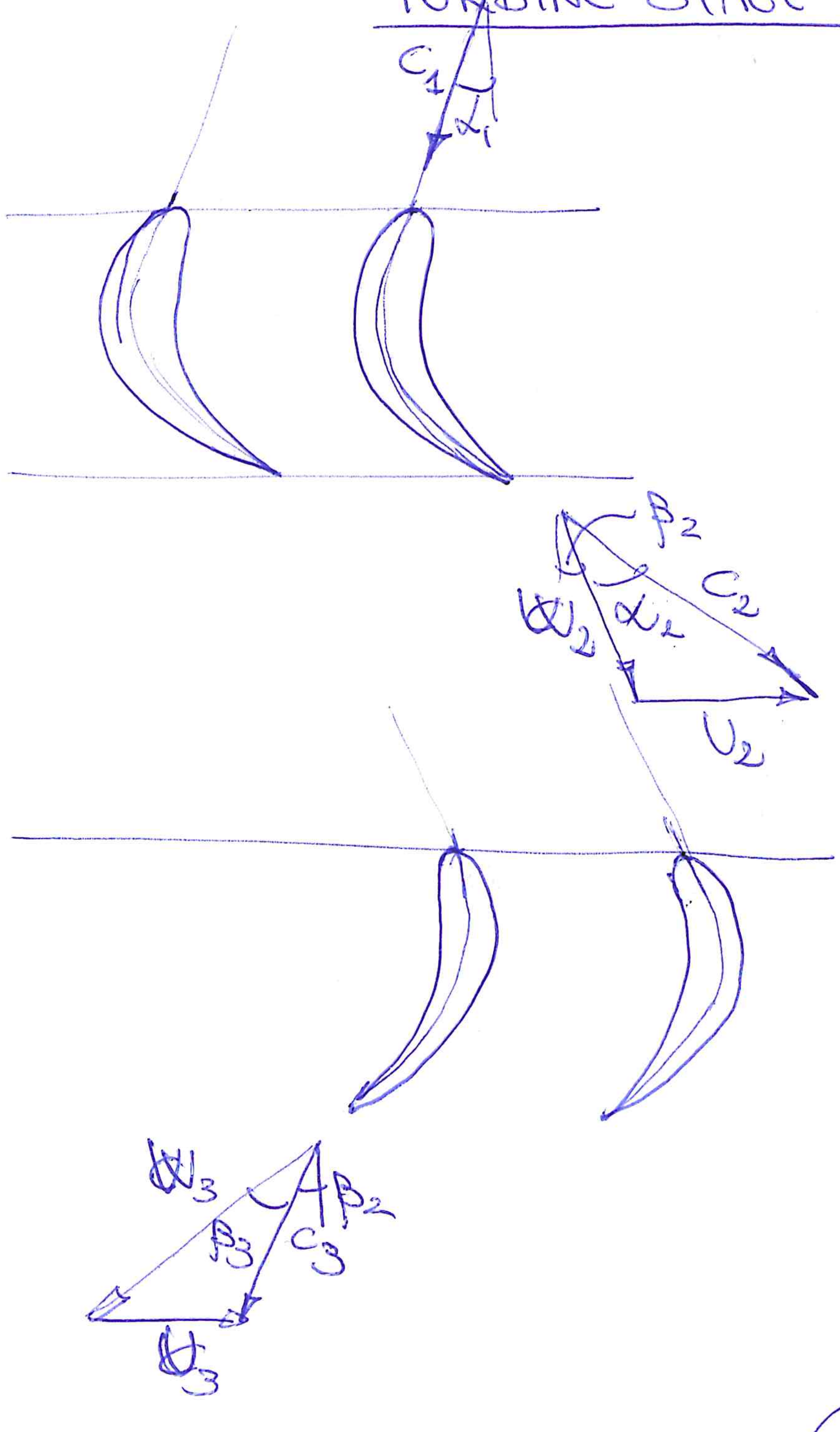
Solving simultaneously

$$M_{2,lim} = \frac{1}{\cos\alpha_2}$$

$$Q(M_{2,lim}) = Q(1)\,\frac{o}{s}\cdot\frac{1}{\cos\alpha_2}$$

One can calculate the maximum exit Mach number.

TURBINE STAGE DESIGN



- FLOW COEFFICIENT: $\phi = \dfrac{\overbrace{c_m}^{\sqrt{c_x^2+c_r^2}}}{U} \simeq \dfrac{c_x}{U}$

<u>continuity requires</u> $\dot{m} = \rho_1 c_{1x} A_{1x} = \rho_2 c_{2x} A_{2x} = \rho_3 c_{3x}$

$= \rho A_x \cdot U \phi$

- STAGE LOADING

$$\psi = \frac{\Delta h_0}{U^2} = \frac{U(\Delta C_\theta)}{U^2} = \frac{\Delta C_\theta}{U}$$

the larger the stage loading, the larger the change in axial velocity across the rotor:

$\Rightarrow \Delta C_\theta = C_{\theta 3} - C_{\theta 2} < 0$ because we are dealing with a turbine

- STAGE REACTION

$$R = \frac{h_2 - h_3}{h_1 - h_3}$$

if the fluid is incompressible $\quad dh = \underbrace{T\cancel{ds}}_{\simeq 0 \text{ (negligible)}} + \frac{dp}{\rho}$

$$R \simeq \frac{p_2 - p_3}{p_1 - p_3}$$

In a turbine

$$w = \frac{\dot{W}}{\dot{m}} = (C_{\theta 2} + C_{\theta 3})\, U$$

(taken positive as in the drawing from the previous page)

In the <u>stator</u>, the total enthalpy is conserved (no work)

$$h_{01} = h_{02}$$

In the <u>rotor</u>

$$h_{02} - h_{03} = U\left(c_{\theta 2} + c_{\theta 3}\right)$$

Hence

$$h_2 + \frac{1}{2}c_{\theta 2}^2 + \frac{1}{2}c_{x2}^2 - h_3 - \frac{1}{2}c_{\theta 3}^2 - \frac{1}{2}c_{x3}^2$$
$$= U\left(c_{\theta 2} + c_{\theta 3}\right)$$

$$c_{\theta 2} = w_{\theta 2} + U$$

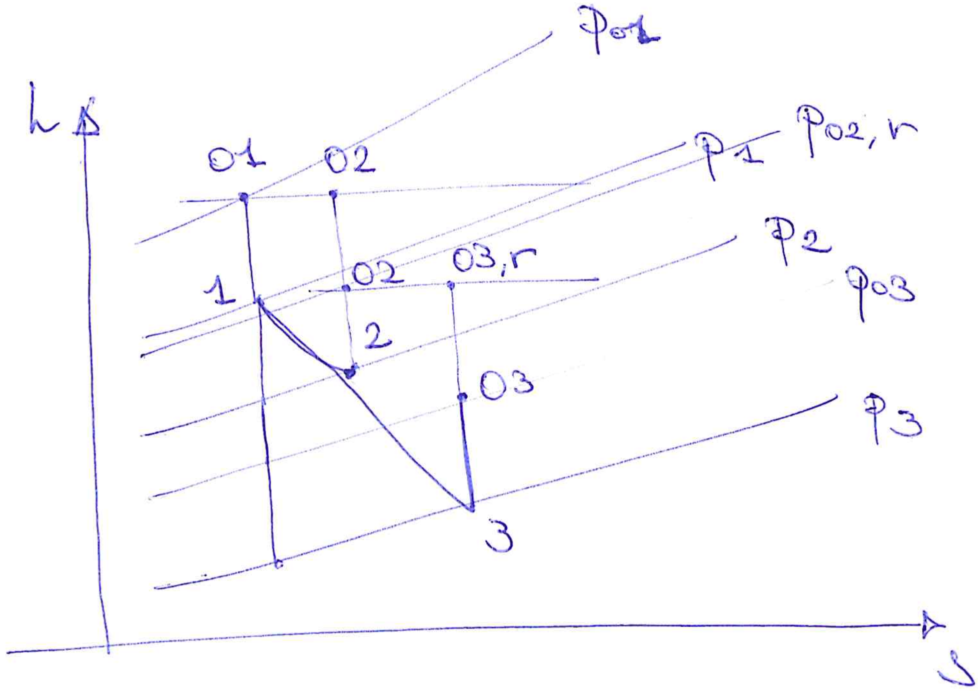
$$c_{\theta 3} = w_{\theta 3} - U$$

$$h_2 - h_3 + \frac{1}{2}\left(c_{x2}^2 - c_{x3}^2\right) + \frac{1}{2}\left(w_{\theta 2}^2 + \cancel{U_2^2} - w_{\theta 3}^2 - \cancel{U_3^2}\right.$$
$$\left. + 2\cancel{U w_{\theta 2}} + 2\cancel{U w_{\theta 3}}\right) = \cancel{U\left(w_{\theta 2} + w_{\theta 3}\right)}$$

$$\rightarrow h_2 - h_3 + \frac{1}{2}\left(c_{x2}^2 - c_{x3}^2\right) + \frac{1}{2}\left(w_{\theta 2}^2 - w_{\theta 3}^2\right)$$

Hence:
$$h_{02,r} - h_{03,r} = 0$$

$$\Delta h_{0,r} = 0$$



REPEATING-STAGE TURBINE

In a repeating-stage turbine, $\alpha_1 = \alpha_3$, $c_x = \text{const}$, $r = \text{const}$

$$R = \frac{h_2 - h_3}{h_1 - h_3} = 1 - \frac{h_1 - h_2}{h_{01} - h_{03}}$$

$$h_1 - h_2 = h_{01} - h_{02} + \frac{1}{2}\left(c_2^2 - c_1^2\right)$$
$$= \left(h_{01} - h_{02}\right) + \frac{1}{2} c_x^2 \left(\tan^2 \alpha_2 - \tan^2 \alpha_1\right)$$

From the definition of Ψ:

$$\Psi = \frac{h_{01} - h_{03}}{U^2} \Rightarrow h_{01} - h_{03} = \Psi U^2$$

$$\Rightarrow R = 1 - \frac{\phi^2}{2\Psi}\left(\tan^2 \alpha_2 - \tan^2 \alpha_1\right)$$

This is true even if the exit stage angle is not equal to the inlet stage angle.

4 $\boxed{\alpha_1 = \alpha_3}$:

$$\psi = \frac{\Delta c_\theta}{U} = \frac{c_{\theta 2} + c_{\theta 3}}{U} = \frac{c_x}{U}(\tan\alpha_2 + \tan\alpha_3)$$

$$= \phi(\tan\alpha_2 + \tan\alpha_3) = \phi(\tan\alpha_2 + \tan\alpha_1)$$

$$\rightarrow R = 1 - \frac{\phi^2}{2\psi}(\tan^2\alpha_2 - \tan^2\alpha_1) =$$

$$= 1 - \frac{\phi}{2}(\tan\alpha_2 - \tan\alpha_1)$$

We can combine the following equation to eliminate α_2:

$$\begin{cases} R = 1 - \frac{\phi}{2}(\tan\alpha_2 - \tan\alpha_1) \\ \psi = \phi(\tan\alpha_2 + \tan\alpha_1) \end{cases}$$

$$2R + \psi = 2 + 2\phi\tan\alpha_1$$

or $\quad \psi = 2(1 - R + \phi\tan\alpha_1)$

A few things can be concluded:

- for high-stage loading, we need to have:
  - low R (reaction)
  - large ϕ (flow coefficient)
  - large α_1 (swirl)

- once loading (Ψ), reaction (R) and flow coefficient (ϕ) are specified, all velocity triangles are set.

We can derive the remaining angles.

$$c_{\theta 2} = w_{\theta 2} + U$$

$$c_x \tan \alpha_2 = c_x \tan \beta_2 + U$$

$$\Rightarrow \quad \tan \beta_2 = \tan \alpha_2 - \frac{1}{\phi}$$

$$c_{\theta 3} = w_{\theta 3} - U$$

$$c_x \tan \alpha_3 = c_x \tan \beta_3 - U$$

$$\Rightarrow \quad \tan \beta_3 = \tan \alpha_3 + \frac{1}{\phi}$$

$$R = 1 - \frac{\phi}{2}\left(\tan \alpha_2 - \tan \alpha_1\right) =$$

$$= 1 - \frac{\phi}{2}\left(\tan \beta_2 + \frac{1}{\phi} - \tan \beta_3 + \frac{1}{\phi}\right) = \frac{\phi}{2}\left(\tan \beta_3 - \tan \beta_2\right)$$

$\Rightarrow$ To chose the turbine design parameters, one can set:

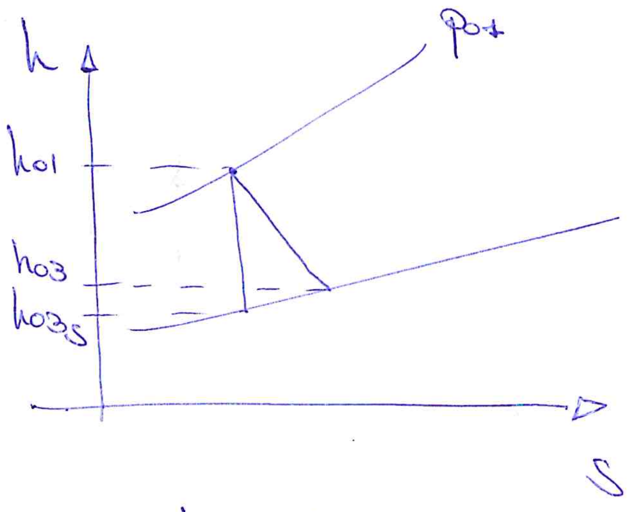
- ϕ, ψ, R

or

- ϕ, ψ, α_1

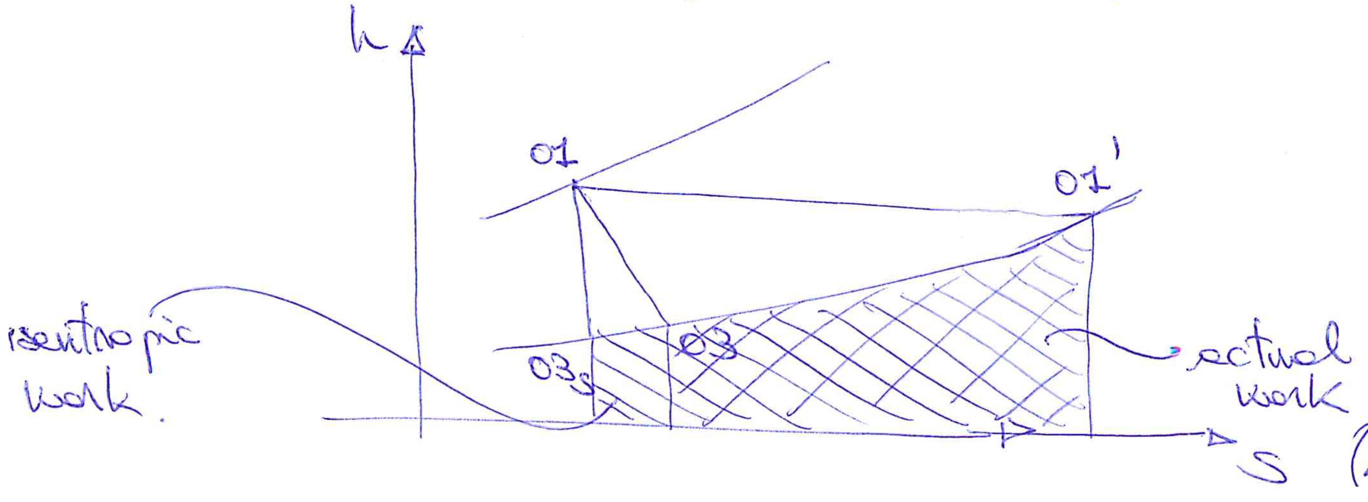
or any othe combination of 3 angles and parameters. Once these are set, the design and hence the losses (mean-line) are set.

STAGE LOSSES AND EFFICIENCY



$$\eta_{tt} = \frac{h_{01} - h_{03}}{h_{01} - h_{03,s}} = \frac{h_{01} - h_{03}}{(h_{01} - h_{03}) + (h_{03} - h_{03,s})}$$

We can show that $(h_{03} - h_{03,s}) \simeq T_{03}(s_3 - s_{3,s}) = T_{03}(s_3 - s$



4

$$-w = dh_0 = T_0\, ds + \frac{dp_0}{\rho_0}$$

Since these are all state functions, I can take the transformation from 01 to 03 to calculate the turbine work. On this transformation (constant pressure)

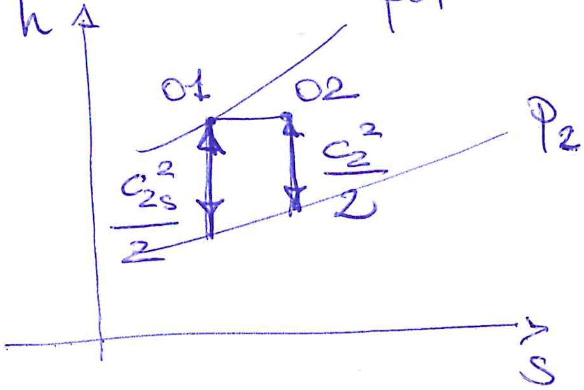
$$dh_0 = T_0\, ds$$

and hence

$$-w = w_t = \Delta h_0 \simeq T_0 \Delta s$$

Recalling the definitions of loss coefficient in the STATOR

$$\zeta_s = \frac{c_{2is}^2 - c_2^2}{c_2^2}$$



$$\frac{c_{2is}^2 - c_2^2}{2} = \frac{c_{2is}^2}{2} + h_{2is} - \frac{c_2^2}{2} - h_2 + h_2 - h_{2is}$$
$$= h_2 - h_{2is} + \cancel{(h_{02is} - h_{02})}$$

$$\zeta_s = \frac{c_{2is}^2 - c_2^2}{c_2^2} = \frac{h_2 - h_{2s}}{\frac{c_2^2}{2}} =$$

Hence

$$(h_2 - h_{2s}) = \frac{c_2^2}{2}\zeta_s$$

Similarly in the ROTOR:

$$(h_3 - h_{3s}) = \frac{1}{2} w_3^2 \zeta_R$$

Going back to the stage efficiency:

$$\eta_{tt} = \left[1 + \frac{T_{03}(s_3 - s_1)}{h_{01} - h_{03}} \right]^{-1}$$

$$s_3 - s_1 = (s_3 - s_2) + (s_2 - s_1) = \frac{h_3 - h_{3s}}{T_3} + \frac{h_2 - h_{2s}}{T_2} =$$

$$= \frac{1}{2} \frac{c_2^2 \zeta_S}{T_2} + \frac{1}{2} \frac{w_3^2 \zeta_R}{T_3}$$

$$\eta_{tt} = \left[1 + \frac{T_{03}}{T_3} \left(\frac{\frac{1}{2} c_2^2 \zeta_S \frac{T_3}{T_2} + \frac{1}{2} w_3^2 \zeta_R}{h_{01} - h_{03}} \right) \right]$$

If the Mach number is low

$$\frac{T_{03}}{T_3} \simeq 1$$

If the stage is a repeating-stage

$$h_{01} - h_{03} \simeq h_1 - h_3$$

$\Rightarrow$ If we want to calculate the efficiency, wer need

ξ_S and ξ_R

(see previous lectures) $\rightarrow$ Soderberg (1949)

Horlock (1966)

Ainley Mathieson (1951)

...

$\Rightarrow$ role of CFD is ever more important, not only for profile losses, but also for 3D losses.

Losses are usually chategorized between 2D on 3D losses. The former are the ones that would be present in a cascade of infinite span (no endwall effects). The 3D losses are the ones that are present because of the three-dimensional geometric features or rotational effects.

<u>2D losses</u> : $\rightarrow$ blade boundary layer

$\rightarrow$ trailing edge mixing

$\rightarrow$ flow separation

$\rightarrow$ shokk waves

<u>3D losses</u> : $\rightarrow$ tip leakage flows

$\rightarrow$ endwall (secondary) flows

$\rightarrow$ cooling.

PRELIMINARY AXIAL-TURBINE DESIGN

The process of choosing the best turbine design for a given application involves compromising among several parameters, such as: mechanical stress, weight, size, efficiency, noise, cost.
The velocity triangles are set when three dimensionless parameters (ϕ, Ψ and R for example) are decided.
The general layout of the turbine is also determined.

1) NUMBER OF STAGES

The specification of the turbine will include the massflow and the required power output. The number of stages can be determined ~~knw~~ knowing the maximum load factor Ψ_{MAX}, which is typically between 1.5 and 2.

$$\frac{\dot{W}}{\dot{m} U^2 n_{stages}} \leq \Psi_{MAX}$$

$$\Downarrow$$

$$n_{stages} \geq \frac{\dot{W}}{\dot{m} U^2 \Psi_{MAX}}$$

Clearly a large blade speed is desirable, but it is usually constrained by stress limits and by losses and noise due to high speed.

2) BLADE HEIGHT AND MEAN RADIUS

Given the mass-flow and assuming a constant axial flow velocity:

$$\rho_1 A_{x1} = \rho_2 A_{x2} = \rho_3 A_{x3} = \text{constant}$$

$$A_x = \frac{\dot{m}}{\rho V_x} = \frac{\dot{m}}{\rho \phi U} = 2\pi r_m H$$

→ usually the mean radius is decided by the need to rotate at a particular speed, or similar considerations, hence

$$H = \frac{\dot{m}}{\rho \phi U \cdot 2\pi r_m}$$

3) NUMBER OF AIRFOILS AND AXIAL CHORD

The blade aspect ratio $\left(\frac{H}{b}\right)$ is usually set to a suitable value (between 1 and 2 for core turbines, significantly larger for LP or steam turbines.)

The pitch-on-chord $\left(\frac{s}{b}\right)$ and therefore the number of blades, are selected using the Zweifel's criterion.

STYLES OF TURBINE

ZERO - REACTION

The advantages:

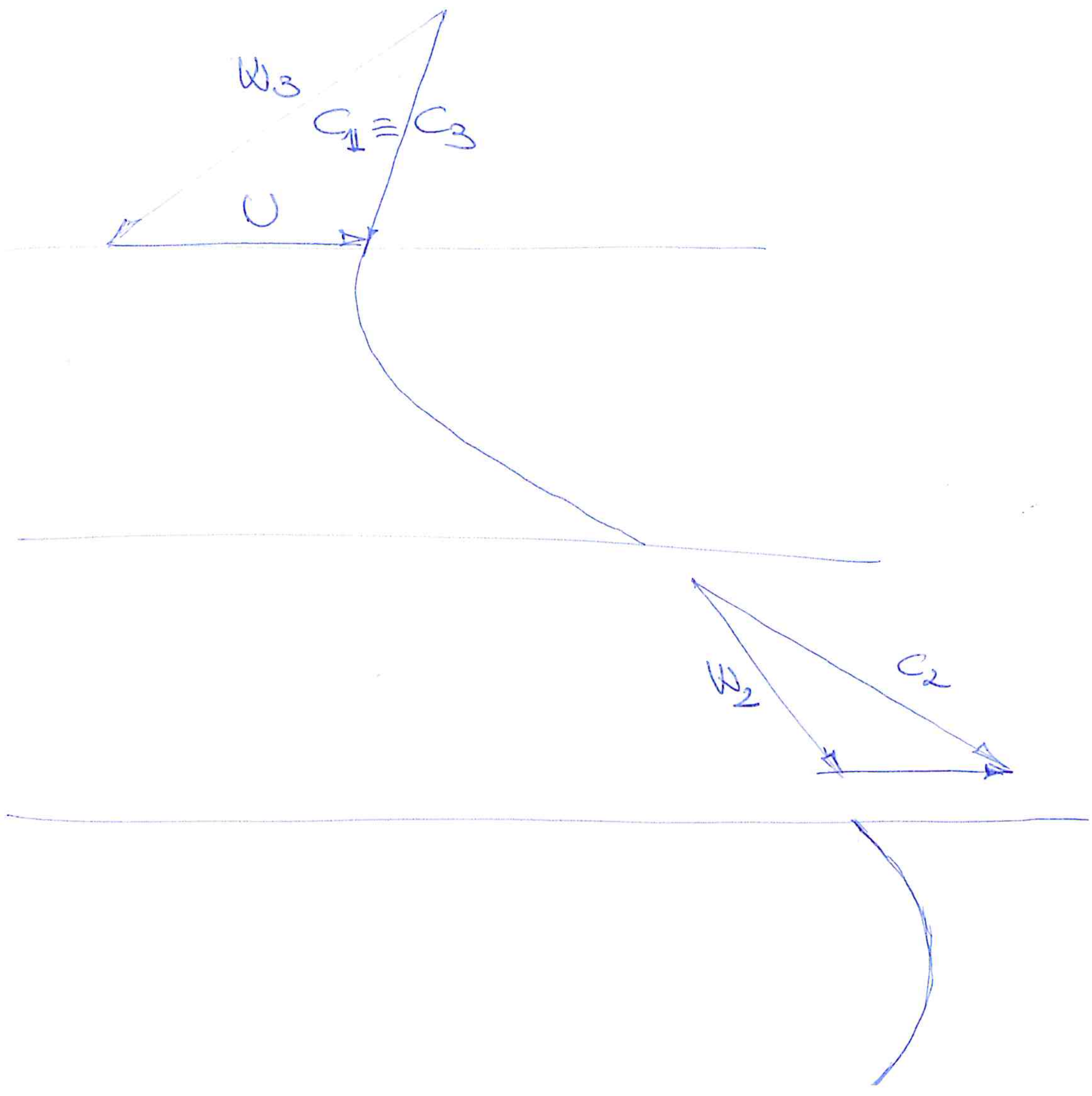
- high loading ($\Psi = 2(1 - R + \phi \tan\alpha_1)$) with low inter-stage swirl
- low thrust on the rotor
- low tip leakage - flow

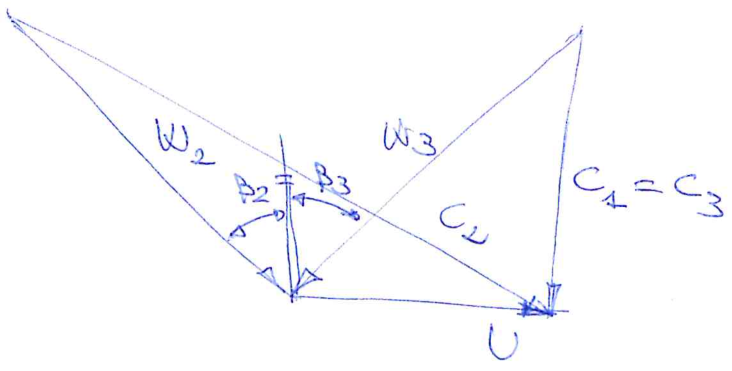
The disadvantages:

- lower efficiency

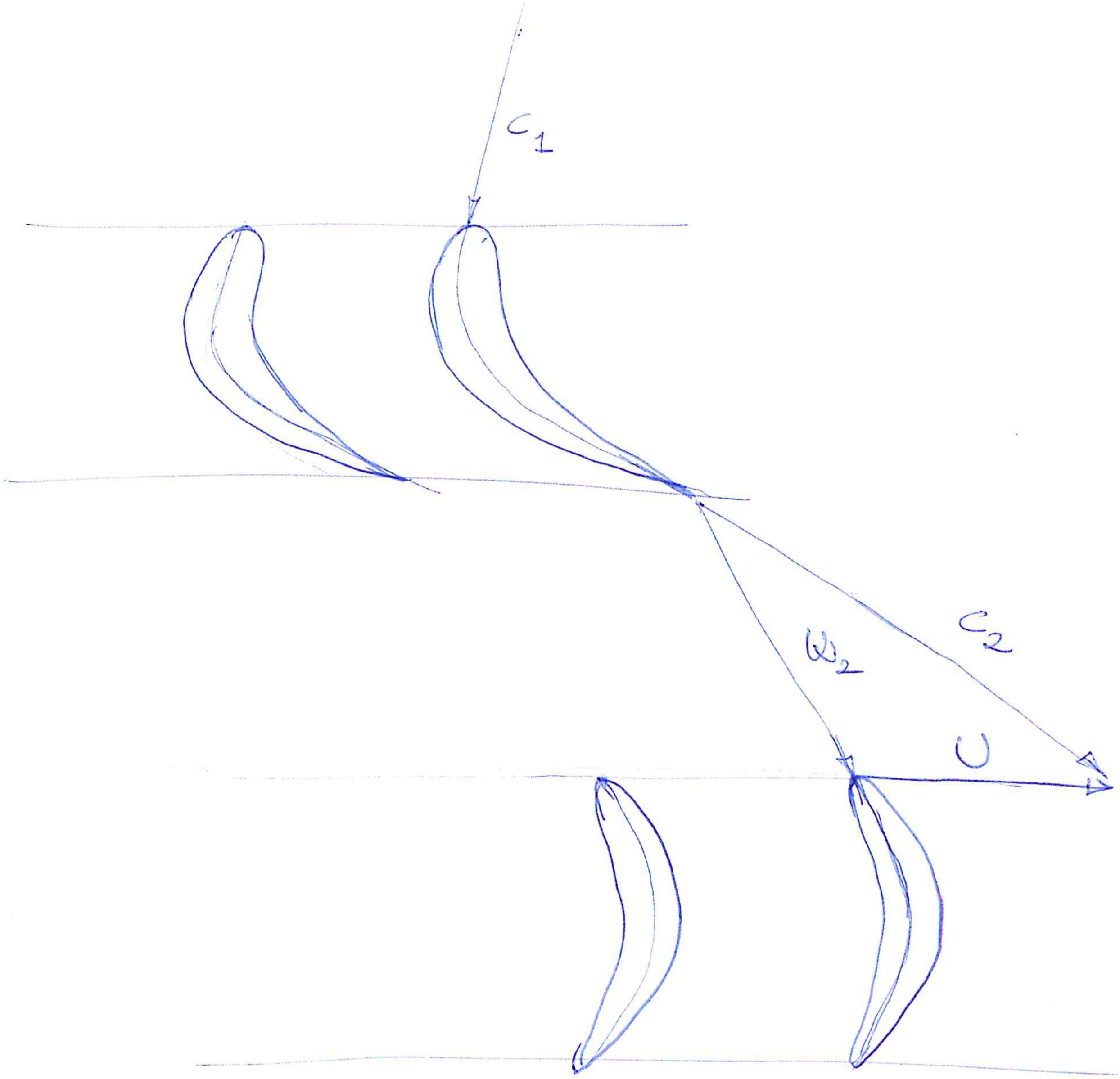
$$R = \frac{\phi}{2}(\tan\beta_3 - \tan\beta_2)$$

If I want $R=0$, I need $\beta_3 = \beta_2$

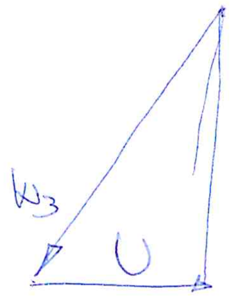




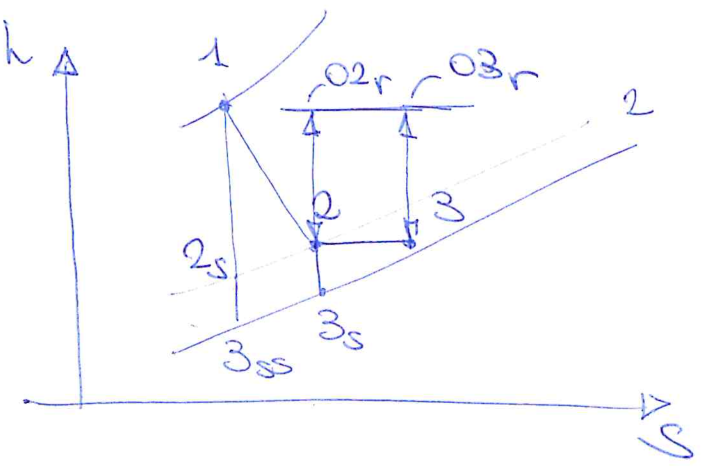
W2
W3
β2
β3
C2
C1 = C3
U

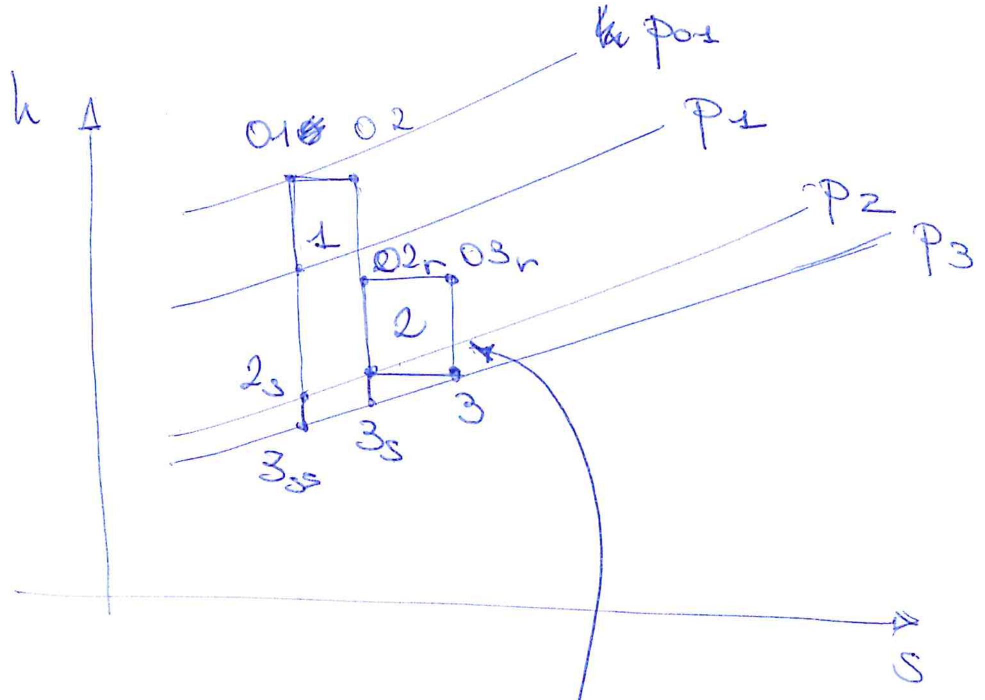


C1
W2
C2
U



W3
U





N.B. In an impulse turbine $p_2 = p_3$, so the point 3 would be here, at a larger h

50% REACTION STAGE

Advantages:

- similar blades (reduced cost)
- low turning
- accelerating flow

} => LOWER LOSSES

Disadvantages

- more stages
- higher rotor structural loads
- higher leakage losses

higher loadings are achieved by increasing interstage swirl angle.

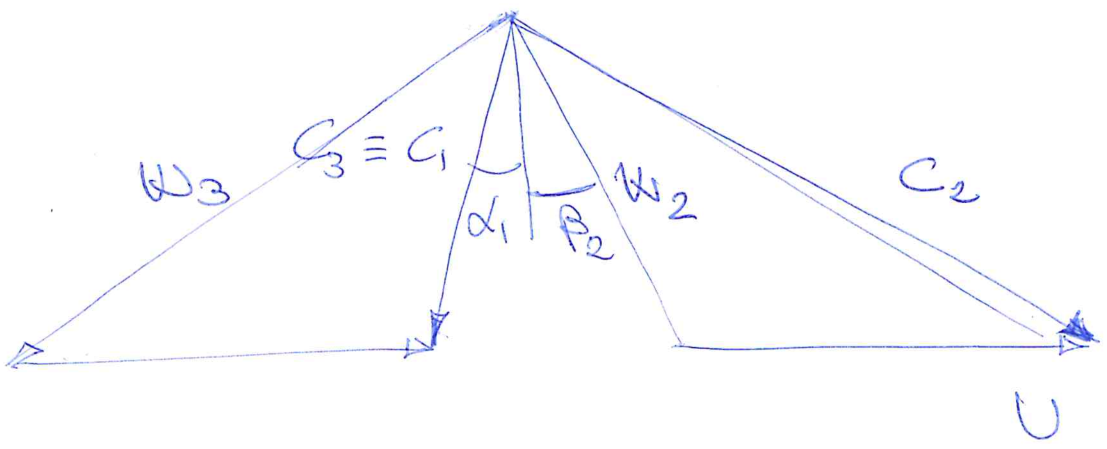
$$R = 1 - \frac{\phi}{2}(\tan\alpha_2 - \tan\alpha_1) =$$

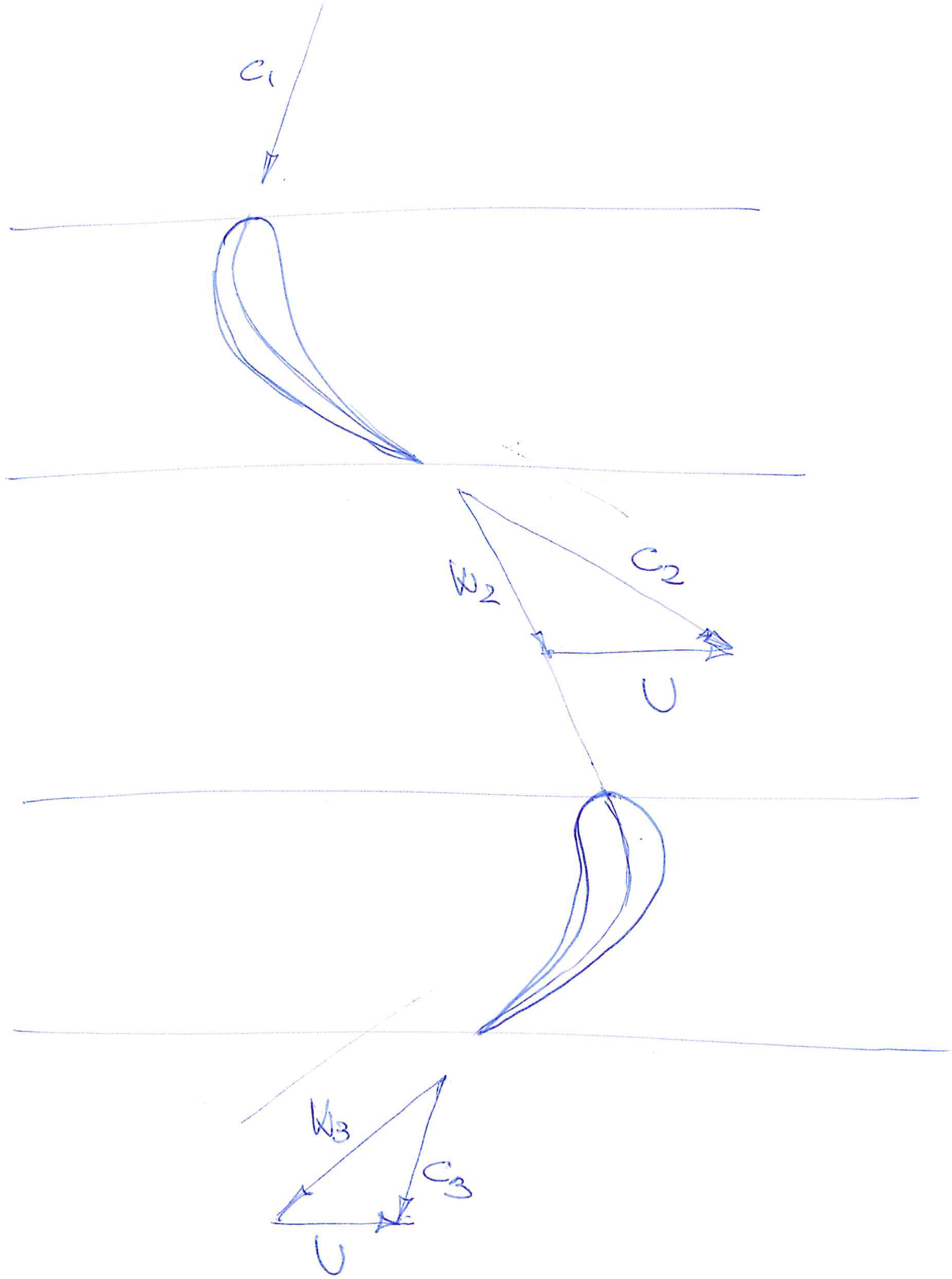
$$\tan\beta_2 = \tan\alpha_2 - \frac{1}{\phi}$$

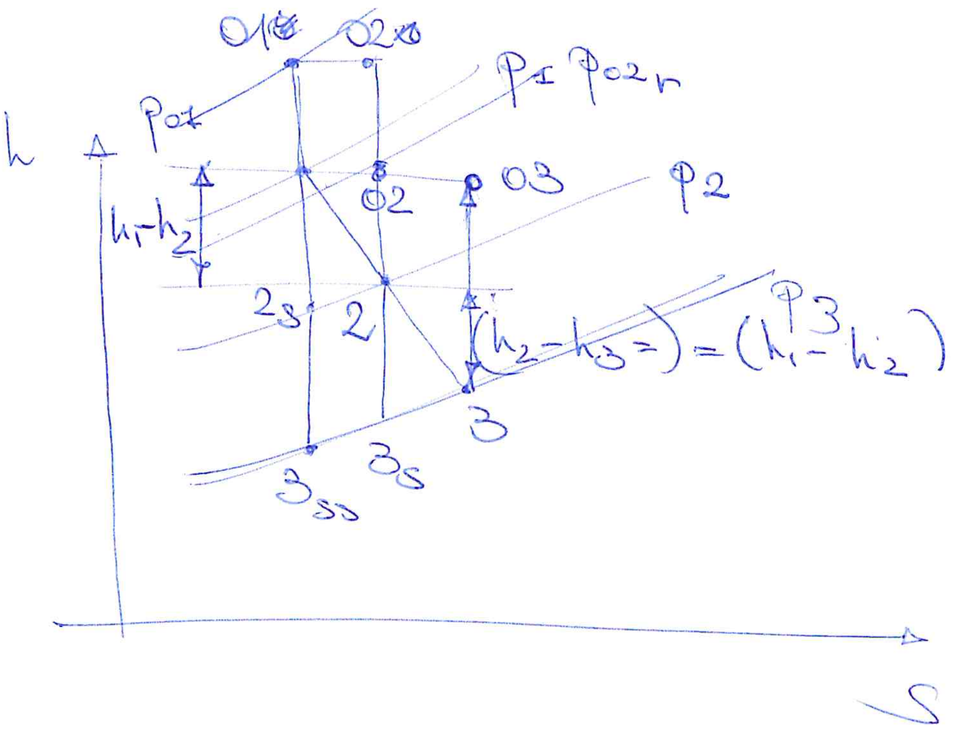
$$R = 1 - \frac{\phi}{2}\left(\tan\beta_2 + \frac{1}{\phi} - \tan\alpha_1\right)$$

$$= \frac{1}{2} - \frac{\phi}{2}(\tan\beta_2 - \tan\alpha_1)$$

$$R = 50\% \Rightarrow \beta_2 = \alpha_1$$


$C_3 \equiv C_1$
W_3
α_1
β_2
W_2
C_2
U


C_1
W_2
C_2
U
W_3
C_3
U



h
s
01
02
p_{01}
p_1
p_{02r}
02
03
p_2
$h_1 - h_2$
2_s
2
p_3
$(h_2 - h_3) = (h_1 - h_2)$
3
3_s
3_{ss}

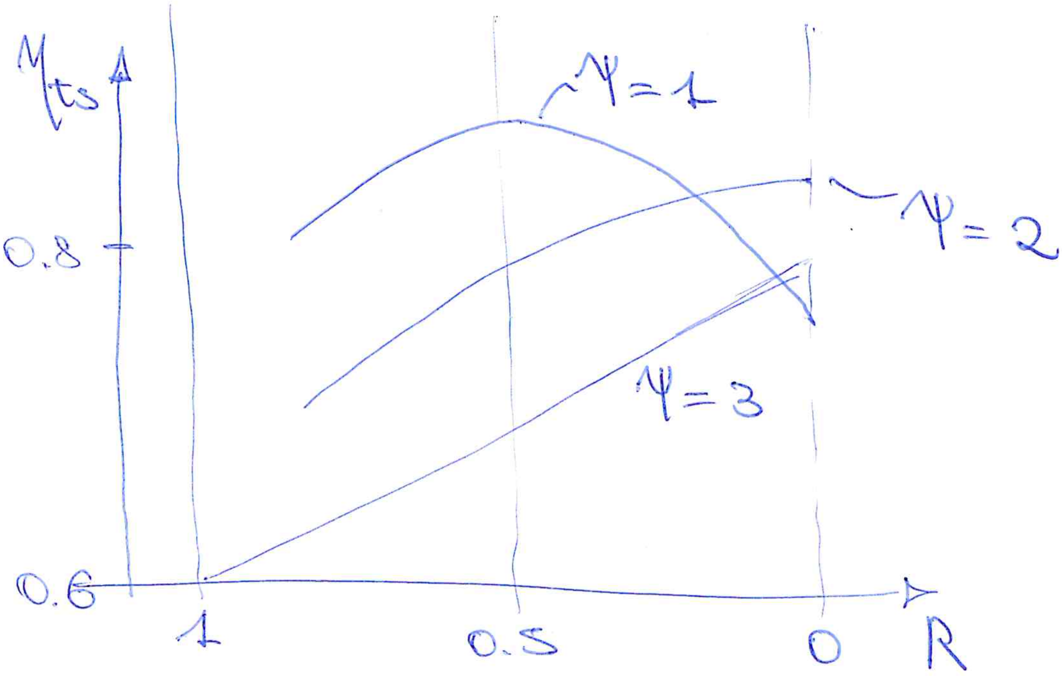
EFFECT OF REACTION ON EFFICIENCY

$$\psi = 2\left(1 - R + \phi \tan \alpha_1\right)$$

Shapiro et al. (1957) did the following: They took a family of turbines each having:

- $\phi = \dfrac{C_x}{U} = 0.4$
- $AR = \dfrac{H}{b} = 3$
- $Re = 10^5$

They calculated the total to static efficiency for different values of ψ, as a function of R (α_1 determined)



$\Rightarrow$ When $\psi = 2$, the maximum value of η_{ts} occurs for zero reaction

$\Rightarrow$ With lighter blade loading, the optimal η_{ts} happens for a higher reaction, and for $\psi = 1$, the optimal R is about 50%

A few considerations : $R = \frac{\phi}{2}\left(\tan\beta_3 - \tan\beta_2\right)$

- If $R<0$ $\tan\beta_3 < \tan\beta_2 \Rightarrow \beta_3 < \beta_2$

Therefore the flow velocity is lower at the exit than at the inlet. Is this a problem? Yes, I have local diffusion, hence separations.

$$c_{\theta 2} = w_{\theta 2} + U \Rightarrow \tan\beta_2 = \tan\alpha_2 - \frac{1}{\phi}$$

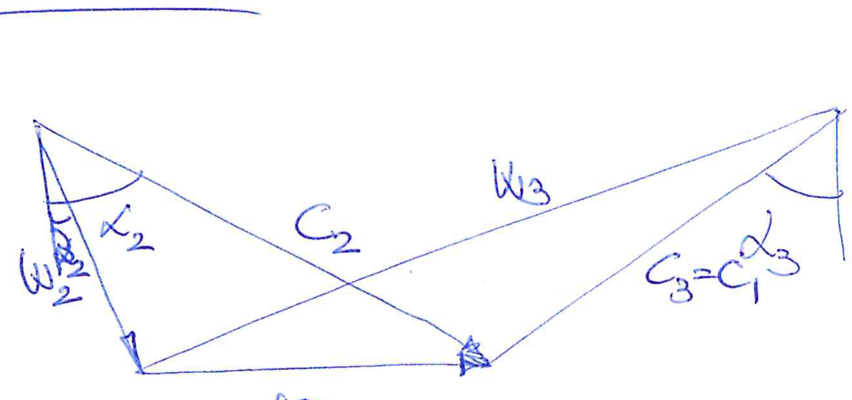
$$c_{\theta 3} = w_{\theta 3} - U \Rightarrow \tan\beta_3 = \tan\alpha_3 + \frac{1}{\phi}$$

$$R = \frac{\phi}{2}\left(\tan\alpha_3 - \tan\alpha_2 + \frac{1}{\phi} + \frac{1}{\phi}\right) =$$

$$= 1 + \frac{\phi}{2}\left(\tan\alpha_3 - \tan\alpha_2\right)$$

- If $R>1 \Rightarrow \tan\alpha_3 > \tan\alpha_2$

For $R=1$ $\boxed{\alpha_3 = \alpha_2}$

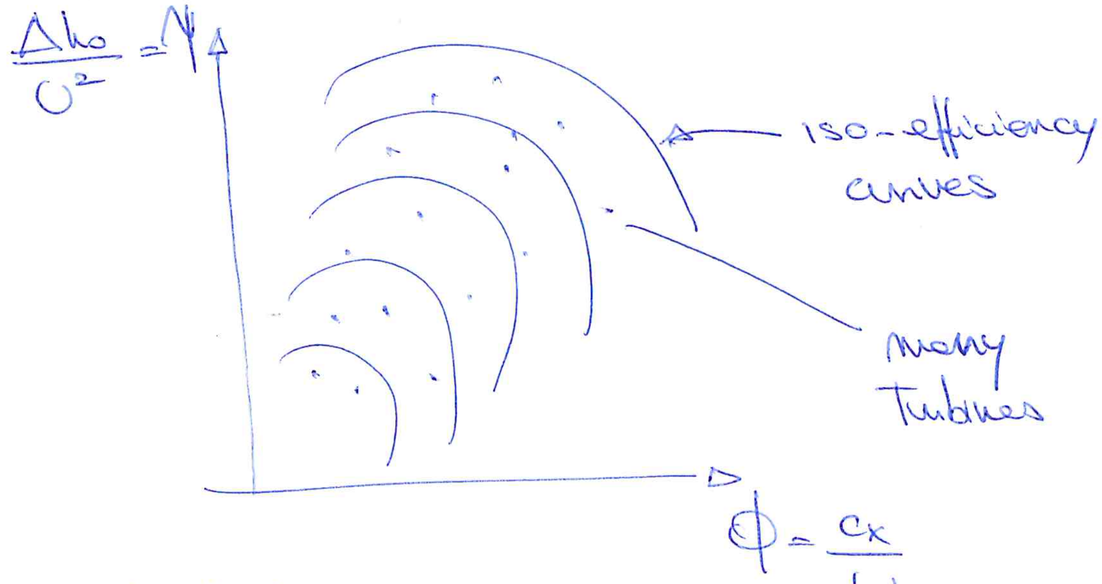


In this case, the nozzle (stator) does not accelerate the flow, if $R>1$, the flow in the nozzle is decelerated

$$R = \frac{h_2 - h_3}{h_1 - h_3} = \frac{h_2 - h_3}{(h_1 - h_2) + (h_2 - h_3)}$$ $(h_1 - h_2 < 0)$

SMITH - CHART (EFFICIENCY)

Smith (1965) devised a correlation chart based upon Rolls-Royce data of turbine efficiency



The turbines had all:

- $0.2 < R < 0.6$
- $3 < AR = \frac{H}{b} < 4$
- $c_x = \text{const}$.

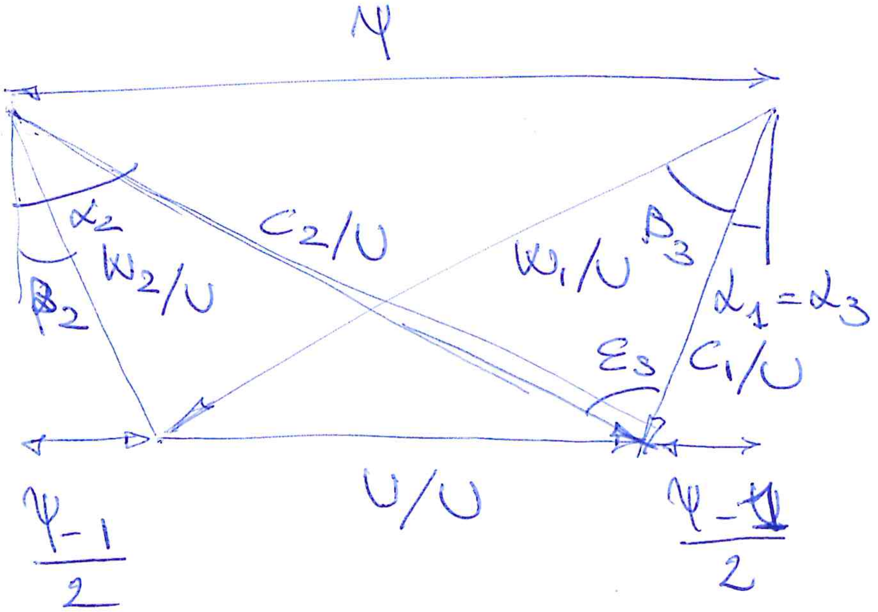
How did Smith explain these curves?

1) He assumed losses to be proportional to the absolute kinetic energy $\left(\frac{1}{2}\left(c_1^2 + c_2^2\right)\right)$

2) He defined a factor f_s

$$f_s = \frac{\Delta h_0}{c_1^2 + c_2^2} = \frac{\Delta h_0 / U^2}{\frac{c_1^2}{U^2} + \frac{c_2^2}{U^2}}$$

3) For $R = 0.5$



$$\tan\alpha_3 = \tan\beta_2 = \frac{\Psi - 1}{2\phi}$$

$$\tan\alpha_2 = \tan\beta_3 = \frac{\Psi + 1}{2\phi}$$

4) $$f_s = \cancel{\frac{\Psi}{\phi^2\left(\tan^2\alpha_3 + \tan^2\alpha_2\right)}} =$$

$$= \frac{\Psi}{2\phi^2 + \phi^2\left(\tan^2\alpha_3 + \tan^2\alpha_2\right)} =$$

$$= \frac{\Psi}{2\phi^2 + \left[\frac{(\Psi-1)^2}{4} + \frac{(\Psi+1)^2}{4}\right]} =$$

$$= \frac{\Psi}{2\phi^2 + \frac{\Psi^2}{2} + \frac{1}{2} +} = \frac{2\Psi}{4\phi^2 + \Psi^2 + 1}$$

5) To find the optimum Ψ for a given Φ:

$$\frac{\partial f_s}{\partial \Psi} = \frac{\partial \left(\frac{2\Psi}{4\Phi^2+\Psi^2+1} \right)}{\partial \Psi} =$$

$$= \frac{2}{4\Phi^2+\Psi^2+1} - \frac{2\Psi \cdot 2\Psi}{(4\Phi^2+\Psi^2+1)^2} =$$

$$= \frac{2(4\Phi^2+\Psi^2+1) - 4\Psi^2}{(4\Phi^2+\Psi^2+1)^2} = \frac{2(4\Phi^2-\Psi^2+1)}{(4\Phi^2+\Psi^2+1)^2}$$

$$\frac{\partial f_s}{\partial \Psi} = 0 \Rightarrow \Psi_{opt} = \sqrt{4\Phi^2+1}$$

From experiments: $\Psi_{opt, exp} = 0.65\sqrt{4\Phi^2+1}$

(remarkably close)

DESIGN POINT EFFICIENCY

We start from $\left(\frac{T_{03}}{T_3} \simeq 1 \right)$

$$\frac{1}{\eta} = 1 + \frac{\zeta_R w_3^2 + \zeta_S c_2^2}{2 \Delta W}$$

$$\Delta W = \psi U^2$$

50% REACTION :

$$w_3^2 = c_2^2$$

$$\zeta_R = \zeta_S = \zeta$$

$$w_3^2 = c_x^2 + w_{\theta 3}^2 = c_x^2 \left(1 + \tan^2 \beta_3 \right)$$

$$\frac{1}{\eta_u} = 1 + \frac{\zeta \phi^2}{\psi} \left(1 + \tan^2 \beta_3 \right)$$

$$= 1 + \frac{\zeta \phi^2}{\psi} \left(1 + \left(\frac{\psi + 1}{2\phi} \right)^2 \right]$$

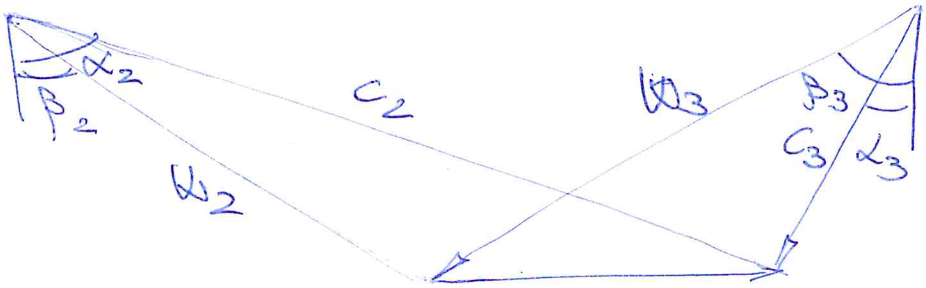
with Soderberg's correlation for losses

$$\zeta = 0.04 + 0.06 \left(\frac{\varepsilon}{100} \right)^2$$

$$\psi = 2 \left(1 - R + \phi \tan \alpha_1 \right)$$

$$\varepsilon = \operatorname{atan} \left(\frac{\psi - 1}{2\phi} \right) + \operatorname{atan} \left(\frac{\psi + 1}{2\phi} \right)$$

0 REACTION



$$\beta_3 = \beta_2$$

$$\frac{w_3^2}{U^2} = \frac{c_x^2}{U^2} + \frac{w_{\theta 3}^2}{U^2} = \phi^2\left(1 + \tan^2\beta_3\right)$$

$$\frac{c_2^2}{U^2} = \frac{c_x^2}{U^2} + \frac{c_{\theta 2}^2}{U^2} = \phi^2\left(1 + \tan^2\alpha_2\right)$$

$$\tan\alpha_2 = \tan\beta_2 + \frac{1}{\phi} \quad ; \quad \tan\alpha_3 = \tan\beta_3 - \frac{1}{\phi}$$

$$\psi = \frac{\Delta W}{U^2} = \frac{c_{\theta 2} + c_{\theta 3}}{U} = \phi\left(\tan\alpha_2 + \tan\alpha_3\right)$$
$$= \phi\left(\tan\beta_2 + \tan\beta_3\right)$$

$$\tan\beta_3 = \frac{\psi}{2\phi}$$

$$\tan\alpha_2 = \frac{\psi}{2\phi} + \frac{1}{\phi}$$

$$\frac{1}{\eta} = 1 + \frac{\zeta_R w_3^2 + \zeta_S c_2^2}{2\Delta W}$$

$$\frac{1}{\eta} = 1 + \frac{1}{2\Psi}\left\{ \zeta_R\left(\phi^2 + \left(\frac{\Psi}{2}\right)^2\right) + \zeta_S\left(\phi^2 + \left(1 + \frac{\Psi}{2}\right)^2\right)\right\}$$

This can be put together with:

- Soderberg's $\quad \zeta = 0.04 + 0.06\left(\frac{\varepsilon}{100}\right)^2$

- $\Psi = 2\left(1 - R + \phi \tan\alpha_1\right)$

~~$\varepsilon_R = 2\tan$~~

$\rightarrow \tan\beta_2 = \tan\beta_3 = \dfrac{\Psi}{2\phi}$

$\rightarrow \tan\alpha_2 = \dfrac{\Psi}{2\phi} + \dfrac{1}{\phi}$

$\rightarrow \tan\alpha_1 = \dfrac{\Psi}{2\phi} - \dfrac{1}{\phi}$

$$\varepsilon_R = 2\,\mathrm{atan}\left(\frac{\Psi}{2\phi}\right) \qquad \varepsilon_S = \mathrm{atan}\left(\frac{\Psi}{2\phi} + \frac{1}{\phi}\right) + \mathrm{atan}\left(\frac{\Psi}{2\phi} - \frac{1}{\phi}\right)$$