Stochastic Models

A.A. 2024/2025

Assignment 6

Exercise 1. Consider a wireless system that provides channels each of which can serve one phone call. There are 100 users making phone calls. Each user makes on average one phone call per hour and the average duration of a phone call is 3 minutes.

(a) What is the total traffic in Erlang that the 100 users generate.

(b) Suppose the system implemented as a M/M/m, allocate the minimum m avoiding the traffic congestion

(c) Suppose the system implemented as a M/M/m/m, determine its blocking probability (the probability a call is rejected by the system), as well as its abandonment rate.

(d) Suppose the abandonment rate obtained at the previous point is not satisfactory. Explain how to improve it.

Hint. Let $\rho = \frac{\lambda}{m\mu}$, the steady-state probabilities of a M/M/m are as follows

$$\pi_{s,0} = \frac{1}{\left(\sum_{i=0}^{m-1} \frac{m^i \varrho^i}{i!}\right) + \frac{m^m}{m!} \sum_{i=m}^{\infty} \rho^i} \quad , \quad \begin{cases} \pi_{s,i} = \dfrac{m^i \varrho^i}{i!} \pi_{s,0} & (i < m) \\[2ex] \pi_{s,i} = \dfrac{m^m}{m!} \varrho^i \pi_{s,0} & (i \geqslant m) \end{cases}$$

Exercise 2. About 80 customers per hour arrive to an ice-cream shop the weekend afternoons.

The manager may hire a very efficient waiter able to serve a customer in 30 seconds, or, at the same cost, 2 waiters each able to serve on average a customer in 1 minute.

Customers, in both cases, would be placed in a single line.

(a) Determines, for the two cases, the average time spent by a customer to buy an ice cream. Arrivals and services are assumed to be independent.

(b) Which of the two solutions would be preferred in terms of quality of the service offered?

(c) For the two cases evaluate the resulting resource utilization factor, then discuss the results.

Exercise 3. In a point-to-point communication link packets arrive to a repeater in a Poisson fashion. The transmission time is proportional to the packets' length $L > 0$.

Determine the Kendall notation of the resulting queue systems, for the following three use-cases:

(a) L is an exponential random variable, and the repeater's buffer has an infinite capacity.

(b) L is fixed and the repeater's buffer may accommodate at most n packets.

(c) $L \sim (S_L, p_L)$ is a nonparametric random variable with sample-space S_L, and probability function p_L, and there is no buffer to store the arrived packets.

Exercise 4. A small Financial Consultancy Office has 3 desks, each offering a different service.

Although the Office opens at 9am, people start queuing from 8am. However, due to COVID-19 contingencies no more than two people may stay at the same time within office, thus people wait in the street and, as soon as a person leave the office another is ready to enter.

The service time of each desk is approximatively the same and equal to 1/4 hours.

Costumers are routed from a desk to another in accordance with the following routing matrix

$$\boldsymbol{R} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.25 & \alpha & 0.75 - \alpha \\ 0.25 & 0.75 - \beta & \beta \end{pmatrix} \qquad \alpha, \beta \in [0,\ 0.75],$$

where r_{ij} denotes the probability a costumer upon the service at desk i is routed to desk j.

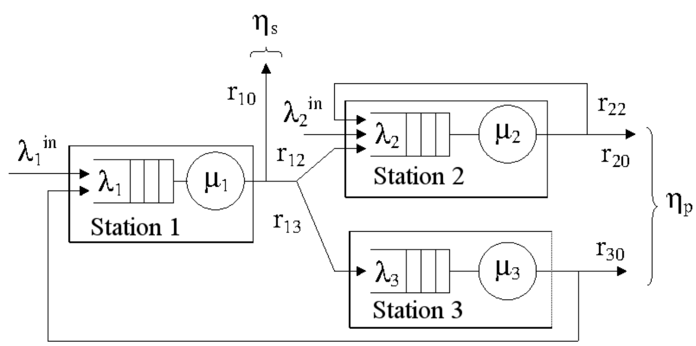
(a) Model the system as a Queuing Network and verify under which conditions it is ergodic.

(b) Model the system as a Continuous-Time Markov Chain (CT-MC) and verify under which conditions it is ergodic.

(c) Set $\alpha = \beta = 0.75$, discuss if the resulting stochastic process is ergodic.
Then discuss how the network's joint steady-state probability distribution can be determined.

Exercise 5. Consider the manufacturing system in the figure where stations are M/M/1 resources.

The routing probabilities are as follows: $r_{12} = 0.4$; $r_{13} = 0.5$; $r_{22} = 0.2$; and $r_{31} = 0.2$, while $\lambda_1^{in} = 2$ and $\lambda_2^{in} = 1$. Finally the service rate of each station are, resp., $\mu_1 = 3$, $\mu_2 = 4$ and $\mu_3 = 2$.



We denote η_s as the mean departure rate of defective pieces, while η_p is the mean system productivity, namely the mean departure rate of correctly processed pieces.

(a) Verify if this queueing network is ergodic.

(b) Determine the probability a piece leave the system after being served in node i, $\forall\, i = 1, 2, 3$.

(c) Determine the number of pieces in the network at the steady-state.

(d) Determine the mean time a piece need to cross the network.

(e) Determine the mean time a piece from outside needs to cross node 2.

(f) Determine η_s and η_p, then compare $\eta_s + \eta_p$ with the external arrival rate.

(g) Let 0.2€ be the cost of a raw piece, whereas a produced piece is sold at 0.5€. Let the cost per hour of each station be 1000€. Evaluate the system's profit per hour.

(h) Determine the steady state probability that both station 1 and 2 are busy.

(i) Determine the steady state probability to have only one pieces is system.

(j) Suppose only one of the three servers will be upgraded to a faster one with a service rate of $\mu' = 5$. Which of the three is the most convenient to upgrade to:

1. reduce the average number of pieces in the system at steady state?
2. reduce the mean time needed to cross the network?
3. reduce the mean time needed to pass through node 2 from outside at steady state?
4. reduce the mean rate of rejected items η_s?
5. increase the mean productivity η_p?

Solutions

Solution of Exercise 1. The concept of amount of traffic is very important for the telecommunications industry. To measure the quantity of traffic, we consider the arrival rate of customer calls as well as the amount of network resources the calls require. The traffic or *traffic intensity* is measured in units called "Erlang" and it is defined as λ/μ for M/M/1 and M/M/∞ queues.

Intuitively, the traffic intensity is λ times the mean service time $1/\mu$. If it is equal to one, it will require on average one busy server forever. If it is equal to two, it will require two busy servers forever, etc. Accordingly, if the traffic is ρ, then ρ is the mean number of servers occupied in steady state by this traffic.

It further results that a single server $\cdot/\cdot/1$ resource cannot satisfies traffic intensities larger then 1, because of it has a single server. On the other hand in a M/M/∞ all the arriving traffic can be served. This is the reason why in a M/M/∞, the mean number of calls in the system is equal to the mean number of servers occupied in steady state, namely $\bar{x} = \bar{x}_s = \rho$.

If instead we consider a multi-server resource with m servers and service time equal to $1/\mu$ (that is widely used in telecommunications network design), it results that it may support a traffic load equal to λ, if and only if $m \cdot \mu > \lambda$. For M/M/m (or M/M/m/m), ρ takes no more the meaning of traffic intensity, in fact in this case $\rho = \frac{\lambda}{m \cdot \mu} < 1$. In some sense for a multi server resource ρ takes the meaning of the traffic that each server have to face off.

Given that, let us now focus again with the given assignment. Since our wireless system is able to provide a channel to each call, then it means the system can be modelled as a M/M/∞ queue. Let us now compute its traffic intensity.

The first step is to choose a consistent time unit. Here it is convenient to choose minutes because the service time is expressed in minutes. Accordingly, the arrival rate of each user is

$$\lambda_i = \frac{1}{60} \text{ calls per minute}$$

The mean service time, that in this consists in the mean call duration, or equivalently in the mean time the serves stay busy, that is called also "holding time" is

$$\frac{1}{\mu} = 3 \text{ minutes.}$$

It follows that each user generate a traffic intensity of $\rho_i = 3/60 = 1/20$ Erlang, and the total traffic is

$$\rho = \sum_{i=1}^{100} \rho_i = 100 \cdot \frac{1}{20} = 5 \text{ Erlangs.}$$

By means of the previous results we can also states that, to face off with this traffic, and if this traffic condition is persistent, in place of a M/M/∞ it is sufficient a M/M/6 system.

In the case instead we were considering a M/M/6/6 (where $k = 6$ is counting also the server number), to determine its blocking there are two ways: the first is to model a M/M/6/6 queue as

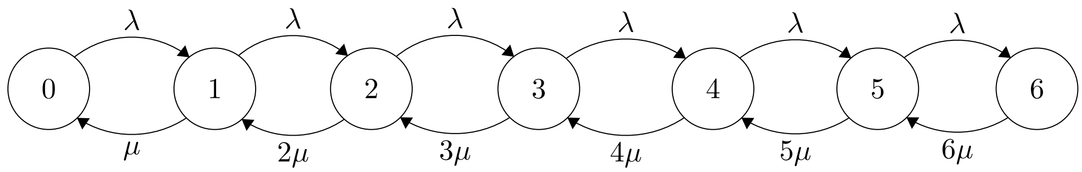


Figure 1: Transition graph of a M/M/6/6 queue.

a finite CT-MC as in the Figure 1, and then solve its balancing equations

$$\begin{cases} \Pi_s \cdot \boldsymbol{Q} = \boldsymbol{0} \\ \Pi_s \cdot \boldsymbol{1} = 1 \end{cases}, \quad \boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & 0 & 0 \\ 0 & 0 & 3\mu & -(\lambda+3\mu) & \lambda & 0 & 0 \\ 0 & 0 & 0 & 4\mu & -(\lambda+4\mu) & \lambda & 0 \\ 0 & 0 & 0 & 0 & 5\mu & -(\lambda+5\mu) & \lambda \\ 0 & 0 & 0 & 0 & 0 & 6\mu & -6\mu \end{pmatrix}$$

where $\Pi_s = (\pi_{s,0}, \pi_{s,1}, \pi_{s,2}, \pi_{s,3}, \pi_{s,4}, \pi_{s,5}, \pi_{s,6})$, and then find $\pi_6(\infty) \equiv \pi_{s,6}$. Alternatively we can derive it from the steady-state probabilities of a M/M/m queue. In particular, since the only difference between a M/M/6 and a M/M/6/6 is that the second has only 6 states, then we have that

$$\pi_{s,i} = \frac{(m\varrho)^i}{i!}\pi_{s,0} = \frac{\left(6 \cdot \frac{5}{6}\right)^i}{i!} = \frac{5^i}{i!} \quad \forall \quad i = 0,1,2,3,4,5,6$$

whereas $\pi_{s,0}$ changes as next

$$\pi_{s,0} = \frac{1}{\left(\sum_{i=0}^{m-1} \frac{m^i \varrho^i}{i!}\right) + \frac{m^m \varrho^m}{m!}} = \frac{1}{\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!} + \frac{5^6}{6!}} = 0.0088$$

in accordance with the constraint that $\Pi_s \cdot \boldsymbol{1} = 1 \equiv \sum_{i=0}^{6} \pi_{s,i} = 1$. From that it follows that the so-called Erlang-B loss formula for a M/M/6/6, that is

$$\pi_6(\infty) = \frac{(6\varrho)^i}{6!} \cdot \pi_0(\infty) = 0.1918 \approx \frac{1}{5}$$

and, it further results that the missed traffic on average is

$$\lambda_{ab} = \lambda \cdot \pi_6(\infty) = \frac{1}{60} \cdot 0.1918 \approx \frac{1}{60} \cdot \frac{1}{5} = \frac{1}{300} \text{calls per minute}$$

Notice that, let $m = 6$ be given, in the case we were interested to higher performances rate in terms of abandonment rates, like $\lambda_{ab} \leqslant \alpha^* \lambda$, and because of μ and m are supposed to be given, we may think to consider a M/M/6/k (where k is counting also the server number), with $k \geqslant 6$ of such that

$$k^* \; : \; \pi_{s,k} = \frac{m^m}{m!} \cdot \rho^k \pi_{s,0} = \frac{m^m}{m!} \cdot \rho^k \left(\frac{1}{\sum_{i=0}^{m-1} \frac{(m\rho)^i}{i!} + \frac{m^m}{m!} \sum_{i=m}^{k} \rho^i} \right) \leqslant \alpha^*, \text{ with } k \geqslant m$$

Solution of Exercise 2. The costumers' arrival rate is

$$\lambda = 80 \frac{\text{clients}}{\text{hours}} \times \frac{1}{60} \frac{\text{hour}}{\text{min}} = \frac{4}{3} \frac{\text{clients}}{\text{min}}$$

Since both arrivals and services follows a Poisson fashion, then the manager may choose between an M/M/1 or an M/M/2 queue system. Let us consider the first case.

The efficient waiter has a service rate

$$\mu' = \frac{1}{30} \frac{\text{clients}}{\text{sec}} \cdot \frac{60}{1} \frac{\text{sec}}{\text{min}} = 2 \frac{\text{clients}}{\text{min}}.$$

It thus results that, in this case the system is subjected to a traffic intensity of

$$\rho' = \frac{\lambda}{\mu'} = \frac{\frac{4}{3}}{2} = \frac{2}{3} < 1.$$

Reminding that an ergodic M/M/1 queues behaves as an ergodic uniform continuous-time birth death process, for which the we have that

$$\pi_{s,i} = \rho^i \cdot (1 - \rho)$$

and the mean population size is equal to

$$\bar{x} = \frac{\rho}{1-\rho} = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

then from the Little's Law it results that the mean time spent by a costumer to buy an ice-creen at the weekend is of

$$\bar{x}' = \lambda \cdot \bar{\theta}' = 2 \quad \to \quad \bar{\theta}' = \frac{\bar{x}'}{\lambda} = \frac{2}{\frac{4}{3}} = \frac{6}{4} = 1.5 \text{ min.}$$

In the second case, and by letting $m = 2$, and

$$\rho'' = \frac{\lambda}{2 \cdot \mu''} = \frac{\frac{4}{3}\frac{\text{clients}}{\text{min}}}{2 \cdot 1 \frac{\text{clients}}{\text{min}}} = \frac{2}{3} < 1$$

we confirms that our M/M/2 system would be ergodic. Moreover from the table of the "Quantities of interest of Markovian Queueing Systems" we can find that for an ergodic M/M/2 resource, the mean number of costumers in the system is

$$\bar{x} = m\rho + \frac{m^m \rho^{m+1}}{m!(1-\rho)^2} \pi_{s,0}$$

where $\pi_{s,0}$ is the so-called *idle rate*, namely, the probability that the resource is empty, that is equal to

$$\pi_{s,0} = \left. \frac{1}{\left(\sum_{i=0}^{m-1} \frac{m^i \rho^i}{i!} \right) + \frac{m^m \rho^m}{m!(1-\rho)}} \right|_{m=2} = \frac{1}{\frac{(2\rho)^0}{0!} + \frac{(2\rho)^1}{1!} + \frac{(2\rho)^2}{2!(1-\rho)}} = \frac{1}{5}.$$

Thus, it further results that

$$\bar{x}'' = 2 \cdot \frac{2}{3} + \frac{2^2 \left(\frac{2}{3}\right)^3}{2! \left(1 - \frac{2}{3}\right)^2} \cdot \frac{1}{5} = \frac{36}{15} = 2.4 \quad \rightarrow \quad \bar{\theta}'' = \frac{\bar{x}''}{\lambda} = \frac{2.4}{\frac{4}{3}} = 1.8 \text{ min}$$

From the above analysis it results that the efficient waiter will be able to drain faster the queue with respect to the second solution. Thus, from the costumer's perspective the first solution should be preferred.

Now for the two cases let us compare, respectively, the resource utilization, the single server utilization and the idle rate, the probability that the service area is busy.

Let us now evaluate the idle rate of the two queues

$$\text{M/M/1} \ : \ \pi_{s,0} = 1 - \rho' = \frac{1}{3} \approx 0.333 \quad , \quad \text{M/M/2} \ : \ \pi_{s,0} = \frac{1}{5} = 0.2,$$

their utilization factors

$$\hat{U}' = 1 - \pi_{s,0} = \rho' = \frac{2}{3} \approx 0.66 \quad , \quad \hat{U}'' = \Pr(x \geqslant 1) = 1 - \pi_{s,0} = 1 - \frac{1}{5} = \frac{4}{5} = 0.8.$$

These results are clearly compliant to each other, in fact since the total demand λ is the same for the two queues, but because in the $M/M/2$ the mean service time is larger than the $M/M/1$, then the $M/M/1$ will be able to serve arrival faster and it will stay in idle for a larger percentage of time.

Solution of Exercise 3. Arrivals are clearly Poissonian in both three cases, whereas since the transmission time is proportional to the packets' length, thus the service process will completely outlined by packet size.

Consider for instance the first case, where the packets' length L in [bits/packet], is assumed exponentially distributed, namely $L \sim Exp(\bar{L})$, where $\bar{L} = \mathsf{E}[L]$. This fact will also implies that, expects for some constants $k > 0$ in [packets/s], the service time θ_s in [sec/packets] has the same distribution of L, i.e.,

$$\theta_s = kL \propto L \sim Exp(\bar{L}) \qquad \Longrightarrow \qquad \theta_s \sim Exp(\bar{\theta}_s)$$

where

$$\bar{\theta}_s = \mathsf{E}[\theta_s] = \mathsf{E}[kL] = k\mathsf{E}[L] = k\bar{L}$$

Then since, the service time is exponentially distributed, then we can further concludes that services follows a Poisson Process which mean is $\mu = 1/\bar{theta}_s$, and thus, it further results that our resource in case (a) is a

$$\text{Case (a): } M/M/1.$$

On the other hand, in the case (b), because the buffer node can accommodate at most n packets, but the packet length is constant, it follows that we are considering a

$$\text{Case (b): } M/D/1/(n+1).$$

Note that in this case we are considering, in the fourth field of the Kendall notation the whole system capacity, not only the buffer size.

Lastly, in case (c), because it is made a zero buffer assumption, but the packets length is a non-parametric random variable, then the Kendall notation for the last system is

$$\text{Case (c): } M/G/1/1.$$

Solution of Exercise 4. The consider social service network can be considered as a closed queueing network consisting of three queues, each with an identical mean service rate $\mu_i = \mu = 4$ costumers per hour, and where the costumers' population at each time t is always equal to $n = 2$.

To evaluated the ergodicity condition for the considered Gordon-Newell network we could simply verify if the networks' routing probability matrix

$$\boldsymbol{R} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.25 & \alpha & 0.75-\alpha \\ 0.25 & 0.75-\beta & \beta \end{pmatrix} \tag{1}$$

has a simple eigenvalue in 1. Thus, by solving

$$\det\left(\mathrm{sI}_{3\times 3} - \mathrm{R}\right) = 0$$

or equivalently by running the following Matlab script

```
>> syms a b s real
>> R=[0.5 0.5 0; 0.25 a 0.75-a; 0.25 0.75-b b]
>> solve(det(s*eye(3)-R)==0)
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one obtains

$$\det\left(\mathrm{sI}_{3\times 3} - \mathrm{R}\right) = 0 \rightarrow \begin{cases} s_1 &= 1 \\ s_2 &= \frac{1}{4} < 1 \\ s_3 &= \alpha + \beta - \frac{3}{4} < 1 \qquad \Longrightarrow \alpha + \beta < \frac{7}{4} \end{cases} \tag{2}$$

Then, because of the two uncertain parameters α and β at most are equal to 0.75, then it results that

$$\alpha + \beta \leqslant 0.75 + 0.75 = 2 \cdot \frac{3}{4} = \frac{6}{4} < \frac{7}{4}.$$

Thus, we can conclude that this network is ergodic for all the possible α and $\beta \in [0, 0.75]$.

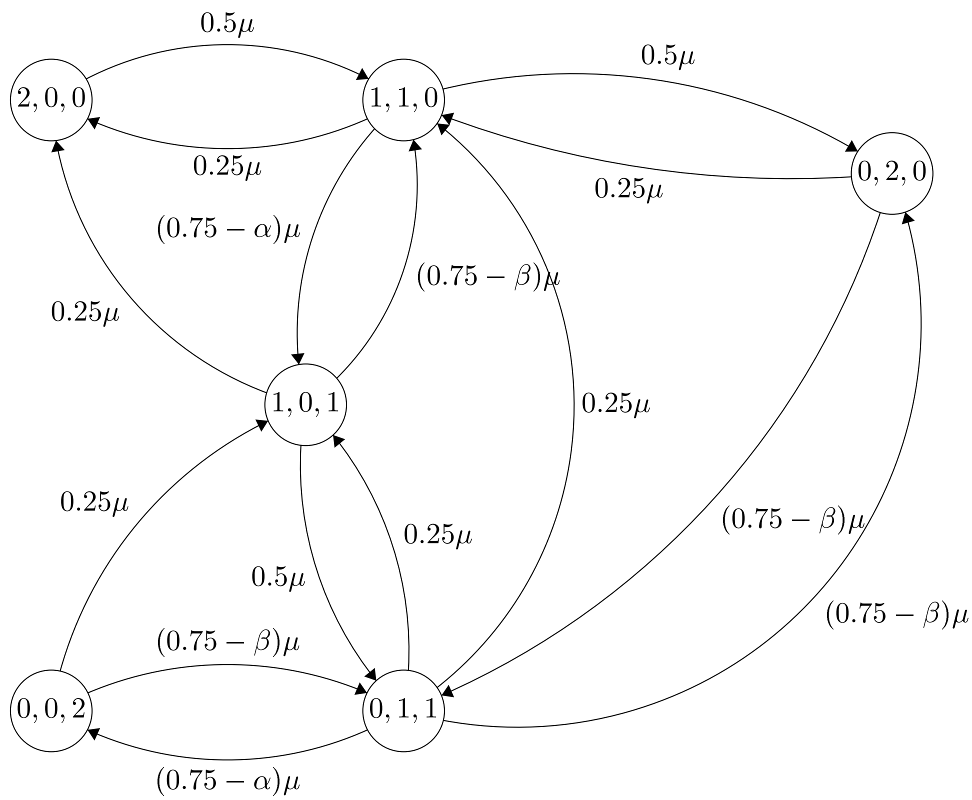
Then, because of the considered network is closed, it consists of only $v = 2$ stations, and $n = 2$, then all the possible arrangement of costumers within the system counts a number of combination that is given by the well-known formula of the number of combination of with repetition, namely, if $N_{v,n}$ denotes the sample space of all the possible arrangement of the $n = 2$ costumers within the $v = 3$ stations, it results that

$$|N_{n,v}| = \mathrm{card}\{N_{n,v}\} = \binom{n+v-1}{n} = \frac{(n+v-1)!}{n!(v-1)!} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2!} = 6.$$

In particular we have that all the possible arrangements are, respectively,

$$N_{2,3} = \{(2,0,0),\ (1,1,0),\ (0,2,0),\ (1,0,1),\ (0,0,2),\ (0,1,1)\} \tag{3}$$

Finally by means of the knowledge of (3), and $\boldsymbol{R}$, and because of $\mu_i = \mu = 5\ \forall\ i = 1,2,3$, we can derive the following transition graph for the CT-MC associated with the considered Gordon-Newell network:



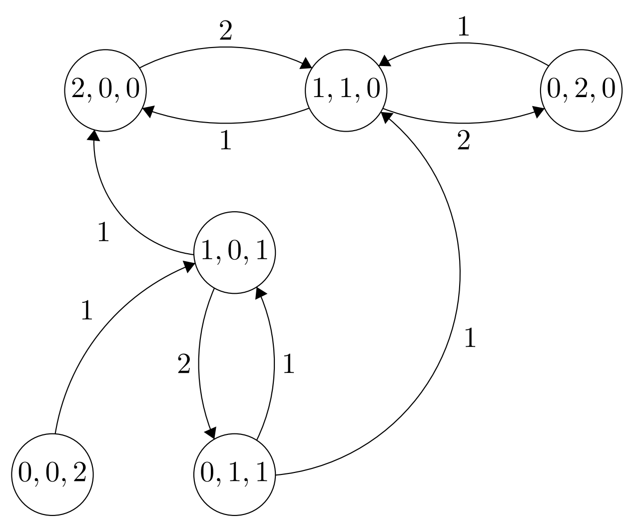
Finally, by letting

$$\alpha = \beta = 0.75 \tag{4}$$

and because of

$$\mu = 4$$

the above transition rate graph degenerates in the following, simpler, transition rate graph



Form that, and by considering the same order for the states as that used in (3) we can derives the

the transition rate matrix of the considered CT-MC as follows

$$\boldsymbol{Q} = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ 1 & -3 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix} \tag{5}$$

Then, by means of the graphical criteria for CT-MC it is easy to note that the considered CT-MC is ergodic because of its graph has a unique strongly connected aperiodic component, that is

$$E = \{(2,0,0),(1,1,0),(0,2,0)\},$$

whereas all the remaining states are transient. This result is clearly compliant to that obtaine by means of equation (2). Such conclusion would be also evident by drawing the routing graph associated with this close queueing network.

Then because of the network is ergodic we can derive its limiting stationary distribution Π_ℓ, by simply evaluating its stationary distribution Π_s (that is unique and independent form $\Pi(0)$ because of the ergodicity) by solving the usual linear system

$$\begin{cases} \Pi_s \cdot \boldsymbol{Q} = \boldsymbol{0} \\ \Pi_s \cdot \boldsymbol{1} = 1 \end{cases} \tag{6}$$

By solving it numerically, we derives that

$$\Pi_\ell \equiv \Pi_s = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} & \frac{4}{7} & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

Note that the above results is coherent we the fact that, after having let $\alpha = \beta = 0.75$, the resulting strongly connected component of the network is degenerated in the following subset of states

$$\{(2,0,0),(1,1,0),(0,2,0)\}$$

The same result could also be obtained by solving the balancing equation associated to the reduce CT-MC consisting of only the above three states, while setting to 0 the stationary probabilities of the remaining states. If so, the problem complexity will be clearly reduced because now we should solve a simpler 3×3 linear system, in place of a 6×6 linear system.

Let us further note that the above stationary probability distribution could be alternatively evaluated by simply invoking of the Gordon-Newell Theorem, that is provided below for the clarity sake.

Gordon-Newell Theorem: *Consider an ergodic Gordon-Newell Network of v $M/M/m_i$ nodes, and n costumers within. Let $\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_v(t))$ be a random vector, and X_i be the discrete random variable counting the number of costumers in the i-th node of the network. Let $N_{v,n}$ be the set of all the possible arrangement of the n costumers is the nodes. Consider*

$\boldsymbol{x} = (x_1, x_2, \dots, x_v) \in N_{v,n}$. *Then, the following joint probability, at the steady state, takes the following product form*

$$\Pr(\boldsymbol{X}(\infty) = \boldsymbol{x}) = \Pr(\ X_1(\infty) = x_1, X_2(\infty) = x_2, \dots, X_v(\infty) = x_v\ ) = \kappa \cdot \prod_{i=1}^{v} \beta_i(x_i) \tag{8}$$

where κ is a normalization constant that is derived by the following constraint

$$\sum_{\forall (x_1, x_2, \dots, x_v) \in N_{v,n}} \kappa \cdot \prod_{i=1}^{v} \beta_i(x_i) = 1 \quad \rightarrow \quad \kappa = \frac{1}{\left(\sum_{\forall (x_1, x_2, \dots, x_v) \in N_{v,n}} \prod_{i=1}^{v} \beta_i(x_i) \right)}$$

whereas the functions $\beta_i(x_i)$ are selected in accordance with the next relations

$$\textit{if the i-th node is a } M/M/1 \ : \quad \beta_i(x_i) \quad = \rho_i^{x_i} \quad \forall \quad x_i \in [0, n]$$

$$\textit{if the i-th node is a } M/M/m_i \ : \quad \beta_i(x_i) \quad = \begin{cases} \dfrac{\rho_i^{x_i}}{x_i!} & \text{if} \quad x_i < m_i \\ \dfrac{\rho_i^{x_i}}{m_i! m_i^{x_i - m_i}} & \text{if} \quad x_i \geqslant m_i \end{cases}$$

and $\rho_i = \lambda_i / \mu_i$, and λ_i is a solution of $\boldsymbol{\lambda} = \boldsymbol{\lambda} \cdot \boldsymbol{R}$, where $\boldsymbol{R} \in \mathbb{R}^{v \times v}$ is routing probability matrix of the network and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_v) \in \mathbb{R}^v_{\geqslant 0}$. ■

Thus by simply solving the traffic equation for the considered Gordon-Newell network we found that

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.75 & 0 \\ 0.25 & 0 & 0.75 \end{pmatrix}$$

$$\rightarrow \begin{cases} \lambda_1 &= 0.5\lambda_1 + 0.25\lambda_2 + 0.25\lambda_3 \\ \lambda_2 &= 0.5\lambda_1 + 0.75\lambda_2 \\ \lambda_3 &= 0.75\lambda_3 \end{cases} \rightarrow \begin{cases} 0.5\lambda_1 &= 0.25\lambda_2 \\ 0.25\lambda_2 &= 0.5\lambda_1 \\ \lambda_3 &= 0 \end{cases} \rightarrow \begin{cases} \lambda_1 &= 0.5\lambda_2 \\ \lambda_3 &= 0 \end{cases}$$

$$\rightarrow \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \gamma & 2\gamma & 0 \end{pmatrix} \quad \forall\, \gamma > 0$$

Notice that the solution of the traffic equation for a closed network in undermined in general because of the matrix $\boldsymbol{R}$ is row-stochastic. However notice that in the computation of $\boldsymbol{\lambda}$ we can just takes one on the possible solutions. For instance by letting $\gamma = \mu = 4$, in this case we have that

$$\boldsymbol{\lambda} = (4, 8, 0)$$

Then, because of $\mu = 4$ we derives the traffic intensity for each node as follows

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{4}{4} = 1 \quad (9)$$
$$\rho_2 = \frac{\lambda_2}{\mu_2} = \frac{8}{4} = 2 \quad (10)$$
$$\rho_3 = \frac{\lambda_3}{\mu_3} = \frac{0}{4} = 0 \quad (11)$$

Let us further note the choice of γ in a closed network does not influences its ergodicity since its population is fixed.

Now, because of our network consists of only M/M/1 nodes, it results that

$$\beta_i(x_i) = \rho_i^{x_i} \quad \forall\ i = 1, 2, 3, \quad \forall\ x_i = 0, 1, 2. \quad (12)$$

Thus, by replacing ρ_i in the above relation one derives that

$$\begin{array}{ccccc} \beta_1(0) = 1 & , & \beta_1(1) = 1 & , & \beta_1(2) = 1 \\ \beta_2(0) = 1 & , & \beta_2(1) = 2 & , & \beta_2(2) = 4 \\ \beta_3(0) = 1 & , & \beta_3(1) = 0 & , & \beta_3(2) = 0 \end{array} \quad (13)$$

Then, thanks to (8) it results that

$$\pi_s(2,0,0) = \kappa \cdot \beta_1(2)\beta_2(0)\beta_2(0) = \kappa \quad (14)$$
$$\pi_s(1,1,0) = \kappa \cdot \beta_1(1)\beta_2(1)\beta_2(0) = \kappa \cdot 2 \quad (15)$$
$$\pi_s(0,2,0) = \kappa \cdot \beta_1(0)\beta_2(2)\beta_2(0) = \kappa \cdot 4 \quad (16)$$

whereas, because of $\beta_3(1) = \beta_3(2) = 0$, it further results that

$$\pi_s(1,0,1) = \pi_s(0,0,2) = \pi_s(0,1,1) = 0. \quad (17)$$

Furthermore by means of (9), one derives that

$$\kappa \cdot (1 + 2 + 4) = 1 \quad \rightarrow \quad \kappa = \frac{1}{7}. \quad (18)$$

and then by substituting it into the state probabilities $\pi_s(x_1, x_2, x_3)$ it finally results that

$$\pi_s(2,0,0) = \kappa = \frac{1}{7} \quad (19)$$
$$\pi_s(1,1,0) = \kappa \cdot 2 = \frac{2}{7} \quad (20)$$
$$\pi_s(0,2,0) = \kappa \cdot 4 = \frac{4}{7} \quad (21)$$

that is compliant to what derived in (7).

Solution of Exercise 5. Because the considered system may accepts arrivals from outside, and because each station behaves as a M/M/1 queue we can conclude we are dealing with an Open Jackson network.

Let us now compute the probabilities a piece leave the system after being served in node i. By simple manipulation one derives that

$$\begin{aligned} r_{10} &= 1 - r_{12} - r_{13} = 1 - 0.4 - 0.5 = 0.1 && (22)\\ r_{20} &= 1 - r_{22} = 1 - 0.2 = 0.8 && (23)\\ r_{30} &= 1 - r_{31} = 1 - 0.2 = 0.8 && (24) \end{aligned}$$

Let us now evaluate if the resulting stochastic process is ergodic. To do that we need to verify if each node behaves as an ergodic queue, namely if $\lambda_i/\mu_i < 1$ for all $i = 1, 2, 3$, where λ_i denotes the effective arrival rate to node i, computed as the solution of their traffic equations

$$\lambda_i = \lambda_i^{in} + \sum_{j=1}^{3} \lambda_j r_{ji} \quad \forall\, i = 1, 2, 3.$$

Then, let $\lambda = [\lambda_1, \lambda_2, \lambda_3]$, and by solving $\lambda = \lambda_{in} + \lambda \cdot \boldsymbol{R}$, it results

$$\left\{\begin{aligned} \lambda_1 &= \lambda_1^{in} + r_{31} \cdot \lambda_3 \\ \lambda_2 &= \lambda_2^{in} + r_{12} \cdot \lambda_1 + r_{22} \cdot \lambda_2 \\ \lambda_3 &= r_{13} \cdot \lambda_1 \end{aligned}\right. \quad \rightarrow \quad \left\{\begin{aligned} \lambda_1 &= 2 + 0.2 \cdot \lambda_3 \\ \lambda_2 &= 1 + 0.4 \cdot \lambda_1 + 0.2 \cdot \lambda_2 \\ \lambda_3 &= 0.5 \cdot \lambda_1 \end{aligned}\right. \qquad (25)$$

from which one has that

$$\lambda_1 = \frac{20}{9} \quad , \quad \lambda_2 = \frac{85}{36} \quad , \quad \lambda_3 = \frac{10}{9} \qquad (26)$$

Notice that the fact that the above system has a unique solution, is only a necessary but not sufficient condition for the network ergodicity. In fact, to observe, on average, the above effective arrival rates to each station it is further needed that the servers in each station are sufficiently fast to support that incoming flows. However since

$$\begin{aligned} \rho_1 &= \frac{\lambda_1}{\mu_1} = \frac{20}{9} \cdot \frac{1}{3} = \frac{20}{27} \approx 0.74 < 1 && (27)\\ \rho_2 &= \frac{\lambda_2}{\mu_2} = \frac{85}{36} \cdot \frac{1}{4} = \frac{85}{144} \approx 0.59 < 1 && (28)\\ \rho_3 &= \frac{\lambda_3}{\mu_3} = \frac{10}{9} \cdot \frac{1}{2} = \frac{5}{9} \approx 0.55 < 1 && (29) \end{aligned}$$

then, we can conclude the queueing network is ergodic.

Let us now focus on the calculus of

$$\bar{x} = \lim_{t\to\infty} \mathsf{E}\left[\sum_{i=1}^{3} x_i(t)\right] = \sum_{i=1}^{3} \lim_{t\to\infty} \mathsf{E}\left[x_i(t)\right] = \sum_{i=1}^{3} \left(\sum_{k=0}^{\infty} k \cdot \Pr(x_i(\infty) = k)\right)$$

From that, and by remaining that from the Jackson theorem, at the steady state each queue behaves as independent to others, where $\Pr(x_i(\infty) = k) = \rho_i^k(1 - \rho_i)$, we can conclude that

$$\bar{x} = \bar{x}_1 + \bar{x}_2 + \bar{x}_3$$

where

$$\bar{x}_1 = \frac{\rho_1}{1-\rho_1} = \frac{20}{7} = 2.86 \quad (30)$$

$$\bar{x}_2 = \frac{\rho_2}{1-\rho_2} = \frac{85}{59} = 1.44 \quad (31)$$

$$\bar{x}_3 = \frac{\rho_3}{1-\rho_3} = \frac{5}{4} = 1.25 \quad (32)$$

From that it further results that

$$\bar{x} = \sum_{i=1}^{3} \bar{x}_i = \bar{x}_1 + \bar{x}_2 + \bar{x}_3 \approx 5.54 \quad (33)$$

By means of the Little's Law for Queuing Networks it results that the mean time spent by a piace to cross the network at the steady state is as follows

$$\bar{\theta} = \frac{\bar{x}}{\lambda^{in}} = \frac{\bar{x}}{\lambda_1^{in} + \lambda_2^{in}} = \frac{5.54}{3} \approx 1.85 \text{ s} \quad (34)$$

Let us further note that, for general open network, it holds that

$$\bar{\theta} \neq \bar{\theta}_1 + \bar{\theta}_2 + \bar{\theta}_3$$

because of the presences of loops. The equality holds only for Tandem networks.

Let us now focus on the evaluation of the mean time to cross node 2 from outside at steady state. This question was clearly a trap, in fact the mean time to cross node 2 is the same if either the item arrives from outside, or from station 1. It thus result that

$$\bar{\theta}_2 = \frac{\bar{x}_2}{\lambda_2} \approx 0.61 \text{ s.} \quad (35)$$

The rate of items that is correctly processed by the network consists instead in the composition of the departures flows from station 2 and 3, after a properly weighted random splitting. In particular, we have that

$$\eta_p = r_{20} \cdot \lambda_2 + r_{30} \cdot \lambda_3 = 0.8 \cdot \frac{85}{36} + 0.8 \cdot \frac{10}{9} \approx 2.778 \ \frac{\text{pieces}}{s} \quad (36)$$

By an analogous reasoning we can derive the mean rate of defective pieces η_s at steady state is as follows

$$\eta_s = r_{10} \cdot \lambda_1 = 0.1 \cdot \frac{20}{9} \approx 0.222 \ \frac{\text{pieces}}{s} \quad (37)$$

The total outgoing rate of items is instead

$$\eta = \eta_p + \eta_2 = 3 \ \frac{\text{pieces}}{s} \quad (38)$$

Let us further note that, because of the network is ergodic, the total arrival flow from outside the system, i.e. $\lambda^{in} = \lambda_1^{in} + \lambda_2^{in}$ and total departure flow $\eta_s + \eta_p$ must be balanced at the steady state, namely on average the arrivals from outside and departures must have the same mean rate. Thus, it must not be a surprise to note that

$$\eta = \lambda^{in} = \lambda_1^{in} + \lambda_2^{in} = 3\ \frac{\text{pieces}}{s} \tag{39}$$

Now, following point (g), since each raw piece costs 0.2€, and by considering that a correctly processed pieces is sold at 0.5€, and because of the hourly cost of each station is of 1000€/h, but each station is busy only a percentage of time equals to $\hat{U}_i = \rho_i$, then the company will earn in an hour a profit of

$$\begin{aligned}\frac{\text{Profit}}{\text{hour}} =& \frac{3600\text{sec}}{\text{hour}}\left(\frac{0.5\text{Euro}}{\text{piece}} \times \frac{2.778\text{piece}}{\text{sec}} - \frac{0.2\text{Euro}}{\text{piece}}\frac{3\text{piece}}{\text{sec}}\right)\\ &- \frac{1000\text{Euro}}{\text{hour}} \times (0.59 + 0.74 + 0.55) \approx \frac{960\text{Euro}}{\text{hour}}\end{aligned}$$

Let us now compute the steady-state probability that both station 1 and 2 are busy at the same time, namely

$$\Pr(x_1 \geqslant 0, x_2 \geqslant 0)$$

In particular, because of from the Jackson's Theorem we know that each queue of an ergodic open network behaves as an independent queue, then the probability that both station 1 and station 2 are busy at the steady state is given by

$$\begin{aligned}\Pr(x_1(\infty) > 0, x_2(\infty) > 0) =& \sum_{k=0}^{\infty} \Pr(x_1(\infty) > 0, x_2(\infty) > 0, x_3(\infty) = k)\\ =&\Pr(x_1(\infty) > 0) \cdot \Pr(x_2(\infty) > 0) \cdot \sum_{k=0}^{\infty} \Pr(x_3(\infty) = k)\\ =&\Pr(x_1(\infty) > 0) \cdot \Pr(x_2(\infty) \geqslant 0) \cdot 1\\ =&(1 - \Pr(x_1(\infty) = 0)) \cdot (1 - \Pr(x_2(\infty) = 0))\\ =&\rho_1 \cdot \rho_2 = \frac{20}{27} \cdot \frac{85}{144} \approx 0.437\end{aligned}$$

The same result can also equivalently be obtained as follows

$$\begin{aligned}\Pr(x_1(\infty) > 0, x_2(\infty) > 0) =&1 - \Pr(x_1 = 0, x_2 = 0) = 1 - \Pr(x_1 = 0 \cup x_2 = 0)\\ =&1 - (\Pr(x_1 = 0) + \Pr(x_2 = 0) - \Pr(x_1 = 0 \cap x_2 = 0))\\ =&1 - (\ (1 - \rho_1)\ +\ (1 - \rho_2)\ -\ (1 - \rho_1)(1 - \rho_2)) = \rho_1 \cdot \rho_2\end{aligned}$$

Let us further note that the probability that at the steady-state a M/M/1 resource is busy corresponds to its utilization factor.

Let us now evaluate the probability that only one pieces is within the network at steady state. Let us first define the random variable $x(t) = x_1(t) + x_2(t) + x_3(t)$, where $x_i(t)$ is the random variable counting the number of piece in node i at time t. Then, by means the total probability law one has that the probability to have only one piece in the system, correspond to the

$$\Pr(x(\infty)=1)=\lim_{t\to\infty}\Pr(A\cup B\cup C)=\Pr(A)+\Pr(B)+\Pr(C)$$

where

$$A=(x_1=1,x_2=0,x_3=0)\ ,\ B=(x_1=0,x_2=1,x_3=0)\ ,\ C=(x_1=0,x_2=0,x_3=1)\ ).$$

Then because of A, B and C are disjoint event, and because of at the steady state each node behaves as independent then,

$$\begin{aligned}\Pr(x(\infty)=1) &= \Pr(A)+\Pr(B)+\Pr(C)\\ &= \Pr(x_1(\infty)=1)\cdot\Pr(x_2(\infty)=0)\cdot\Pr(x_3(\infty)=0)\\ &\quad +\Pr(x_1(\infty)=0)\cdot\Pr(x_2(\infty)=1)\cdot\Pr(x_3(\infty)=0)\\ &\quad +\Pr(x_1(\infty)=0)\cdot\Pr(x_2(\infty)=0)\cdot\Pr(x_3(\infty)=1)\\ &= \rho_1(1-\rho_1)(1-\rho_2)(1-\rho_3)+(1-\rho_1)\rho_2(1-\rho_2)(1-\rho_3)\\ &\quad +(1-\rho_1)(1-\rho_2)\rho_3(1-\rho_3)\\ &= (1-\rho_1)(1-\rho_2)(1-\rho_3)(\rho_1+\rho_2+\rho_3)\approx 0.0891\end{aligned}$$

Suppose now that one of the three servers could be replaced with a faster one which mean service rate is $\mu'=5\ \mathrm{s}^{-1}$. If we were interested to reduce the average number of pieces in network at steady state, by observing (33) and (30)-(32), it straightforwardly results that, due to the independence of network at steady we have to simply replace the server corresponding to

$$\max_i\{\ \bar{x}_1,\bar{x}_2,\bar{x}_3\ \}=\max\{2.86,\ 1.44,\ 1.25\}=\bar{x}_1 \tag{40}$$

that is that of station 1. After that, it results that the mean number of costumers in station 1 becomes

$$\rho_1'=\frac{20}{9}\cdot\frac{1}{5}=\frac{4}{9}\qquad\to\qquad\bar{x}_1'=\frac{\rho_1'}{1-\rho_1'}\approx 0.8 \tag{41}$$

then the mean number of pieces in the system reduces form

$$\bar{x}=5.5\qquad\mapsto\qquad\bar{x}'=3.48$$

Note that, due to the independence of networks at the steady regime, the fact of having replaced serve 1 do not modifies the behaviours of stations 2 and 3.

If instead we were interested to reduce the mean time needed by an item to cross the network and leave the system, because of this quantities, linearly depended by $\bar{x}$ in (33), then even in this case the choice falls in replacing the server of station 1. After that, it results that the mean time to cross the network reduces from

$$\bar{\theta}\approx 1.85\qquad\mapsto\qquad\bar{\theta}'=\frac{\bar{x}_1'+\bar{x}_2+\bar{x}_2}{\lambda^{in}}=\frac{0.8+1.44+1.25}{3}\approx 1.163 \tag{42}$$

On the other hand, to reduce only the mean time needed to pass through node 2 the choice falls in replacing the server of station 2.

Finally, because of the calculus of the effective arrival (or departure[1]) rates to each node do not depend on how faster the service is, then replacing one of the servers will not help to improve neither the rate of rejected items η_s, nor the rate of correctly processed pieces.

Let us further note that to improve these quantities, namely η_s and η_p, the system engineer may only act on the routing probabilities, that means improve the quality of the service offered by each station in a such a way that the rate of discharged $r_{10}\lambda_1$ or reprocessed pieces $r_{31}\lambda_3$ decrease.

[1]Do not forget that at the steady state, if the network is ergodic, then arrivals and departure in each node are balanced, thus $\eta_i = \lambda_i$