Stochastic Models

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Assignment 5

Exercise 1. Glaucoma is a disease of the eyes that usually occurs in old aged patients. Its progression can be modelled as a time-homogeneous CT-MC which states are: mild ($x(t) = 0$), severe ($x(t) = 1$) and near blindness ($x(t) = 2$). In the absence of treatment the evolves from mild to severe with a rate λ_1 and from severe to near blindness with a rate λ_2. The direct progression from mild to severe blindness was never observed.

(a) Draw a transition graph illustrating the disease progression, and write out the transition rate matrix $\boldsymbol{Q} \in \mathbb{R}^{3\times 3}$.

(b) Assume at $t = 0$ the first row of the transition probability matrix $\boldsymbol{P}(t) \in \mathbb{R}^{3\times 3}$ is $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. Discuss how to approach the calculus of the transition probabilities $p_{00}(t)$ and $p_{01}(t)$.

(c) Show the average time necessary to jump from state 0 to 2 is given by

$$\mathsf{E}[T] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Exercise 2. Consider a single-server resource where tasks to be solved have a mean arrival rate $\lambda \geqslant 0$, while the mean service rate is $\mu \geqslant 0$. Suppose the resource is equipped with a waiting line, aimed to store, when the server is busy, at most 1 further task in a buffer.

(a) Model the system as a uniform birth-death process.

(b) Determine which conditions on λ and μ makes the process ergodic.

(c) Under those conditions calculate its limiting distribution.

Exercise 3. A production process features a single server station comprising of a robot responsible for painting and polishing car bodyworks. The car bodyworks are received based on a Poisson process with an average rate $\lambda > 0$.

In case the robot is occupied, the car bodyworks are redirected to another location (rejected).

The time required for either painting or polishing a car bodywork is exponentially distributed with an average value of $1/\mu$.

The painting process success is Bernoulli distributed with a parameter of $p \in (0, 1)$, while the polishing process always terminate correctly.

Consider the next three use-cases:

- S1: The painting process continues until it is successfully completed, after that, the bodywork is polished.
- S2: The painting process is carried out a maximum of two times. If it is successfully completed, the car bodywork is polished, otherwise it is rejected.
- S3: The painting process is carried out a maximum of two times. The second attempt is guaranteed to end successfully, and after that, the car bodywork is polished.

(a) For each scenario draw the transition graph and discuss if the processes are ergodic.

(b) Suppose you solved the balancing equations for the three CT-MC, thus the vectors of the limiting probability distributions $\Pi(\infty)$ are in your hand.

  (i) Provide the long term probability expression an incoming car bodywork is rejected.

  (ii) For scenario S2, provide the expression of the probability a car bodywork is rejected after having failed its painting processes twice.

Exercise 4. The production system of a factory consists of a service station with 2 servers denoted by M_1 and M_2 and a waiting line.

The system capacity is of 3 pieces and it counts also the pieces under work.

The process of arrivals is Poisson with rate $\lambda = 10$ while yhe service time of the two servers is exponentially distributed with mean rates μ_1 and μ_2, respectively.

A piece may be either be processed by one of the machines and then leave the system, or wait for while in the buffer if there is place. Otherwise the piece is rejected.

Draw the graph of a CT-MC modelling the system, under the next forwarding policies:

(a) a regular splitting (if both servers are in idle, the next piece is forwarded to M_1)

(b) a random splitting (if both the servers are in idle, the next piece is forwarded to M_1 or to M_2 with the same probability)

(c) suppose $\mu_1 = \mu_2 = 10$. Are the two processes ergodic?

(d) suppose you solved the balancing equations for the two CT-MC in (a) and (b), obtaining

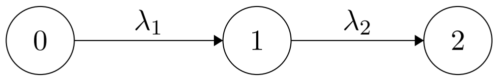
$$\Pi^{(a)} = \begin{pmatrix} 0.3636 & 0.2727 & 0.0909 & 0.1818 & 0.0909 \end{pmatrix}$$
$$\Pi^{(b)} = \begin{pmatrix} 0.3636 & 0.1818 & 0.1818 & 0.1818 & 0.0909 \end{pmatrix}$$

provide an interpretation of this result.

Then, calculate the expected number of pieces in the system at the steady state.

Solutions

Solution of Exercise 1. The transition graph for this process has three states as shown in the graph below:



The corresponding transition rate matrix is:

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ 0 & 0 & 0 \end{pmatrix}$$

For a generic CT-MC the transition probabilities p_{ij} are time-varying functions of the actual time t_k and the time passed w.r.t. to t_k, namely $h = dt$. Indeed, it results that

$$\frac{\partial \boldsymbol{P}(t_k, h)}{\partial h} = \boldsymbol{P}(t_k, h) \cdot \boldsymbol{Q}(h), \quad \forall\ t_k \in \mathbb{R}_{\geqslant 0}, h \geqslant 0 \tag{1}$$

However because of the considered stochastic process is assumed to be time-homogenous (i.e. matrix $\boldsymbol{Q}$ is a costant matrix), then the transition probabilities results to be function only of the time-interval length h, instead of the pair (t_k, h). This means that, although in general for each pair (t_k, h) there is a different matrix $\boldsymbol{P}(t_k, h)$ that solves (1), on the other hand if the stochastic process is time-homogenous then, there is not dependence by t_k, thus $\boldsymbol{P}(t_k, h) = \boldsymbol{P}(h)$. Thus, if the process is time-homogenous we can rewrite (1) as follows, by replacing for simplicity h with t, as shown next

$$\frac{d\boldsymbol{P}(t)}{dt} = \boldsymbol{P}(t) \cdot \boldsymbol{Q} \tag{2}$$

or equivalently

$$\frac{d}{dt}\begin{pmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) \\ p_{10}(t) & p_{11}(t) & p_{12}(t) \\ p_{20}(t) & p_{21}(t) & p_{22}(t) \end{pmatrix} = \begin{pmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) \\ p_{10}(t) & p_{11}(t) & p_{12}(t) \\ p_{20}(t) & p_{21}(t) & p_{22}(t) \end{pmatrix} \cdot \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

that is the so-called "Kolmogorov forward equation" for CT-MC. Because of we are interested in the evaluation of $p_{00}(t)$ and $p_{01}(t)$, under the constraint that a person starts with a mild glaucoma disease, namely $p_{00}(0) = 1$, then we are interested to get to solve the following system of differential equations

$$\begin{pmatrix} \frac{dp_{00}(t)}{dt} & \frac{dp_{01}(t)}{dt} \end{pmatrix} = \begin{pmatrix} p_{00}(t) & p_{01}(t) \end{pmatrix} \cdot \overbrace{\begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix}}^{\bar{\boldsymbol{Q}}} \tag{4}$$

There are many ways to solve (4), among them we mention the method involving the resolvent matrix that employs the Laplace transformation where

$$\begin{pmatrix} p_{00}(t) & p_{01}(t) \end{pmatrix} = \boldsymbol{\mathcal{L}}^{-1}\left\{\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \left(s \cdot \boldsymbol{I} - \bar{\boldsymbol{Q}}\right)^{-1}\right\}$$

or that involving the Matrix Exponential and the diagonalization of matrix $\bar{\boldsymbol{Q}}$, where we have that

$$\begin{pmatrix} p_{00}(t) & p_{01}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \boldsymbol{V} \cdot \mathrm{e}^{\overbrace{\boldsymbol{V}^{-1}\bar{\boldsymbol{Q}}\boldsymbol{V}}^{\Lambda}} \cdot \boldsymbol{V}^{-1} \tag{5}$$

and Λ and $\boldsymbol{V}$ are respectively the eigenvalues and eigenvector matrices of $\bar{\boldsymbol{P}}$. Here, it is easy derive that $\Lambda = \mathrm{diag}(-\lambda_1, -\lambda_2)$, whereas $\boldsymbol{V}$ can be derived by solving

$$\begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = -\lambda_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \rightarrow \begin{cases} -\lambda_1 v_{11} + \lambda_2 v_{21} = -\lambda_1 v_{11} \\ -\lambda_2 v_{21} = -\lambda_1 v_{21} \end{cases} \rightarrow \begin{cases} v_{11} = a_1 \in \mathbb{R} \\ v_{21} = 0 \end{cases} \tag{6}$$

and

$$\begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = -\lambda_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \rightarrow \begin{cases} -\lambda_1 v_{12} + \lambda_2 v_{22} = -\lambda_2 v_{12} \\ -\lambda_2 v_{22} = -\lambda_2 v_{22} \end{cases} \rightarrow \begin{cases} v_{11} = \frac{\lambda_1}{\lambda_1+\lambda_2} a_2 \\ v_{22} = a_2 \in \mathbb{R} \end{cases} \tag{7}$$

from which one derives that

$$\boldsymbol{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\lambda_1}{\lambda_1-\lambda_2} \\ 0 & 1 \end{pmatrix}_{a_1=1, a_2=-1}$$

thus recalling that

$$\mathrm{adj}\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\} = \mathrm{cof}\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathsf{T}}\right\} = \mathrm{cof}\left\{\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right\} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

than the inverse of V is

$$\boldsymbol{V}^{-1} = \frac{\mathrm{adj}\{\boldsymbol{V}\}}{\det\{\boldsymbol{V}\}} = \frac{\mathrm{cof}\{\boldsymbol{V}^{\mathsf{T}}\}}{\det\{\boldsymbol{V}\}} = \frac{\mathrm{cof}\left\{\begin{pmatrix} 1 & 0 \\ \frac{\lambda_1}{\lambda_1-\lambda_2} & 1 \end{pmatrix}\right\}}{1} = \begin{pmatrix} 1 & -\frac{\lambda_1}{\lambda_1-\lambda_2} \\ 0 & 1 \end{pmatrix}$$

Notice that matrices Λ and $\boldsymbol{V}$ could also be easily evaluated by means of the symbolic toolbox of Matlab as follows

```
syms lambda1 lambda2 t real positive
Q=[-lambda1 lambda1 0; 0 -lambda2 lambda2;0 0 0]
barQ=Q(1:2,1:2)
[V,Lambda]=eig(barQ) % compute the eigenvectors and the eigenvalues matrices
simplify(V^-1*barQ*V)   % check the diagonalization
exp_Lambdat=[exp(-lambda1*t) 0;0 exp(-lambda2*t)] % matrix exponential
simplify([1 0]*V*exp_Lambdat*V^-1) % solution
```

From (5), Λ and $\boldsymbol{V}$, one derives that

$$\begin{pmatrix} p_{00}(t) & p_{01}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \overbrace{\begin{pmatrix} 1 & \frac{\lambda_1}{\lambda_1-\lambda_2} \\ 0 & 1 \end{pmatrix}}^{\boldsymbol{V}} \cdot \overbrace{\begin{pmatrix} \mathrm{e}^{-\lambda_1 \mathrm{t}} & 0 \\ 0 & \mathrm{e}^{-\lambda_2 \mathrm{t}} \end{pmatrix}}^{\mathrm{e}^{\Lambda \mathrm{t}}} \cdot \overbrace{\begin{pmatrix} 1 & -\frac{\lambda_1}{\lambda_1-\lambda_2} \\ 0 & 1 \end{pmatrix}}^{\boldsymbol{V}^{-1}} = \begin{pmatrix} 1 & \frac{\lambda_1}{\lambda_1-\lambda_2} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{e}^{-\lambda_1 \mathrm{t}} \\ \mathrm{e}^{-\lambda_2 \mathrm{t}} \end{pmatrix}$$

where

$$\begin{aligned} p_{00}(t) &= \mathrm{e}^{-\lambda_1 \cdot t} \\ p_{01}(t) &= \frac{\lambda_1}{\lambda_2 - \lambda_1} \cdot \left(\mathrm{e}^{-\lambda_1 \cdot t} - \mathrm{e}^{-\lambda_2 \cdot t} \right) \end{aligned}$$

From which we can further derive that

$$p_{02}(t) = 1 - p_{00}(t) - p_{01}(t) = \frac{1}{\lambda_2 - \lambda_1} \cdot \left(\lambda_2 - \lambda_1 - \lambda_2 \cdot \mathrm{e}^{-\lambda_1 \cdot \mathrm{t}} + \lambda_1 \cdot \mathrm{e}^{-\lambda_2 \cdot \mathrm{t}} \right)$$

Notice that such result could be also be derived on Matlab by means of the following set of commands:

```
syms lambda1 lambda2 t real positive
Q=[-lambda1 lambda1 0; 0 -lambda2 lambda2;0 0 0]
simplify([1 0 0]*expm(Q*t))  % see differences between command exp and expm
```

Let us now derive the final result. In particular here we are interested to proof that the mean first time to jump from state 0 to state 2 is

$$\mathsf{E}[T] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

To prove that we can use different approaches. One is that of the calculus of the mean first time of x_2, i.e.,

$$\mu_i = \mathsf{E}[\min\{t \geqslant 0 \ : \ X_t = 2 | X_0 = i\}] \quad i = 0$$

In particular, since x_2 is an absorbing state, we can directly compute its mean absorbing time by means of the identity

$$\left(\sum_j q_{ij} \right) \mu_i = 1 + \sum_k q_{ik} t_k$$

from which the next well-posed linear system takes place

$$\begin{cases} \mu_0 = \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_1} \mu_1 \\ \mu_1 = \frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_2} \mu_2 \\ \mu_2 = 0 \end{cases} \rightarrow \begin{cases} \mu_0 = \frac{1}{\lambda_1} + \mu_2 \\ \mu_1 = \frac{1}{\lambda_2} \\ \mu_2 = 0 \end{cases} \rightarrow \begin{cases} \mu_0 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \\ \mu_1 = \frac{1}{\lambda_2} \\ \mu_2 = 0 \end{cases}$$

Alternatively the same result can be derived by noticing that the transition from state 0 to 1, and that from 1 to 2 are independent processes each with rate, resp., λ_1 and λ_2. Thus, it follows that the number of event enabling the transition from state 0 to 1, and that from 1 to 2 are poisson distributed with rate, resp. λ_1 and λ_2.

Thus, because of to from state 0 to 2, we can pass only through state 1, we have that the average time taken to pass from state 0 to state 2 is thus the sum of the inter-event of each of the two process, namely

$$\mathsf{E}[T] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}.$$

More formally, let $T' \sim Exp(\lambda_1)$ be randon variable associated to the inter-event time from the from $x(0) = 0$ to $x(t) = 1$. Clearly, it results that if the transition is not occurred yet on $[0, t]$, hence $T' > t$, which implies that

$$\Pr(T' > t) = \Pr(x(t) = 0 | x(0) = 0) = \frac{(\lambda_1 t)^0}{0!} \mathrm{e}^{-\lambda_1 t} \equiv p_{00}(t) = \mathrm{e}^{-\lambda_1 \cdot \mathrm{t}}.$$

From than, one further derive that

$$\Pr(T' \leqslant t) = 1 - \mathrm{e}^{-\lambda_1 \cdot \mathrm{t}}$$

that is the cumulative probability distribution function (cdf) of an exponential random variable with rate λ_1, which mean is

$$\mathsf{E}[T'] = \frac{1}{\lambda_1}.$$

Then after having reached 1, we need to note that the amount of time spent in state 1 does not affect the amount of time it takes to transition from state 1 to 2. Thus, the time T'' taken to transition from state 1 to 2 is also exponentially distributed with mean

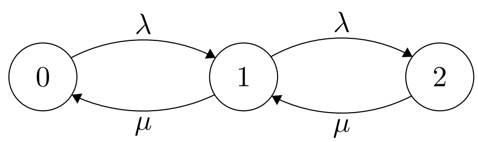
$$\mathsf{E}[T''] = \frac{1}{\lambda_2}.$$

Finally, by letting T be the random variable associated with the time needed for the transition from $x(0) = 0$ to state $x(t) = 2$, due to the independence between the event of moving from state 0 to 1, and that from 1 to 2, and because of there exist only one path from 0 to 2, then one has that $T = T' + T''$, and thus the total expected time associated with T is given as follows

$$\mathsf{E}[T] = \mathsf{E}[T' + T''] = \mathsf{E}[T'] + \mathsf{E}[T''] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}.$$

Just for information, in probability theory the sum of independent but not identical exponential random variable is known as "hypoexponential distribution" or " generalized Erlang distribution" which find many application in queueing theory, and teletraffic engineering, because of it is more general case of the Erlang distribution. The main difference between the Erlang distribution and the Generalized Erlang distribution is that the first consider the sum of i.i.d. exponential random variables, whereas in the second one those random variable exponential and independent but not identicals, namely they may have different rates λ_i.

Solution of Exercise 2. Since the system may has a the same time at most 2 tasks, this status of this resurce can be modelled as a three states uniform birth-death process as follows:



Since this continuous-time birth death process (CT-BDP) has finite number of states its ergodicity can be evaluated has it was a simpler continuous-time markov chain (CT-MC) with a finite number of states. Thus we have to simply evaluate the conditions on λ and μ such that its associated transition graph has a single absorbing component. In this case the necessary and sufficient condition are that either $\lambda > 0$ or $\mu > 0$. In this case it is clearly sufficient one of the two.

Another way it could be by means of the Eigenvalues' Criteria, and than by means of the Routh-Hurwitz Criteria find the conditions on μ and λ such that the system has a single null eigenvalue, and the remaining eigenvalues have strictly negative real part. For the sake of comparison, it results that

$$\det\{\mathrm{sI}-\mathrm{Q}\} = \det\left\{\begin{pmatrix} s+\lambda & -\lambda & 0 \\ -\mu & s+\mu+\lambda & -\lambda \\ 0 & -\mu & s+\mu \end{pmatrix}\right\} = \mathrm{s}(\mathrm{s}^2 + 2(\mu+\lambda)\mathrm{s} + (\lambda+\mu)^2) = 0$$

from which it is evident that analogous consideration can be concluded.

Let us finally evaluate its stationary distribution Π_s, by solving the following linear system

$$\begin{cases} \Pi_s \cdot \boldsymbol{Q} = \mathbf{0} \\ \Pi_s \cdot \mathbf{1} = 1 \end{cases} \equiv \begin{cases} \pi_{s,i+1} = \left(\frac{\lambda}{\mu}\right)\pi_{s,i}, \; i = 0,1,2,\dots \\ \sum_i \pi_{s,i} = 1 \end{cases} \to \begin{cases} \pi_{s,1} = \left(\frac{\lambda}{\mu}\right)\pi_{s,0} \\ \pi_{s,2} = \left(\frac{\lambda}{\mu}\right)^2 \pi_{s,0} \\ \pi_{s,0} + \pi_{s,1} + \pi_{s,2} = 1 \end{cases}$$

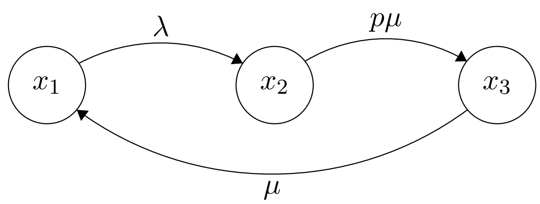
$$\to \begin{cases} \pi_{s,1} = \left(\frac{\lambda}{\mu}\right)\pi_{s,0} \\ \pi_{s,2} = \left(\frac{\lambda}{\mu}\right)^2 \pi_{s,0} \\ \left(1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2\right)\pi_{s,0} = 1 \end{cases}$$

from which it results that

$$\Pi_\ell \equiv \Pi_s = \frac{1}{1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2} \cdot \left(1, \left(\frac{\lambda}{\mu}\right), \left(\frac{\lambda}{\mu}\right)^2\right).$$

Solution of Exercise 3. Let us first note that since arrivals are Poisson distributed, then cars bodyworks arrives in continuous time, and the inter-arrival times are exponentially distributed with rate λ and average arrival time $1/\lambda$. Moreover note that the service time are exponentially distributed with average service time $1/\mu$, thus with rate μ.

For the scenario "S1", the sample spaces of the process describing the state of the robot is as follows $X \in \{x_1, x_2, x_3\}$, where "x_1" denotes the system in idle mode, "x_2" denotes that the robot is painting a car bodywork and "x_3" denotes that the robot is polishing a car bodywork. Since for "S1" the painting process can be repeated until it terminates correctly, and then the car bodywork is polished, the resulting CT-MC can be represented by the following transition graph:



Remember that in CT-MC arcs do not represent probabilities but transition rates and that in some sense self-loops are implicit in the graph. Let us further note that matrix $\boldsymbol{Q}$ not only describes the dynamical evolution of the marginal (uncoditional) probabilities of being in a given states in accordance with

$$\frac{d}{dt}\Pi(t) = \Pi(t) \cdot \boldsymbol{Q}$$

but, by means of the Kolgomorov forward equation it further describes the evolution of the conditional probabilities of moving from a state to another as time passes accordingly with

$$\frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{Q},$$

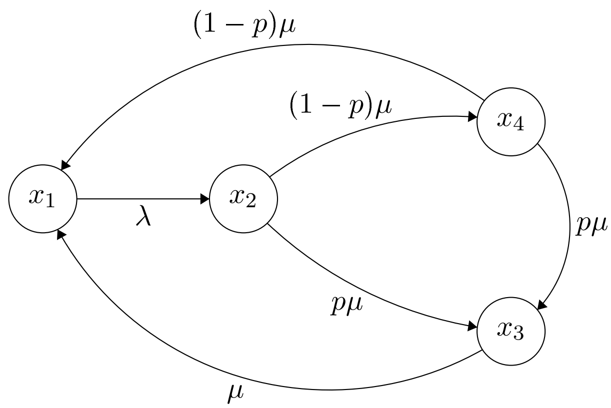
where $q_{ij}dt$ is the probability of moving from state i to state j after a time interval dt, with $i \neq j$.

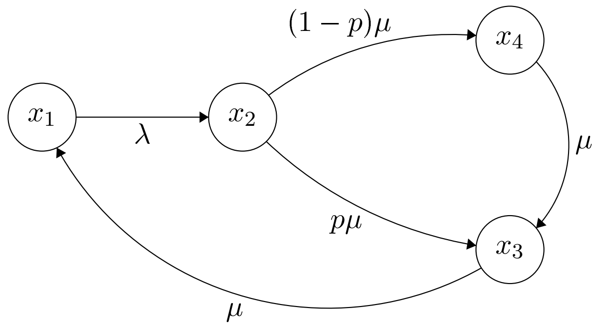
Let us now consider the scenario "S2". In this case, since the painting process can be performed at most twice, after that, if the process is not terminated correctly the car bodywork is discharged. Then, the resulting transition graph is as follows

where process sample spaces is $X \in \{x_1, x_2, x_3, x_4\}$, where "$x_1$" denotes the system in idle mode, "x_2" denotes that the robot is painting for the first time a bodywork, "x_3" denotes that the robot is polishing a car bodywork, whereas "x_4" denotes that the robot is painting for the second time a bodywork.

It is evident by observing the arc from x_4 to x_0 that the rate of discharging a car bodywork is proportional to the probability that to fail the paint process, that is $(1 - p)$. Thus, if the process is failed twice the robot moves in the idle state with rate $(1 - p)\mu$.

Finally, if we assume that the painting process is reliable, in the sense that it can fail only one time, then we have the following transition graph





In this case we can observe that the arc from x_4 to x_0 do not exist anymore, and that the service rate of the second painting process is again equal to μ.

Let us now provide the transition rate matrices for the three scenario, denoted here for simplicity as Q_i, for $i = 1, 2, 3$. In particular we have:

$$\boldsymbol{Q}_1 = \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -p\mu & p\mu \\ \mu & 0 & -\mu \end{pmatrix}, \quad \boldsymbol{Q}_2 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -\mu & p\mu & (1-p)\mu \\ \mu & 0 & -\mu & 0 \\ \mu(1-p) & 0 & p\mu & -\mu \end{pmatrix},$$

$$\boldsymbol{Q}_3 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -\mu & p\mu & (1-p)\mu \\ \mu & 0 & -\mu & 0 \\ 0 & 0 & p\mu & -\mu \end{pmatrix}.$$

Now, to calculate the probability that an incoming car bodywork is not accepted within the system at the steady regime, the stationary probability distribution for each of the three scenarios are needed. Let us first solve for each of them the following linear system

$$\begin{cases} \Pi_s^{(i)} \cdot \boldsymbol{Q}_i = \boldsymbol{0} \\ \Pi_s^{(i)} \cdot \boldsymbol{1} = 1 \end{cases} \quad \forall \quad i = 1, 2, 3$$

For $i = 1$ we have that

$$\begin{cases} -\lambda\pi_{s,0}^{(1)} + \mu\pi_{s,2}^{(1)} = 0 \\ \lambda\pi_{s,0}^{(1)} - p\mu\pi_{s,1}^{(1)} = 0 \\ p\mu\pi_{s,1}^{(1)} - \mu\pi_{s,2}^{(1)} = 0 \\ \pi_{s,0}^{(1)} + \pi_{s,1}^{(1)} + \pi_{s,2}^{(1)} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,2}^{(1)} = \left(\frac{\lambda}{\mu}\right)\pi_{s,0}^{(1)} \\ \pi_{s,1}^{(1)} = \left(\frac{\lambda}{p\mu}\right)\pi_{s,0} \\ \pi_{s,0}^{(1)} = \frac{1}{1+\frac{\lambda}{\mu}+\frac{\lambda}{p\mu}} \end{cases}$$

from which it results that

$$\Pi^{(1)} = \frac{1}{1+\frac{\lambda}{\mu}+\frac{\lambda}{p\mu}} \cdot \begin{pmatrix} 1, & \frac{\lambda}{p\mu}, & \frac{\lambda}{\mu} \end{pmatrix} = \frac{1}{p\mu + p\lambda + \lambda} \cdot \begin{pmatrix} p\mu, & \lambda, & p\lambda \end{pmatrix}$$

By similar computation it results that

$$\Pi^{(2)} = \frac{1}{\mu + 2\lambda - \lambda p^2 + \lambda p} \cdot \begin{pmatrix} \mu, & \lambda, & 2p\lambda + \lambda p^2, & (1-p)\lambda \end{pmatrix}$$

$$\Pi^{(3)} = \frac{1}{1 + 3\lambda - p\lambda} \cdot \begin{pmatrix} \mu, & \lambda, & \lambda, & \lambda(1-p) \end{pmatrix}$$

From that, it results that the probability that an incoming car bodywork is rejected is, resp.:

$$\Pr(\text{1 piece is rejected in S1}) = \pi_{s,2}^{(1)} + \pi_{s,3}^{(1)} = 1 - \pi_{s,1}^{(1)} = \frac{p\lambda + \lambda}{p\mu + p\lambda + \lambda}$$

$$\Pr(\text{1 piece is rejected in S2}) = \pi_{s,2}^{(2)} + \pi_{s,3}^{(2)} + \pi_{s,4}^{(2)} = 1 - \pi_{s,1}^{(2)} = \frac{2\lambda - \lambda p^2 + \lambda p}{\mu + 2\lambda - \lambda p^2 + \lambda p}$$

$$\Pr(\text{1 piece is rejected in S3}) = \pi_{s,2}^{(3)} + \pi_{s,3}^{(3)} + \pi_{s,4}^{(2)} = 1 - \pi_{s,1}^{(3)} = \frac{1 + 3\lambda - p\lambda - \mu}{1 + 3\lambda - p\lambda}$$

that is the probability, for each of the three scenario, that the robot is busy, thus either in state x_2 or in state x_3, or in x_4, if we consider the last two scenarios. Notice that these results are conditioned to the fact that our service station has no waiting line.

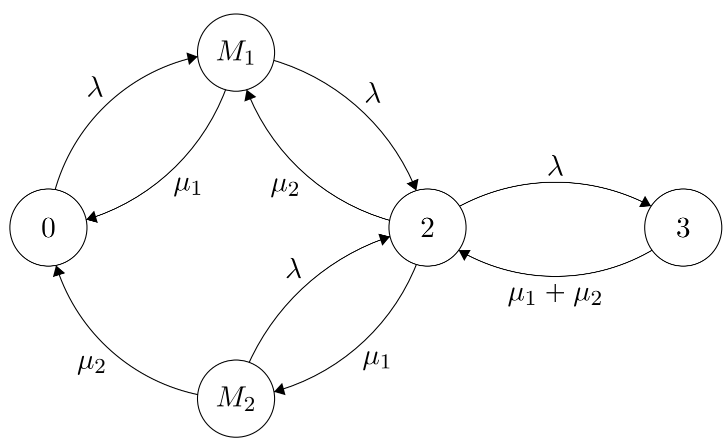
Finally, limited to the "S2" use-case we have that the probability of discarding a car bodywork, at the steady state, after having failed two painting processes is

$$\Pr(\text{discarding a car bodywork}) = (1-p) \cdot \pi_{s,4}^{(2)}.$$

Solution of Exercise 4. Because the system capacity includes also the pieces under works, then when the system is full there are three pieces in total. Thus all the possible status for the service station are:

- State 0 denote that the system is empty and both the serves are in idle;
- State M_1 denote that there is one work piece to M_1;
- State M_2 denote that there is one work piece to M_2;
- State 2 denote that there are two pieces in the system, both under works;
- State 3 denote that there are three pieces in the system, two under works and one in the waiting area.

Given that, under the regular splitting policy, the resulting CT-MC can be described by the following transition graph:



It is evident, that if the system is empty, i.e. $x(t) = 0$, we can reach only M_1 with a rate equal to λ. Then, if another pieces arrives with a rate λ the system is state moves to $x(t) = 2$. From that, if the machine M_1 which service rate is μ_1 terminates its task for first, the CT-MC moves to state M_2, whereas if M_2 terminates for first, the CT-MC jumps to "M_1".

Finally, if the system is full, i.e. $x(t) = 3$ we can reach 2 with a rate that is the sum of the service rate of the two machines. This straightforwardly derive by the fact that the superposition of two independent processes with mean rate μ_1 and μ_2 has mean rate equal to $\mu_1 + \mu_2$.

Indeed, as known the time T passed until the occurrence of the first, over the two possible service processes each with rate μ_i is exponentially distributed with mean $1/(\mu_1+\mu_2)$. Indeed, if we have a competition between the two exponential random variable $T^{(1)}$ and $T^{(2)}$, and we are interested in the time of the winner whichever it is. Then we have that

$$\Pr(T > t) = \Pr(\min\{T^{(1)}, T^{(2)}\} > t) = \Pr(T^{(1)} > t, T^{(2)} > t).$$

Then due to the independence of these random variable we have that

$$\Pr(T > t) = \Pr(T^{(1)} > t) \cdot \Pr(T^{(2)} > t) = (1 - \Pr(T^{(1)} \leqslant t)) \cdot (1 - \Pr(T^{(2)} \leqslant t))$$

$$= \left(1 - \int_0^t \mu_1 \cdot e^{-\mu_1 \tau} d\tau\right) \cdot \left(1 - \int_0^t \mu_1 \cdot e^{-\mu_2 \tau} d\tau\right) = e^{-\mu_1 t} \cdot e^{-\mu_2 t} = e^{-(\mu_1+\mu_2)t}.$$

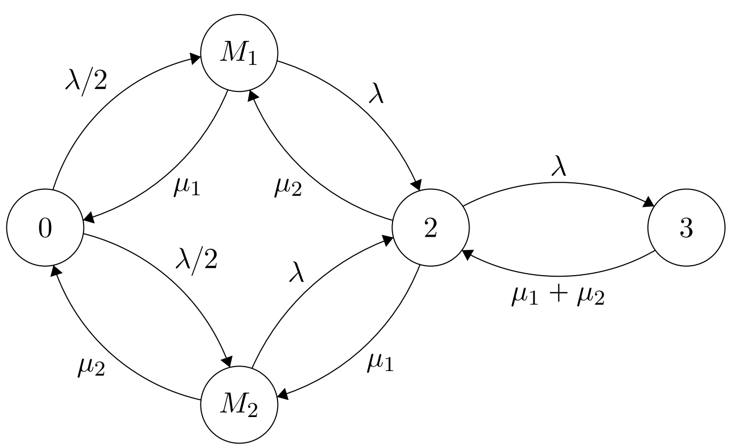
Thus, we have that $\Pr(T \leqslant t) = 1 - e^{-(\mu_1+\mu_2)t}$, and then by differentiating with respect to t we can derive the probability density function of T as follows

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{d}{dt}\left(1 - e^{-(\mu_1+\mu_2)t}\right) = (\mu_1 + \mu_2) \cdot e^{-(\mu_1+\mu_2)t}$$

that is an exponential distribution with parameter $\mu_1 + \mu_2$.

Let us further note that, although this use-case consider a regular splitting policy, since the arrival process to each machine M_i does not depend only to λ but also to μ_1 and μ_2, then the arrival time of each machine result to be more complex than simply a $Erl(2, \lambda)$.

Let us now consider the second discipline. Under a random splitting" in which, if both the two servers are free, the new arrival forwarded either to M_1 or M_2 with the same probability. In this case the resulting transition graph is as follows:



In the end, let $\mu_1 = \mu_2 = 10$ and $\lambda = 10$, let us evaluated the average number of pieces in the system. The transition rate matrix for the two scenarios be, resp.,

$$\boldsymbol{Q}_1 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda) & 0 & \lambda & 0 \\ \mu_2 & 0 & -(\mu_2+\lambda) & \lambda & 0 \\ 0 & \mu_2 & \mu_1 & -(\mu_1+\mu_2+\lambda) & \lambda \\ 0 & 0 & 0 & \mu_1+\mu_2 & -(\mu_1+\mu_2) \end{pmatrix},$$

$$\boldsymbol{Q}_2 = \begin{pmatrix} -\lambda & \lambda/2 & \lambda/2 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda) & 0 & \lambda & 0 \\ \mu_2 & 0 & -(\mu_2+\lambda) & \lambda & 0 \\ 0 & \mu_2 & \mu_1 & -(\mu_1+\mu_2+\lambda) & \lambda \\ 0 & 0 & 0 & \mu_1+\mu_2 & -(\mu_1+\mu_2) \end{pmatrix},$$

Thus by solving

$$\begin{cases} \Pi_s^{(i)} \cdot \boldsymbol{Q}_i = \boldsymbol{0} \\ \Pi_s^{(i)} \cdot \boldsymbol{1} = 1 \end{cases}$$

the following stationary probability distributions take place

$$\Pi^{(1)} = \begin{pmatrix} 0.3636 & 0.2727 & 0.0909 & 0.1818 & 0.0909 \end{pmatrix}$$

$$\Pi^{(2)} = \begin{pmatrix} 0.3636 & 0.1818 & 0.1818 & 0.1818 & 0.0909 \end{pmatrix}$$

Finally let $N^{(i)}$ be the number of pieces in the system, it results that its long term expected number, for the two cases is, resp.,

$$\begin{aligned} \mathsf{E}[N^{(1)}] &= 0 \cdot \pi_{s,0}^{(1)} + 1 \cdot (\pi_{s,M_1}^{(1)} + \pi_{s,M_2}^{(1)}) + 2 \cdot \pi_{s,2}^{(1)} + 3 \cdot \pi_{s,3}^{(1)} = \\ &= 1 \cdot (0.2727 + 0.0909) + 2 \cdot 0.1818 + 3 \cdot 0.0909 = 1 \\ \mathsf{E}[N^{(2)}] &= 0 \cdot \pi_{s,0}^{(2)} + 1 \cdot (\pi_{s,M_1}^{(2)} + \pi_{s,M_2}^{(2)}) + 2 \cdot \pi_{s,2}^{(2)} + 3 \cdot \pi_{s,3}^{(2)} = \\ &= 1 \cdot (0.1818 + 0.1818) + 2 \cdot 0.1818 + 3 \cdot 0.0909 = 1. \end{aligned}$$

As you can see the two policies do not change the average number of pieces in the system, but what changes is the utilization of the two. From the example arises that regular splitting force machine M_1 to work more. Such a policy is useful whenever we want to guarantee a backup server, whereas the random splitting uses to balance their utilization, and ageing.

For the sake of completeness, hereinafter it is provided the Matlab script that solves these two points

```
n=5;  % number of states
B=[zeros(1,n-1) 1]  % known term

mu_1=10; % Service rate of M1
mu_2=10; % Service rate of M2
lambda=10; % Arrival rate
% Transition rate matrices
Q1=[-lambda  lambda  0  0  0
mu_1  -(mu_1+lambda) lambda  0 0
mu_2  0 -(mu_2+lambda) lambda  0
0  mu_2 mu_1  -(mu_1+mu_2+lambda)  lambda
0 0 0 mu_1+mu_2  -(mu_1+mu_2) ]

Q2=[-lambda  lambda/2  lambda/2  0  0
mu_1  -(mu_1+lambda) lambda  0 0
mu_2  0 -(mu_2+lambda) lambda  0
0  mu_2 mu_1  -(mu_1+mu_2+lambda)  lambda
```

```
0 0 0 mu_1+mu_2  -(mu_1+mu_2) ]
% Matrix of coefficient of the linear system to be solved
A1=[Q1(:,1:n-1), ones(n,1)]
A2=[Q2(:,1:n-1), ones(n,1)]

Pi1_staz=B*inv(A1) % Staz. distr. for the regular splitting
Pi2_staz=B*inv(A2) % Staz. distr. for the random splitting
outcomes=[0 1 1 2 3]'
mean_regular_splitting=Pi1_staz*outcomes
mean_random_splitting=Pi2_staz*outcomes
```