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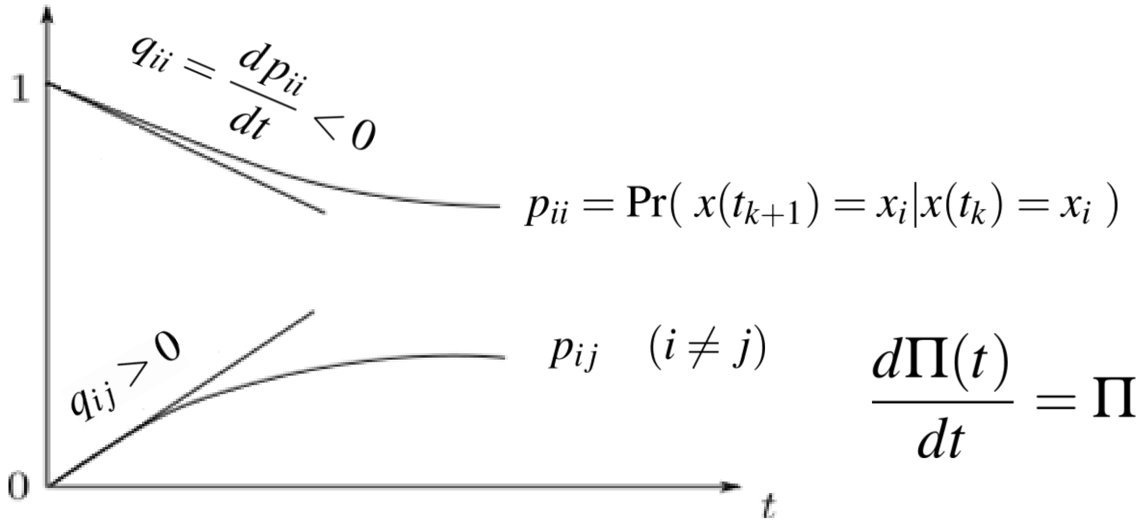
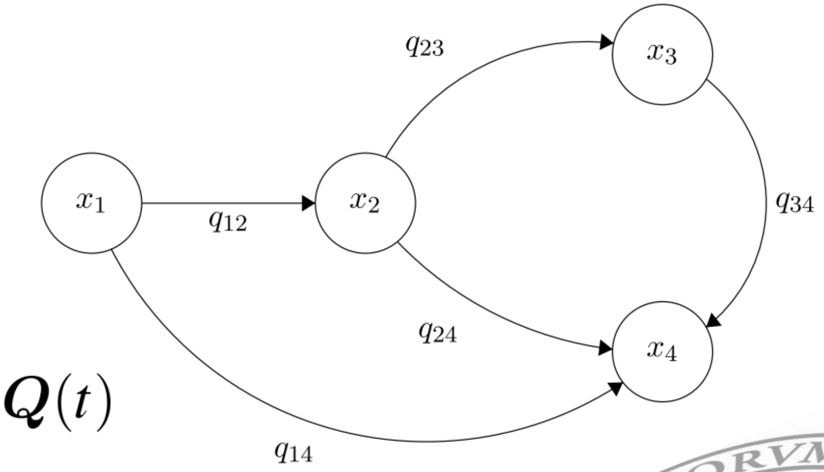
DIEE - Department of Electrical and Electronic Engineering

STOCHASTIC MODELS
-
Continuous-Time Markov Chains



$$\frac{d\Pi(t)}{dt} = \Pi(t)\boldsymbol{Q}(t)$$



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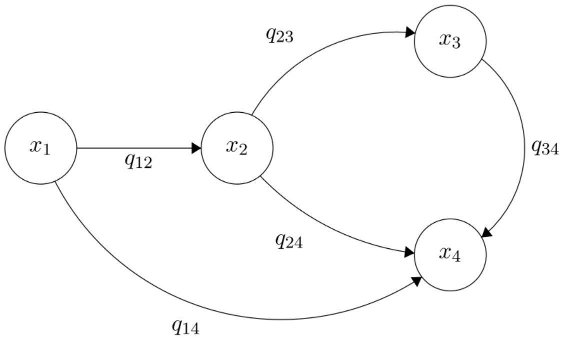
Continuous-time Markov Chain



- A Continuous-time Markov Chain (CT-MC) is a SP $\{X_t, t \in T\}$ satisfying

  a) The sample space $X = \{x_0, x_1, x_2, \dots\}$ can be either **finite** or **countably infinite**

  b) Transition from x_i to x_j may occur at each time t with a give **rate of** $q_{ij}(\mathrm{t})$, with $t \in \mathbb{R}_{\geq 0}$

  c) The transition probabilities satisfy the **Markov Property**

$$p_{ij}(t_k, t_{k+1}) = \Pr(x(t_{k+1}) = x_j \mid x_{[0,t_k]}) = \Pr(x(t_{k+1}) = x_j \mid x(t_k) = x_i), \quad t_k \leq t_{k+1}$$

- The Markov property implies the random time $\Delta T_k = t_{k+1} - t_k$ we need to wait to leave the actual state $x(t_k)$ at time t_k is **exponentially distributed** (thus **memoryless**) such that:

$$\Delta T_k = \min\{dt \geq 0 \,:\, X_{t_k+dt} \neq x_i | X_{t_k} = x_i\} \sim Exp\left(\left(\sum_j q_{ij}(t_k)\right) dt\right)$$

$$\mathrm{E}[\Delta T_k] = \frac{1}{\sum_j q_{ij}(t_k)}$$

Holding time of state i

IMPORTANT: It follows that the flow of events to jump out of a state x_i are $\mathrm{Poiss}\left(\Delta t_k \cdot \sum_j q_{ij}(t_k)\right)$.
Consequently, as the time Δt_k spent in a state without a jump increases.
Follows that the probability $p_{ii}(t_k, t_k + \Delta t_k)$ of remaining in the same state decrease.

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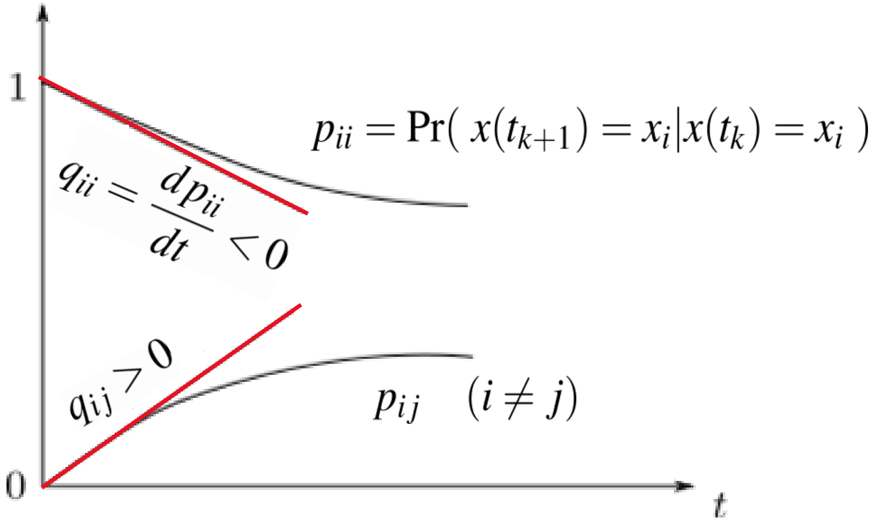
Continuous-time Markov Chain (cont'd)

- Since the time of leaving x_i at time t_k is $Exp\left(\sum_j q_{ij}(t_k)\right)$, the transition probabilities depend on t_k and t_{k+1} (or equivalenty by t_k and $\Delta t_k = t_{k+1} - t_k$)

$$p_{ij}(t_k, t_{k+1}) = \Pr(x(t_{k+1}) = x_j \mid x(t_k) = x_i), \quad t_k \leq t_{k+1}$$

- Let $q_{ij}(t_k) > 0$ be the rate of events to jump from $x_i \to x_j$ this implies

$$q_{ij}(t_k) = \lim_{t_{k+1} \to t_k} \frac{\Pr(\, x(t_{k+1}) = x_j \mid x(t_k) = x_i)}{t_{k+1} - t_k}$$



- Equivalently, it results that

$$p_{ij}(t_k, t_{k+1}) = \int_{t_k}^{t_{k+1}} q_{ij}(t_k) dt_k \approx q_{ij}(t_k) \cdot \overbrace{(t_{k+1} - t_k)}^{\Delta t_k}$$

(**Note:** CT-MCs is thus GSMP)

- This implies, as $\Delta t_k \to 0$ then $p_{ii}(t_k, t_k) = 1 - \sum_i p_{ij}(t_k, t_k) \to 1$, while as the time passes it monotonically decreases to zero

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Evolution of a CT-MC: Connection between P and Q

- We have seen that

$$p_{ij}(t_k, t_{k+1}) \approx q_{ij}(t_k) \cdot \overbrace{(t_{k+1} - t_k)}^{\Delta t_k}$$

- Moreover it also results that

$$\sum_{x_j \in X} p_{ij}(t_k, t_{k+1}) \; = \; 1 \quad \forall\, x_i \in X \quad , \forall\, t_k, t_{k+1} \in \mathbb{R}_{\geq 0}$$

- Thus, the transition probability matrix takes the next expression

$$\boldsymbol{P}(t_k, t_{k+1}) = \begin{pmatrix} p_{11}(t_k, t_{k+1}) & p_{12}(t_k, t_{k+1}) & \cdots \\ p_{21}(t_k, t_{k+1}) & \ddots & \ddots \\ \vdots & \ddots & p_{ij}(t_k, t_{k+1}) \end{pmatrix} \quad \forall\, t_k \leq t_{k+1}$$

- Thus, it is straightforward note that

$$p_{ij}(t_k, t_k) = \int_{t_k}^{t_k} q_{ij}(\tau) d\tau = 0 \quad \forall j \neq i$$

$$p_{ii}(t_k, t_k) = 1 - p_{ij}(t_k, t_k) = 1$$

➡

$$\boldsymbol{P}(t_k, t_k) = \boldsymbol{I} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \quad \forall\, t_k$$

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Evolution of a CT-MC: Connection between P and Q (cont'd)

- By using the same notation of DT-MC we define the marginal (unconditional) probability of being in state x_i at time t as

$$\pi_i(t) = \Pr(x(t) = x_i) \quad \Longrightarrow \quad \Pi(t) = [\pi_0(t), \pi_1(t), \dots]$$

- Moreover, by means of the Total Probability Law it yields

$$\pi_j(t+\Delta t) = \sum_{x_i \in X} \pi_i(t) \cdot p_{ij}(t, t+\Delta t) \qquad\qquad p_{ij}(t_k, t_{k+1}) \approx q_{ij}(t_k) \cdot \overbrace{(t_{k+1} - t_k)}^{\Delta t_k}$$

- From that, and by the exploiting the transition probability matrix one derives

$$\Pi(t+\Delta t) = \Pi(t) \cdot \boldsymbol{P}(t, t+\Delta t), \quad \forall\, t \geq 0$$

- Problem: There are infinite many transition probabilities matrices, one for each pair of t and Δt (function of 2 variables)!!! How to find $P(t, t+\Delta t)$ and $\Pi(t)$?

- **Answer:** We require a method to represent the marginal probability dynamics as a function of the transition rates instead of the transition probabilities

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Evolution of a CT-MC: Connection between P and Q (cont'd)

- As for DT-MC also for CT-MC the Markov Property allows to write

$$p_{ij}(t_k,t_{k+1}) = \sum_{x_r\in X} p_{ir}(t_k,t)\cdot p_{rj}(t,t_{k+1}) \qquad t_k \le t < t_{k+1}$$

- in matrix form: $\boldsymbol{P}(t_k,t_{k+1}) = \boldsymbol{P}(t_k,t)\cdot \boldsymbol{P}(t,t_{k+1})$ $\quad t_k \le t < t_{k+1}$
- or equivalently $\boldsymbol{P}(t_k,t+\Delta t_k) = \boldsymbol{P}(t_k,t)\cdot \boldsymbol{P}(t,t+\Delta t_k)$ $\quad 0 \le \Delta t_k = t - t_k \le t_{k+1} - t_k$
- Subtracting both side $\boldsymbol{P}(t_k,t)$, and dividing by $\Delta t_k \to 0$, and because $\boldsymbol{P}(t,t) = \boldsymbol{I}$, then

$$\frac{\boldsymbol{P}(t_k,t+\Delta t_k)-\boldsymbol{P}(t_k,t)}{\Delta t_k} = \frac{\boldsymbol{P}(t_k,t)\cdot\boldsymbol{P}(t,t+\Delta t_k)-\boldsymbol{P}(t_k,t)}{\Delta t_k} \implies \boldsymbol{P}(t_k,t)\cdot\frac{\boldsymbol{P}(t,t+\Delta t_k)-\boldsymbol{I}}{\Delta t_k}$$

As $\Delta t_k \to 0$ it results the Kolmogorov forward PDE

$$\boldsymbol{P}(t_k,t)\cdot\boxed{\boldsymbol{Q}(t)}$$

$$\frac{\partial \boldsymbol{P}(t_k,t)}{\partial t} = \boldsymbol{P}(t_k,t)\cdot\overbrace{\lim_{\Delta t_k\to 0}\frac{\boldsymbol{P}(t,t+\Delta t_k)-\boldsymbol{I}}{\Delta t_k}}$$

Transition rate matrix

(or Intensity matrix)

This Partial Differential Equation describes the time-evolutions of matrix $\boldsymbol{P}$ starting from time t_k

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Marginal probability evolution of CT-MCs

- Moroever, note that $\Pi(t+\Delta t) = \Pi(t) \cdot \boldsymbol{P}(t, t+\Delta t), \quad \forall\, t \geq 0$

- Thus, by subtracting $\Pi(t)$ on both sides, and by dividing $\Delta t \to 0$, one has

Chapman-Kolgomorov
Equation for CT-MC

$$\boxed{\frac{d\Pi(t)}{dt} = \Pi(t) \cdot \boldsymbol{Q}(t)}$$

$$\boldsymbol{Q}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{P}(t, t+\Delta t) - \boldsymbol{I}}{\Delta t}$$

- Each entry $q_{ij}(t)$ measures the **events rate** enabling to jump from x_i at time t to x_j after an arbitrarily small time-interval $\Delta t \to 0$

$$q_{ij}(t) = \lim_{\Delta t \to 0} \frac{\Pr(\, x(t+\Delta t) = x_j \mid x(t) = x_i)}{\Delta t}$$

- Clearly, when p_{ij} increases, p_{ii} decreases

- Indeed, notice that $q_{ii}(t)$ is always **negative** for each time $t \geq 0$

$$q_{ii}(t) = \lim_{\Delta t \to 0} \frac{\overbrace{1 - \sum_{\forall j \neq i} q_{ij}(t) \cdot \Delta t}^{p_{ii}(t, t+\Delta t)} - 1}{\Delta t} = -\sum_{\forall\, j \neq i} q_{ij}(t)$$

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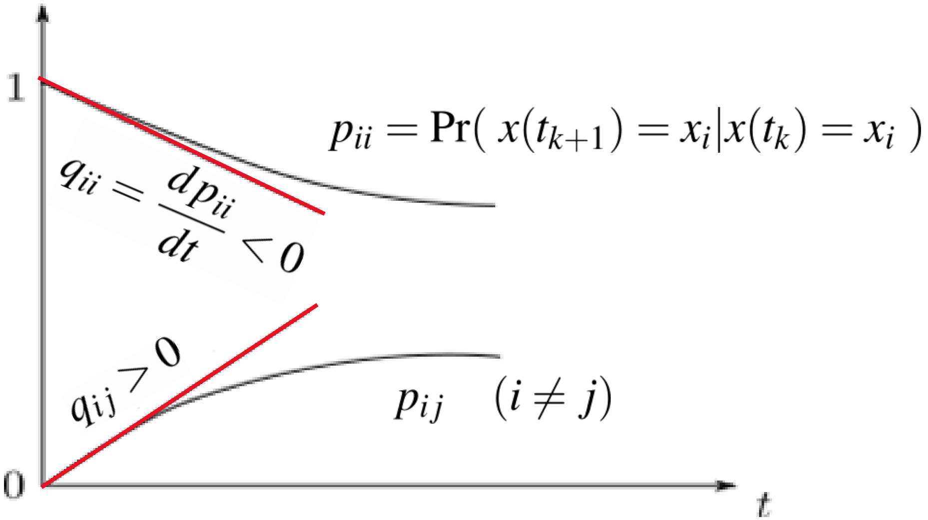
Transition rate matrix: $\boldsymbol{Q}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{P}(t, t+\Delta t) - \boldsymbol{I}}{\Delta t}$

- By construction the matrix $\boldsymbol{Q}(t)$ satisfies the following properties:



$$\frac{d\Pi(t)}{dt} = \Pi(t)\boldsymbol{Q}(t)$$

$$q_{ij}(t) \geq 0 \quad \forall\, x_i,\, x_j \in X,\, i \neq j$$

$$q_{ii}(t) \leq 0 \quad \forall\, x_i \in X$$

$$\sum_{\forall\, x_j \in X} q_{ij}(t) = 0 \quad \forall\, x_i \in X$$

Since $\boldsymbol{Q}(t)$ satisfies:

$$\boldsymbol{Q}(t) \cdot \vec{1} = 0 \cdot \vec{1}$$

It has at least one **zero eigenvalue**

Since $-\boldsymbol{Q}(t)$ is diagonally dominant, it yields that $\boldsymbol{Q}(t)$ has non positive eigenvalues

$$\Re\{\mathrm{eig}\{\boldsymbol{Q}(t)\}\} \in \mathbb{R}_{\leq 0}$$

- A **finite-state CT-MC** is always **WSS** (indeed as $t \to \infty$ both $p_{ij}(\infty)$ and $\Pi(\infty)$ converges to a pmf distribution), but a MC is not necessarily ergodic, namely $\Pi(\infty)$ may not be unique $\forall\ \Pi(0)$

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Continuous-time Markov Chains

- **Definition:** A CT-MC is mathematical model for stochastic processes defined by the triplet

$$\boxed{C = (X, \boldsymbol{Q}(t), \Pi(0))}$$

- where

1. X : is the discrete sample space of the CT process
2. $\boldsymbol{Q}(t)$: is the transition rate matrix at time $t \in \mathbb{R}_{\geq 0}$

$$\boldsymbol{Q}(t) = \begin{pmatrix} q_{11}(t) & q_{12}(t) & \cdots \\ q_{21}(t) & \ddots & \ddots \\ \vdots & \ddots & q_{ij}(t) \end{pmatrix}$$

$$q_{ij}(t) = \lim_{\Delta t \to 0} \frac{\Pr(\ x(t+\Delta t) = x_j \mid x(t) = x_i\ )}{\Delta t}$$

$$\forall\ x_i,\ x_j \in X,\ \forall\ t \in \mathbb{R}_{\geq 0}$$

3. $\Pi(0)$: is the initial marginal probability distribution

$$\Pi(0) = [\pi_0(0), \pi_1(0), \ldots]$$

$$\pi_i(0) = \Pr(x(0) = x_i)$$

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Time-Homogeneous CT-MC

- The study of CT-MCs is simpler when the matrix $\boldsymbol{P}(t, t+\Delta t)$ depends only by Δt and not by the actual time t. In this case the transition rate matrix $\boldsymbol{Q}$ is constant

$$\boldsymbol{Q}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{P}(t, t+\Delta t) - \boldsymbol{I}}{\Delta t} \quad \Rightarrow \quad \boldsymbol{Q}(t) = \boldsymbol{Q} \ (\text{cost.}), \quad \forall\, t \geq 0$$

- Thus the CT-MC is called time-homogeneous and its evolution is governed by a linear time-invariant (LTI) systems, where $eig(Q) \leq 0$

$$\frac{\partial \boldsymbol{P}(t_k, t)}{\partial t} = \boldsymbol{P}(t_k, t) \cdot \boldsymbol{Q}(t) \quad \Rightarrow \quad \frac{d\boldsymbol{P}(t)}{dt} = \boldsymbol{P}(t) \cdot \boldsymbol{Q}$$

Kolmogorov forward equation

$$\frac{d\Pi(t)}{dt} = \Pi(t) \cdot \boldsymbol{Q}$$

Chap.-Kolg. equation

- Remark: Physical processes are sometimes not time-homogeneous, however, by restricting the analysis over a small time-horizon may provide faithful predictions as well, e.g., *whether prediction models are re-calibrated periodically*
- **Such approximation is equivalent to a linearization for a LTI dynamical system**

Transition graph of a Homogeneous CT-MC

- A time-homogeneous CT-MC can be described a **directed graph** with constant weights

$$G = (V, E)$$

V is the vertex set

$E \subseteq V \times V$ is the edge set of G

- Each vertex of the graph corresponds to a possible outcome of the CT-MC

$$V = \{x_0, x_1, x_2, \dots\}$$

- An edge $(x_i, x_j) \in E$ at time t exists if

$$q_{ij}(t) = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{q_{ij}\Delta t}{\Delta t} = q_{ij} > 0$$

(for time-homogenous CT-MC rates are constants $\forall\, t$)

- **Self-loops** in **CT-MCs** are **implicit**, indeed we can always remain for a while in the actual state. Since negative rates make little sense, they are **omitted** in the graph. A **CT-MC** is always **aperiodic**.

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Examples

- In this course we focus our attention on time-homogenous CT-MCs, thus $\boldsymbol{Q}$ will be always time-invariant.

- Example 1: A machine can be in two states

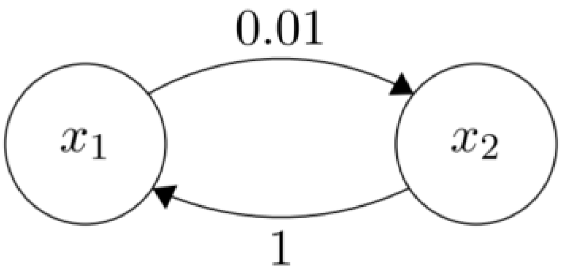
$$X = \{x_1, x_2\} = \begin{cases} x_1 & : \quad \text{healthy} \\ x_2 & : \quad \text{faulty} \end{cases}$$

- The failure rate is of a fault every 100 days of work, whereas the repair time is of 1 day, thus

$$q_{12} = \frac{1}{100}\,\frac{\text{fault}}{\text{day}} \qquad\qquad q_{21} = 1\,\frac{\text{reparation}}{\text{day}}$$

- Determine the transition rate matrix $\boldsymbol{Q}$ and its transition graph are

$$\boldsymbol{Q} = \begin{pmatrix} -0.01 & 0.01 \\ 1 & -1 \end{pmatrix}$$



CT Chapman Kolgomorov Equation’s solution

- In contrast with the DT-MC, the CT CK equation cannot be solved easily. This makes the study of probabilities in **transient-regime (in the short time)** a bit harder
- Fortunately, probabilities in a time-homogeneous CT-MC follows the solutions of an LTI system:

CT Chapman-Kolmogorov Eqn.
$$\frac{d\Pi(t)}{dt}=\Pi(t)\cdot Q$$

Its analytical solution is:
$$\Pi(t)=\Pi(0)\cdot e^{Q\cdot t}$$

"Matrix Exponential operator"

- **Note:** Matrix Exponential operator, to be computed, requires the diagonalization of $\mathbf{Q}$ thought the eigenvalues matrix $\boldsymbol{\Lambda}(\boldsymbol{Q})=\mathrm{diag}(\lambda_1,\lambda_2,\dots)\preccurlyeq 0$ and eigenvectors matrix $\boldsymbol{V}=[v_i]$ of $\mathbf{Q}$

$$\boldsymbol{Q}\boldsymbol{v}_i=\lambda_i\boldsymbol{v}_i$$

- Then, because of $\boldsymbol{\Lambda}=\mathbf{V}^{-1}\mathbf{QV}$ it yields

$$\Pi(t)=\Pi(0)\cdot \boldsymbol{V}\cdot e^{\overbrace{\boldsymbol{V}^{-1}\boldsymbol{Q}\boldsymbol{V}}^{\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)}\cdot t}\cdot \boldsymbol{V}^{-1}$$

- **Alternatively:** it can be solved numerically for a small sampling time $\tau\ll|\max\{eig\{Q\}^{-1}\}|$

DT Chapman-Kolmogorov Eqn.

$$\frac{d\Pi(t)}{dt}\approx\frac{\Pi(t+\tau)-\Pi(t)}{\tau}=\Pi(t)\cdot\boldsymbol{Q}\implies\Pi(t+\tau)=\Pi(t)\overbrace{(\boldsymbol{I}+\tau\boldsymbol{Q})}^{\boldsymbol{P}}$$

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CT Chapman Kolgomorov Equation’s solution (cont’d)

- Another way to solve the CT Chapman Kolgomorov Equation is by resorting to the concept of **Laplace transformation** for CT LTI systems.
- In particual let s be the **Laplace variable**, and $\Pi(0)$ be the initial probability distribution, one has

$$\mathcal{L}\left\{\frac{d\Pi(t)}{dt}\right\} = \mathcal{L}\left\{\Pi(t)\cdot \boldsymbol{Q}\right\} \quad \Rightarrow \quad s\cdot \Pi(s) - \Pi(0) = \Pi(s)\cdot \boldsymbol{Q}$$

$$\Rightarrow \quad \Pi(s)\cdot(s\cdot \boldsymbol{I} - \boldsymbol{Q}) = \Pi(0)$$

$$\Rightarrow \quad \Pi(s) = \Pi(0)\cdot(s\cdot \boldsymbol{I} - \boldsymbol{Q})^{-1}$$

(see “*resolvent matrix*”)

$$\Rightarrow \quad \boxed{\Pi(t) = \mathcal{L}^{-1}\left\{\Pi(0)\cdot(s\cdot \boldsymbol{I} - \boldsymbol{Q})^{-1}\right\}}$$

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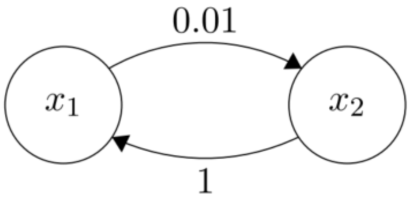
Examples (cont’d)

- **Example 1:** A machine can be in two states $X = \{x_1, x_2\} = \begin{cases} x_1 & : & \text{healthy} \\ x_2 & : & \text{faulty} \end{cases}$



$$\boldsymbol{Q} = \begin{pmatrix} -0.01 & 0.01 \\ 1 & -1 \end{pmatrix} \qquad \Pi(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

- Then, one has

$$\Pi(s) = \Pi(0) \cdot (s \cdot \boldsymbol{I} - \boldsymbol{Q})^{-1} \qquad (s\boldsymbol{I} - \boldsymbol{Q}) = \begin{pmatrix} s + 0.01 & -0.01 \\ -1 & s+1 \end{pmatrix}$$

$$(s \cdot \boldsymbol{I} - \boldsymbol{Q})^{-1} = \frac{\operatorname{adj}\{s\boldsymbol{I} - \boldsymbol{Q}\}}{\det\{s\boldsymbol{I} - \boldsymbol{Q}\}} = \frac{1}{s(s+1.01)} \begin{pmatrix} s+0.01 & 0.01 \\ 1 & s+1 \end{pmatrix}$$

$$\boxed{\Pi(s) = \Pi(0) \cdot (s \cdot \boldsymbol{I} - \boldsymbol{Q})^{-1} = \begin{pmatrix} \frac{s+1}{s(s+1.01)} & \frac{0.01}{s(s+1.01)} \end{pmatrix}}$$

Examples (cont'd)

- Then, by means of the partial fraction decomposition

$$\Pi(s) = \left(\frac{R_1^1}{s-p_1} + \frac{R_2^1}{s-p_2} \quad , \quad \frac{R_1^2}{s-p_1} + \frac{R_2^2}{s-p_2}\right)_{p_1=0,p_2=-1.01} \qquad [R_j^1, R_j^2] = \lim_{s\to p_j}(s-p_j)\cdot\Pi_i(s)$$

- As an example:

$$[R_1^1, R_1^2] = \lim_{s\to 0}\cancel{(s-0)}\cdot\left(\frac{s+1}{\cancel{s}(s+1.01)} \quad \frac{0.01}{\cancel{s}(s+1.01)}\right) = \left(\frac{1}{1.01} \quad \frac{0.01}{1.01}\right) = \left(\frac{100}{101} \quad \frac{1}{101}\right)$$

$$[R_2^1, R_2^2] = \lim_{s\to -1.01}\cancel{(s+1.01)}\cdot\left(\frac{s+1}{s\cancel{(s+1.01)}} \quad \frac{0.01}{s\cancel{(s+1.01)}}\right) = \left(\frac{-0.01}{-1.01} \quad \frac{0.01}{-1.01}\right) = \left(\frac{1}{101} \quad \frac{-1}{101}\right)$$

- Then:

$$\Pi(s) = \left(\frac{100}{101}\cdot\frac{1}{s} + \frac{1}{101}\cdot\frac{1}{s+1.01} \quad , \quad \frac{1}{101}\cdot\frac{1}{s} - \frac{1}{101}\cdot\frac{1}{s+1.01}\right)$$

- Finally, by the **Laplace inverse transformation method**, one derives

$$\Pi(t) = \mathcal{L}^{-1}\left\{\Pi(0)\cdot(s\cdot \boldsymbol{I} - \boldsymbol{Q})^{-1}\right\} =$$

$$= \left(\frac{100}{101} + \frac{1}{101}\cdot e^{-1.01t} \quad , \quad \frac{1}{101} - \frac{1}{101}\cdot e^{-1.01t}\right)\cdot\delta_{-1}(t), \qquad \forall\, t \geq 0$$

Stationary and limiting distribution of CT-MC

- Let us consider the evolution of a time-homogeneous CT-MC

$$\frac{d\Pi(t)}{dt} = \Pi(t) \cdot \boldsymbol{Q}$$

- The CT-MC reaches an equilibria, i.e. it has a limiting distribution $\Pi_\ell = \Pi(\infty)$ if and only if its probability rate becomes null $\dot{\Pi} = 0$, thus $\Pi(t)\boldsymbol{Q} = \boldsymbol{0}$, as $t \to \infty$

- **Definition:** A CT-MC admits a limiting distribution Π_ℓ if, for the given initial probability distribution $\Pi(0) = [\pi_1(0), \pi_2(0), \dots]$, the following limit exists

$$\Pi_\ell = \lim_{t \to \infty} \Pi(t) = \lim_{t \to \infty} \Pi(0) \cdot \mathrm{e}^{\boldsymbol{Q} \cdot t}$$

- Definition: The marginal probability distribution Π_s is called stationary if and only if it satisfies the so-called balance equations for CT-MCs:

$$\Pi_s \boldsymbol{Q} = \boldsymbol{0} \qquad \Pi_s \cdot \boldsymbol{1} = 1 \implies \sum_i \pi_{i,s} = 1$$

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Considerations on Stationary and Limiting distribution

- Remark 1: A limiting distribution Π_ℓ is also a stationary distribution Π_s, thus it solves the balancing equations

  The viceversa, is not true.

- Remark 2: This limit does not exist if $\mathbf{Q}$ has more than 1 eigenvalue in the origin "0" or if it has eigenvalue with pure imaginary part

$$\nexists\ \Pi_\ell = \lim_{t\to\infty} \Pi(0)\cdot \mathrm{e}^{\boldsymbol{Q}\cdot t}$$

- Remark 3: The above limit may exists but it is not necessarily unique, thus Π_ℓ may be influenced by the considered initial marginal probability distribution

$$\Pi(0) = [\pi_1(0), \pi_2(0), \ldots]$$

- Remark 4: If the limit exists, once Π_ℓ is reached, then Π_ℓ will remain unchanged, and constant, for all the subsequent instant of time.

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Ergodic CT-MC

- Definition: A CT-MC is said to be ergodic if and only if:

1) Exists a limiting probability distribution Π_ℓ (stationarity prerequisite)

$$\exists\, \Pi_\ell = \lim_{t \to \infty} \Pi(t)$$

2) This limit is unique, thus independent by

$$\Pi(0) = [\pi_1(0), \pi_2(0), \ldots]$$

If these conditions are verified, to understand the behavior of the CT-MC

we can simply study one, sufficiently long, possible realizations

- Remark: If a CT-MC is ergodic, the calculus of its limiting distribution Π_ℓ reduces to the calculus of its stationary component Π_s (thus, to the resolution of a linear system).

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Criteria for evaluating the ergodicity of CT-MC

Eigenvalue Criteria

- Theorem: Let $\lambda_i \in \mathbb{C}_{\leq 0}$ with $i = 1, \ldots, n$, be the eigenvalues of the transition rate matrix $\boldsymbol{Q}$ of a time-homogeneous CT-MC.
- Since $\boldsymbol{Q}$, by construction, diagonally dominant, it satisfies that

$$\lambda_1 = 0, \quad \mathrm{Re}\{\lambda_i\} \leq 0 \qquad \forall \quad i = 2, 3, \ldots$$

- The homogeneous CT-MC is ergodic **if and only if**

$$\mathrm{Re}\{\lambda_i\} < 0 \qquad \forall \quad i = 2, 3, \ldots \quad , \quad \lambda_1 = 0$$

Graph's Criteria

- Theorem: A homogeneous CT-MC is ergodic **if and only if** its associated graph $G(V, E)$ has a single "ergodic" Strongly Connected Component (SCC).

- Remark: The classification of the states and of a SSC follows that for DT-MCs without changes, the only exception regards the concept of periodicity.
- Here SCC are always aperiodic, since self-loops are implicit in each state

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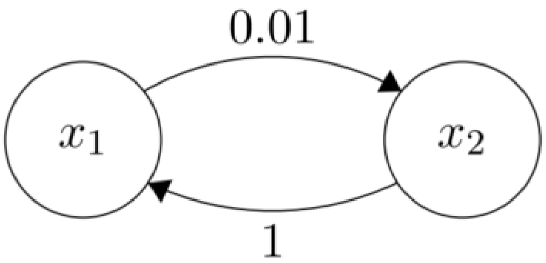
Examples

- Example 1:



$$Q = \begin{pmatrix} -0.01 & 0.01 \\ 1 & -1 \end{pmatrix}$$

a) By the Eigenvalues' criteria one has

$$\det\{\lambda I - Q\} = \det\left\{\begin{pmatrix} \lambda + 0.01 & -0.01 \\ -1 & \lambda + 1 \end{pmatrix}\right\} = \lambda(\lambda + 1.01)$$

$$\lambda_1 = 0 \quad , \lambda_2 = -1.01$$

This CT-MC is ergodic

b) By the Graph's criteria one has that $G(V,E)$ has a single ergodic component, namely, $\{x_1, x_2\}$.

This graph is irriducible and thus CT-MC is ergodic

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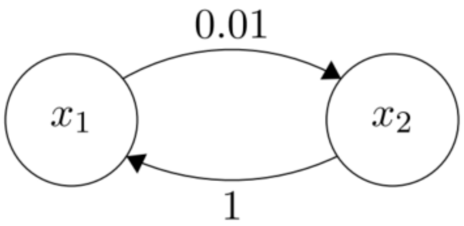
Examples (cont'd)

- Since the CT-MC is ergodic its limiting distribution Π_ℓ coincides with its stationary distribution Π_s, that is thus unique and independent by $\Pi(0)$.



$$\begin{cases} \Pi_s \boldsymbol{Q} = \boldsymbol{0} \\ \sum_i \pi_{i,s} = 1 \end{cases} \Rightarrow \quad \Pi_\ell = \Pi_s = \left(\frac{100}{101} \quad , \quad \frac{1}{101}\right)$$

- Remark 1: The result is coherent with that obtained by the Laplace Transformation

$$\Pi(t) = \left(\frac{100}{101} + \frac{1}{101} \cdot \mathrm{e}^{-1.01t} \quad , \quad \frac{1}{101} - \frac{1}{101} \cdot \mathrm{e}^{-1.01t}\right), \qquad \forall\, t \geq 0$$

- Remark 2: In a CT-MC transition occur in CT with given mean rates
- This mean we also can continuously leave and return to a state as the time passes
- It follows $G(V,E)$ is always aperiodic.
- Probability gives an info about the proportion of time to be in each state

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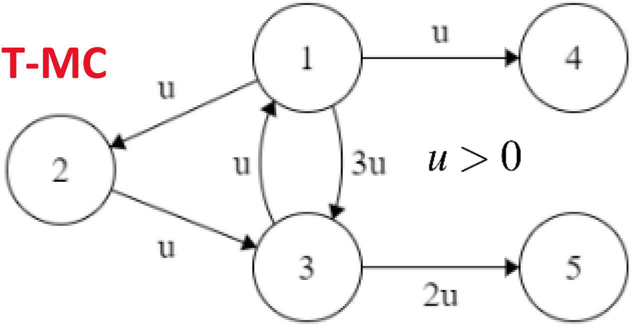
Absorption probability: $a'_i = \Pr(\exists\ t \geq 0 : x(t) \in E_2 | x(0) = i)$

- **Example:** Given a CT-MC, how to compute $a'_i = \Pr(\exists\ t \geq 0 : x(t) \in E_2 | x(0) = i)$?

CT-MC



$u > 0$

$E_1 = \{4\}$

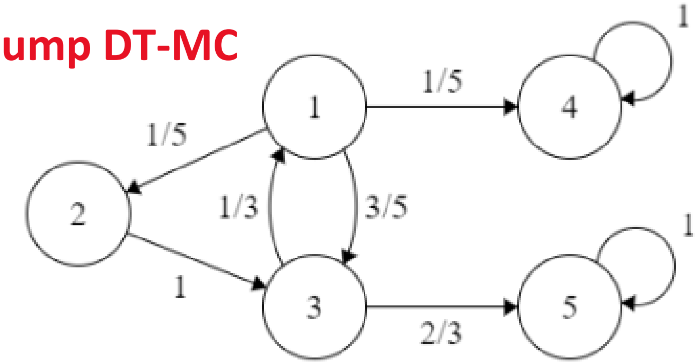
$E_2 = \{5\}$

To answer this question we can invoke the concept of **Jump DT-MC of a CT-MC**, that is a **DT-MC** where

$$p'_{ii} = 1 \text{ iff } q_{ii} = 0 \quad , \quad p'_{ij} = \frac{q_{ij}}{\sum_j q_{ij}} \quad j \neq i$$

- Then, the calculus of a'_i follow that for **standard DT-MCs**. Thus one derives that

Jump DT-MC



$$\begin{aligned} a'_1 &= \frac{1}{5}a'_2 + \frac{3}{5}a'_3 + \frac{1}{5}a'_4 \\ a'_2 &= a'_3 \\ a'_3 &= \frac{1}{3}a'_1 + \frac{2}{3}a'_5 \\ a'_4 &= 0\ ,\ a'_5 = 1 \end{aligned}$$

$$a'_1 = \frac{8}{11}$$

$$a'_2 = \frac{10}{11}$$

$$a'_3 = \frac{10}{11} > p'_{35}$$

- From that, let $b'_i = \Pr(\exists t \geq 0 : x(t) \in E_1 | x(0) = i)$ one can further conclude that

$$b'_1 = 1 - a'_1 \approx 0.27 > p_{14} \qquad b'_2 = 1 - a'_2 \approx 0.09 \qquad b'_3 = 1 - a'_3 \approx 0.09$$

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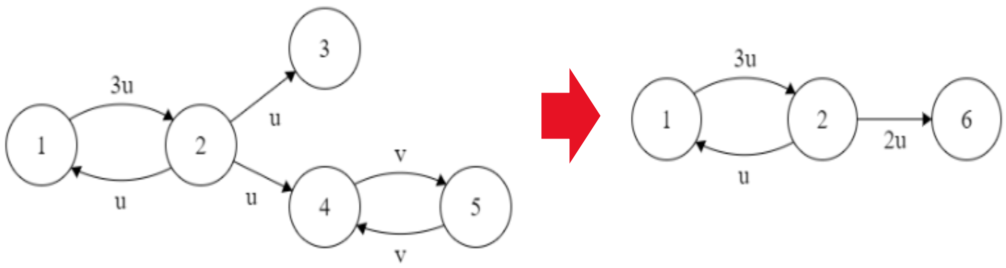
Expected time to absorption: $\mu_i = E[T|X_0 = i]$

- As for CT-MCs the expected time to absorption can be derived iff it has a single absorbing state, otherwise $\mu_i \to \infty$. If **not**, the chain **need to be rearranged, e.g., as follows**:



Then: Let T be the **min time to reach**, e.g. $s = 6$

$$T = \min\{t \geq 0 \,:\, X_t = 6 | X_0 = i\} \sim Exp\left(\mu_i^{-1}\right)$$

we can look forward $\mu_i = \mathsf{E}[T|X_0 = i]$

- The expected time to absorption of state "s" given $X_0 = i$, is the solution of a linear system where $\mu_s = 0$

- T^* is the mean time to leave i

$$T^* = \min\{t \geq 0 \,:\, X_t \neq i | X_0 = i\}$$

- dt is the time to reach s from state X_{T^*}

$$\mu_i = \overbrace{\frac{1}{\sum_j q_{ij}}}^{\mathsf{E}[T^*]} + \overbrace{\sum_{\forall\ j \in X} \overbrace{\mu_j}^{\mathsf{E}[dt|X_{T^*}=j]} \cdot \overbrace{\frac{q_{ij}}{\sum_j q_{ij}}}^{\Pr(X_{T^*}=j)}}^{\mathsf{E}[dt] \leftarrow \text{Law of Total Expectation}} \qquad \forall \quad i \neq s$$

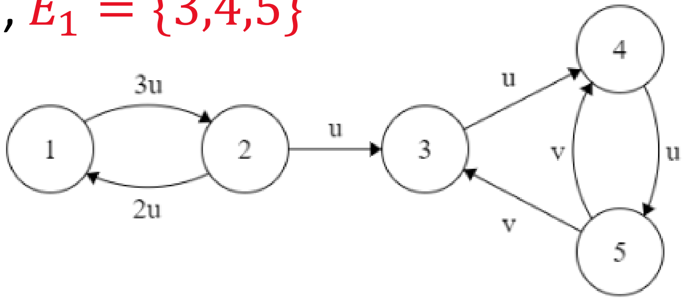
- Example: Compute $\mu_i = E[T|X_0 = i]\ \forall\ i \neq s$:

$$\begin{cases} \mu_6 = 0 \\ \mu_1 = \frac{1}{3u} + \frac{3u}{3u}\mu_2 \\ \mu_2 = \frac{1}{3u} + \frac{u}{3u}\mu_1 + \frac{2u}{3u}\mu_6 \end{cases} \Rightarrow \begin{cases} \mu_6 = 0 \\ \mu_1 = \frac{1}{3u} + \mu_2 \\ \mu_2 = \frac{1}{3u} + \frac{1}{3}\left(\frac{1}{3u} + \mu_2\right) \end{cases} \Rightarrow \begin{cases} \mu_6 = 0 \\ \mu_1 = \frac{1}{u} \\ \mu_2 = \frac{2}{3u} < \mu_1 \end{cases}$$

Mean first time & Mean Recurrence time

- Example: Consider a CT-MC with 1 ergodic class, e.g., $E_1 = \{3,4,5\}$

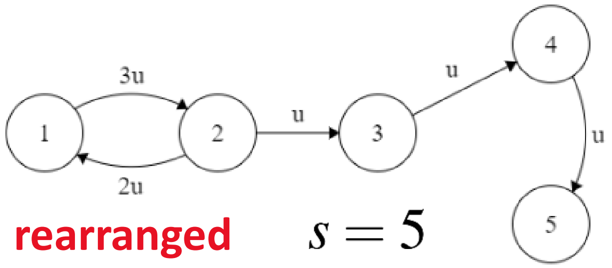
original CT-MC



The **mean first time** from i to a **target state** s is

$$t_i = \mathrm{E}[\min\{t \geq 0 \,:\, X_t = s \mid X_0 = i\}]$$

- Consider, e.g., $s = 5$, since we don't care where the chain will jump after reaching s, then this is as asking for the expected absorbing time of state $s = 5$ for the next **CT-MC**



rearranged CT-MC

$$s = 5$$

$$t_s = t_5 = 0$$

$$t_i = \frac{1}{\sum_j q_{ij}} + \sum_k t_j \cdot \frac{q_{ij}}{\sum_j q_{ij}}$$

$$t_4 = 1/u$$

$$t_3 = 1/u + t_4 = 2/u$$

$$t_2 = \frac{1}{3u} + \frac{2}{3} t_1 + \frac{1}{3} t_3 = 3.67/u$$

$$t_1 = \frac{1}{3u} + t_2 = 4/u$$

- The **mean recurrence time** of s can be computed as

$$t_s^* = \mathrm{E}[\min\{t > 0 \,:\, X_t = s \mid X_0 = s\}] \quad \Rightarrow \quad t_s^* = \frac{1}{\sum_j q_{ij}} + \sum_k \frac{q_{ij}}{\sum_j q_{ij}} \cdot t_j$$

- Example (cont'd): Let $s = 5$

$$\Rightarrow \quad t_5^* = \frac{1}{2v} + \frac{v}{2v} t_3 + \frac{v}{2v} t_4 = \frac{1}{2v} + \frac{3}{2u}\bigg|_{u=v=1} = 2 \text{ sec}$$

Continuous-time Birth-Death Processes

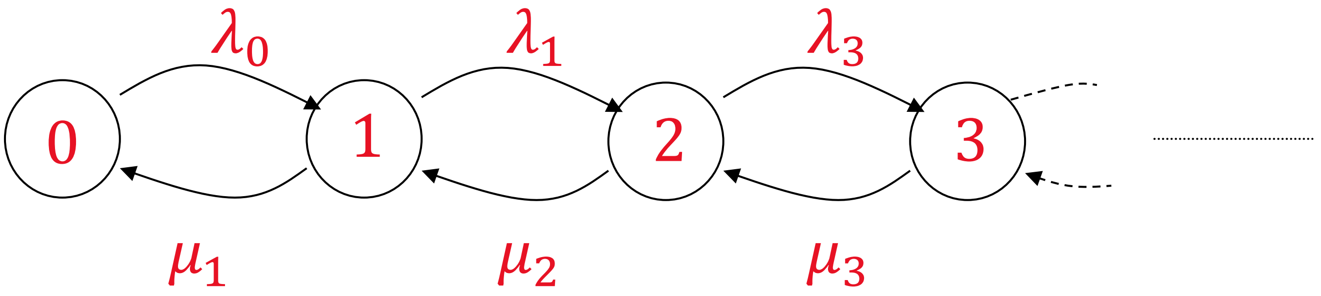
- Definition: A continuous-time Birth-Death Process (CT-BDP) is a special class of CT-MC which are characterized by:

a) The sample space X can only take integer values

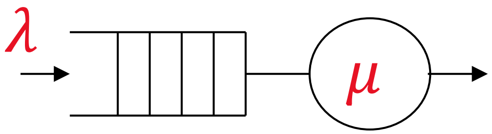
$$X = \{\, 0 \,,\, 1 \,,\, 2 \,, \dots \,\} \in \mathbb{I}$$

b) Only transitions between adjacent states are allowed

c) Transitions may be observed at any instant of time $t_n \in \Re_{\geq 0}$



CT-BDP are often used to model queues



- Remark: The sample space X may denotes the population of packects to be delivered by a router, or costumers in a queue, token in service system, infected people in a geographic area, etc…

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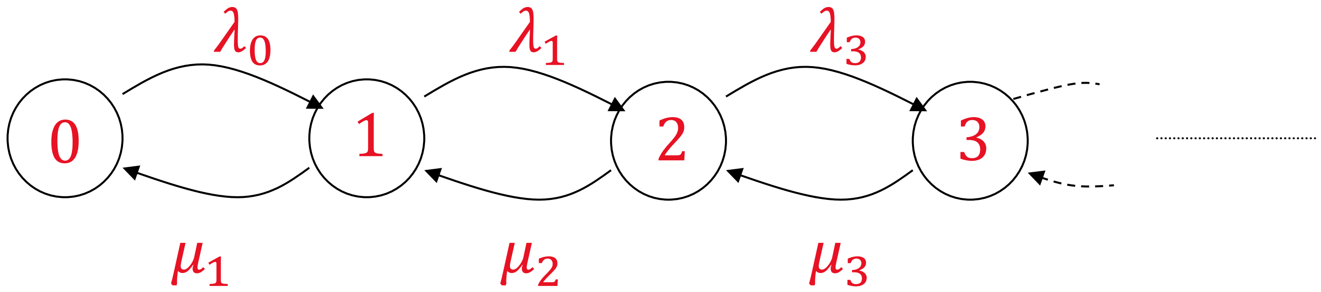
Continuous-time Birth-Death Processes (cont'd)

- The transition graph of CT-BDP is as follows



- State transitions are of only two types

a) Births which increase the state by one

λ_i denotes birth rate

b) Deaths which decrease the state by one

μ_i denotes the death rate

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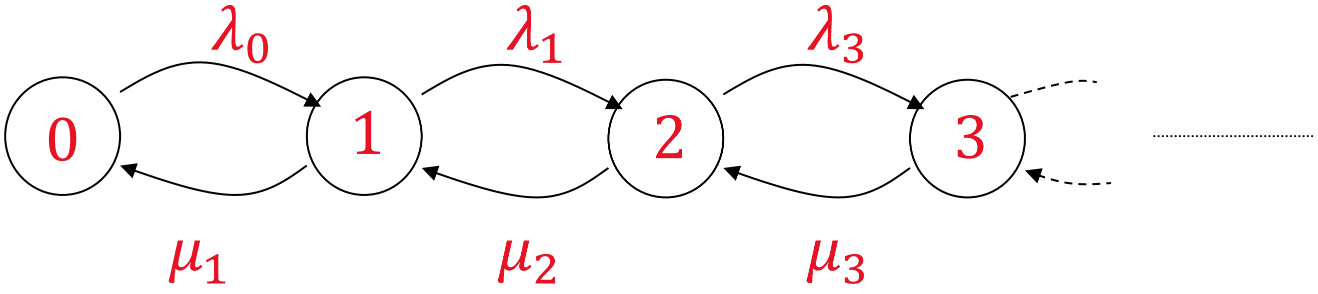
Transition rate matrix of a CT-BDP



- The transition rate matrix $\boldsymbol{Q}$ of a CT-BDP is **three-diagonal** as follows

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \mu_1 & -\lambda_1-\mu_1 & \lambda_1 & 0 & \cdots & \cdots & \cdots \\ 0 & \mu_2 & -\lambda_2-\mu_2 & \lambda_2 & 0 & \cdots & \cdots \\ 0 & 0 & \mu_3 & -\lambda_3-\mu_3 & \lambda_3 & \ddots & \ddots \\ \vdots & \vdots & 0 & \mu_4 & -\lambda_4-\mu_4 & \lambda_4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Remark: If the number of is limited it can be studied as finite-state CT-MC

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Continuous-time Birth-Death Processes (cont'd)

Differently with the DT-BDPs, CT-BDPs are by construction aperiodic

- Then, let $\lambda_i = \lambda_i(t)$ and $\mu_i = \mu_i(t)$ one has that:

a) If λ_i and μ_i are time-invariant, i.e., constant for each t:

The CT-BDP is called homogenous, and matrix $\boldsymbol{Q}$ has constant entries

b) If $\lambda_i = \lambda$ and $\mu_i = \mu$ are constant for each i, and t:

The CT-BDP is called uniform.

c) If $\lambda_i > 0$ and $\mu_i > 0$ for each i

The CT-BDP is irriducible
(all states are mutually reachable)

Remark: Not enough to say ergodic, because we have infinite many states… ☹

Stationary distribution of CT-BDP

- Consider an uniform CT-BDP

$$\lambda_i = \lambda \quad , \quad \mu_i = \mu , \quad \forall i \in X$$

- The stationary distribution Π_s for a **CT-BDP with infinite many state** can be evaluated by solving the **balance equation**:

$$\begin{cases} \Pi_s \cdot \boldsymbol{Q} = \boldsymbol{0} \\ \sum_i \pi_{i,s} = 1 \end{cases}$$

$$\begin{cases} \pi_{s,1} & = & \frac{\lambda_0}{\mu_1} \cdot \pi_{s,0} \\ \pi_{s,2} & = & \frac{\lambda_1}{\mu_2} \cdot \pi_{s,1} \\ \vdots & = & \vdots \\ \pi_{s,i} & = & \frac{\lambda_{i-1}}{\mu_i} \cdot \pi_{s,i-1} \\ \vdots & = & \vdots \\ \sum_i \pi_{s,i} & = & 1 \end{cases}$$

$$\rho = \frac{\lambda}{\mu}$$

$$\Rightarrow$$

$$\begin{cases} \pi_{s,1} & = & \rho \cdot \pi_{s,0} \\ \pi_{s,2} & = & \rho^2 \cdot \pi_{s,0} \\ \vdots & = & \vdots \\ \pi_{s,i} & = & \rho^i \cdot \pi_{s,0} \\ \vdots & = & \vdots \\ \sum_i \pi_{s,i} & = & 1 \end{cases}$$

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Stationary distribution of CT-BDP (cont'd)

- By simple manipulation one has

$$\pi_{s,0} + \rho\pi_{s,0} + \rho^2\pi_{s,0} + \cdots = 1 \qquad \Longrightarrow \qquad \pi_{s,0}\sum_i \rho^i = 1$$

$$\rho = \frac{\lambda}{\mu} < 1$$

If this geometric series is convergent (i.e. $\rho < 1$), then $\boldsymbol{\Pi_s}$ exists unique

$$\sum_{i=0}^{n}\rho^i = \left.\frac{1-\rho^{n+1}}{1-\rho}\right|_{n\to\infty} = \frac{1}{1-\rho}$$

- The irreducibility and aperiodicity of the graph along with the **above condition** implies the ergodicity of the CT-DBP.
- Thus:

$$\pi_{s,0}\cdot\frac{1}{1-\rho} = 1 \quad \Longrightarrow \quad \Pi_\ell \equiv \Pi_s = \begin{cases} \pi_{s,0} & = & 1-\rho \\ \pi_{s,i} & = & \rho^i\cdot\pi_{s,0} = \rho^i\cdot(1-\rho), \qquad \forall\, i>0 \end{cases}$$

- Remark: Clearly if instead the number of states is finite, the CT-MC is always ergodic regardless of the value of ρ

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Metrics of interest of CT-BDPs

- As for **DT-BDP** the long term expected population size as well as its long term variance can be derived by exploting the Moment generating function for discrete r.v.

$$M_\ell(z) \quad = \quad \sum_{i=0}^{n} z^{-1} \cdot \pi_{\ell,i} = (1-\rho)\cdot\sum_{i=0}^{\infty}\left(\frac{\rho}{z}\right)^i = (1-\rho)\cdot\frac{1}{1-\frac{\rho}{z}} = \frac{(1-\rho)\cdot z}{z-\rho}$$

- In particular, it results both long term expectation as well as the long term variance takes the same expression seen for a DT-BDP

$$\mu_X(\infty) \quad = \quad -\frac{dM_\ell(z)}{dz}\bigg|_{z=1} = -\frac{d}{dz}\left(\frac{(1-\rho)z}{z-\rho}\right)\bigg|_{z=1} = \frac{(1-\rho)(z-\rho)-(1-\rho)z\cdot 1}{(z-\rho)^2}\bigg|_{z=1} = -\frac{\rho(1-\rho)}{(z-\rho)^2}\bigg|_{z=1} = \frac{\rho}{1-\rho}$$

$$\mathrm{Var}[X_\infty] \quad = \quad \frac{d^2M_\ell(z)}{dz^2}\bigg|_{z=1} - \mu_X(\infty) - (\mu_X(\infty))^2 = \frac{2\rho}{(1-\rho)^2}$$

Remark: If $\rho > 1$ the population size of the BDP will tends $+\infty$

$\mu_X(\infty) \to \infty$ $\Rightarrow$ the DT-BDP is **not stationary** $\Rightarrow$ **not ergodic**

Remark: For a DT-BDP all the techniques for DT-MC can be also applied by properly reducing/rearranging the transition graph as a finite-state CT MC

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