Stochastic Models

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Assignment 3

Exercise 1. Consider 20 transmission sources, each transmitting at a peak rate of 10 Mbps with a probability of $p_1 = 0.1$, while the remaining time it is in idle mode. Additionally, there are 80 sources, each transmitting at a peak rate of 1 Mbps with a probability of $p_2 = 0.05$, or it is in idle mode.

A service provider aims to allocate the minimal capacity C_{opt} such that no more than 1.5 minute per hour the traffic demand exceeds the available capacity. Evaluate C_{opt}.

Exercise 2. A teletraffic channel with a capacity of $C = 200$ Mbps serves a rural area consisting of 4 apartments. Through observation, the service provider noticed that most of the time each apartment requires 15 Mbps of bandwidth.

First, determine the percentage of time the total traffic exceeds the available capacity.

Next, determine the maximal standard deviation σ tolerated for the total traffic that satisfies the demand and exceeds the available capacity no more than 10% of the time.

Exercise 3. A supermarket offers a LEGO minifigure for every 20€ of purchases. There are $n = 4$ different minifigures, each equally likely to be received. Let C be the amount of euros spent until a full set is obtained. Compute the expected cost $\mathsf{E}[C]$ of completing the collection.

Exercise 4. Let us consider the stochastic process $\{X_\tau, \tau = 0, 1, 2, \dots\}$ of rolling a dice at time $\tau = 0$, then, if we get 2, 3, 4 or 5 at time $\tau > 0$, the dice is rolled again at time $\tau + 1$; if we get 1 or 6 the dice is left in the previous position for the remaining of the time.

Solve the following problems without the using markov chains.

(a) By means of the Total probability Law determine the update rule for the marginal probabilities, namely $\pi_i(\tau + 1) = \Pr(X_{\tau+1} = i)$, valid $\forall\ \tau > 0$, where $i = 1, \dots, 6$.

(b) Let $\pi_i(0) = 1/6$ for all i, by induction, determine the explicit expression of $\pi_i(\tau)$, i.e. for the generic time step $\tau > 0$, as a function of $\pi_i(0)$.

(c) For the given $\pi_i(0)$, calculates its mean $\mu_X(\tau) = \mathsf{E}[X_\tau]$, then derives the corresponding long-term mean value $\mu_X(\infty)$, namely at the process steady state, if there exists.

(d) Provides 3 feasible movies (i.e. realizations) of the process, each with 10 samples, then try to concludes if the process is ergodic or not.

(e) Calculate the following Joint probabilities, resp., $\Pr(X_0 = 2, X_1 = 6, X_2 = 6)$ and $\Pr(X_0 = 1, X_1 = 6, X_1 = 6)$.

(f) Discuss if this process is, respectively, weak-sense stationary, ergodic, independent, markovian.

Hint: $\sum_{i=0}^{\tau} \rho^\tau = \frac{1-\rho^{\tau+1}}{1-\rho}, \qquad \forall\ \rho < 1$

Exercise 5. Let Z_1, Z_2, ..., be a family of independent and identically distributed random variables with probabilities $\Pr(Z_n = 1) = p$ e $\Pr(Z_n = -1) = 1 - p$ for each $n = 1, 2, \dots$.

Let us further define the random process

$$\{X_n, n = 0, 1, 2, \dots\}\ :\ X_0 = 0, \qquad X_n = \sum_{k=1}^{n} Z_k, \qquad n = 1, 2, \dots$$

The resulting stochastic process is commonly referred to as a *1-D random walk*.

(a) Determine the sample space of X_n.

(b) Is $\{X_n,\ n \in \mathbb{N}\}$ a counting process?

(c) Is $\{X_n,\ n \in \mathbb{N}\}$ a markov process?

(d) Let $p = 0.5$, provides a possible process realization for the first 10 instants of time

(e) Let $p = 0.5$, calculate $\mathsf{E}[X_n]$ and $\mathsf{Var}[X_n]$.

(f) Let $p = 0.5$, is $\{X_n,\ n \in \mathbb{N}\}$ a weak stationary process?

Exercise 6. Consider the MatLab code for generating an exponential random variable $Y \sim Exp(\lambda)$, by means of a uniform random variable $X \sim Uni_c(0,1)$, given below

```
n=1e4;                        % the length is arbitrary
lambda=40;                    % Inter-even time of Poisson 40 arrivals/sec
x=rand(n,1);                  % x~Uniform(0,1)
y = -lambda^-1*log(1-x);      % y~Exp(lambda)
mu_y=(1/n)*sum(y);            % Sampled mean of X (approx. E[X]=1/lambda)
```

(a) Reminding the meaning of the Erlang distribution, from `y` generate on MatLab a vector of occurrences `yE2` that is $Erlang(2,\lambda)$ distributed, where $\lambda = 40$.

(b) Reminding that for a Poisson process $\{Z_t, t \in T\}$, the inter-event time is $Y \sim Exp(\lambda)$. By means of `y` and the command `poissrnd` (see `help poissrnd`), generates on MatLab, a Poisson process which mean event rate is of 40arrivals/sec, and where the arrivals are scheduled accordingly with the inter-event random times `y`. Then verify if its sampled mean is equivalent to its expected value $\mathsf{E}[Z_t] = \lambda = 40$arrivals/sec.

Solutions

Solution of Exercise 1. In this assignment we have to calculate the optimal capacity $C = C_{opt}$ for a communication channel to serve all the 100 sources such that no more than 3 minute per our, on average, the traffic exceed the total link capacity. It follows our QoS parameter is:

$$\alpha = \frac{1.5\text{min}}{60\text{min}} = 0.025 \equiv 5\%$$

.

This problem can be solved by simply defining a random variable Σ_R which account the whole generated traffic, and then evaluates the smaller value of $C = C_{opt}$ such that the following inequality is in force

$$\Pr(\Sigma_R > C) \leqslant \alpha.$$

Sources can can be clustered into two families, the first one consists of $N_1 = 20$ sources, each transmitting with a rate $R_1 = 10\text{Mbps}$ with probability $p_1 = 0.1$, otherwise they stay in idle mode with probability $(1 - p_1) = 0.9$. The second family consists instead of $N_2 = 80$ sources, each transmitting with a rate $R_2 = 1\text{Mbps}$ with probability $p_2 = 0.05$, otherwise they stay in idle mode with probability $(1 - p_2) = 0.95$.

More formally, let the link capacity be a scalar $C > 0$, and let R_{X_i} be a random variable representing the rate transmitted by source i. Then we are interested to calculate the optimal channel capacity $C = C_{opt}$ such that the probability that the random variable associated with the demand generated by the 100 sources, denoted by $\Sigma_R = \sum_{i=1}^{100} R_{X_i}$ does not exceed C, namely

$$\Pr(\Sigma_R > C) \leqslant \alpha. \tag{1}$$

Let us now note that the probability that each source R_i transmits, can be seen as a Bernoulli (thus independent) trial with parameter p_i. Moreover, note that the demand generate by each family, since described by independent and identically distributed random variables, it can be modelled by a binomial distribution with parameters N_1 and p_1, for the family of sources counting $i = 1, 2, \ldots, 20$, and with parameters N_2 and p_2, for the family of sources $i = 21, 22, \ldots, 100$.

If the two families of sources were, not only independent, but also identically distributed, then by invoking the "Central Limit Theorem" we could approximated the distribution of $\Sigma_R = \sum_i R_{X_i}$ by a Gaussian random variable with mean and variance, N times greater than that of the single random variable R_{X_i}. However this is not our scenario, because the two families are not identical.

In particular, each family of sources is characterized by the following expected transmission rate and variance, respectively,

$$\begin{array}{rcll} \mathsf{E}[R_{X_i}] & = & p_1 \cdot R_1 + (1-p_1)\cdot 0 = 0.1 \cdot 10 = 1\text{Mpbs} & \\ \mathsf{Var}[R_{X_i}] & = & \mathsf{E}[(R_{X_i} - \mathsf{E}[R_i])^2] = \mathsf{E}[R_{X_i}^2] - \mathsf{E}[R_i]^2 & \forall\ i = 1, 2, \cdots, 20, \\ & = & [R_1^2 \cdot p_1 + 0 \cdot (1-p_1)] - R_1^2 \cdot p_1^2 = R_1^2 \cdot p_1(1-p_1) & \\ & = & 10^2 \cdot 0.1 \cdot 0.9 = 9\text{Mbps}^2 & \end{array} \quad (2)$$

and

$$\begin{array}{rcll} \mathsf{E}[R_{X_i}] & = & p_1 \cdot R_2 + (1-p_2)\cdot 0 = 0.05 \cdot 1 = 0.05\text{Mpbs} & \\ \mathsf{Var}[R_{X_i}] & = & \mathsf{E}[(R_{X_i} - \mathsf{E}[R_i])^2] = \mathsf{E}[R_{X_i}^2] - \mathsf{E}[R_i]^2 & \forall\ i = 21, 22, \cdots, 100. \\ & = & [R_2^2 \cdot p_2 + 0 \cdot (1-p_2)] - R_2^2 \cdot p_1^2 = R_2^2 \cdot p_2(1-p_2) & \\ & = & 1^2 \cdot 0.05 \cdot 0.95 = 0.0475\text{Mbps}^2 & \end{array} \quad (3)$$

Fortunately, since both the expected values and variances of the random variables associated to each R_{X_i} are bounded (they are not infinite), by invoking a generalized version of the "Central Limit Theorem" called "Lyapunov's central limit theorem" it results that the probability distribution of the total demand Σ_R has a Gaussian distribution with mean and variance given by

$$\begin{array}{rcl} \mathsf{E}[\Sigma_R] & = & \sum_{i=1}^{100} \mathsf{E}[R_{X_i}] = N_1 \cdot R_1 \cdot p_1 + N_2 \cdot R_2 \cdot p_2 \\ & = & 20 \cdot 1 + 80 \cdot 0.05 = 20 + 4 = 24\text{Mpbs} \\ \mathsf{Var}[\Sigma_R] & = & \sum_{i=1}^{100} \mathsf{Var}[R_{X_i}] \\ & = & 20 \cdot 9 + 80 \cdot 0.0475 = 183.8\text{Mbps}^2 \end{array} \quad (4)$$

Now, since a Gaussian random variables with mean $\mu = \mathsf{E}[\Sigma_R]$ and standard deviation $\sigma = \sqrt{\mathsf{Var}[\Sigma_R]}$ obey the so-called "$68-95-99.7\%$ Rule", given below

$$\begin{array}{rcl} \Pr(|\Sigma_R - \mu| \leqslant \sigma) = \Pr(|\Sigma_R - \mathsf{E}[\Sigma_R]| \leqslant \sqrt{\mathsf{Var}[\Sigma_R]}) & = & 0.68\% \\ \Pr(|\Sigma_R - \mu| \leqslant 2\cdot\sigma) = \Pr(|\Sigma_R - \mathsf{E}[\Sigma_R]| \leqslant 2\cdot\sqrt{\mathsf{Var}[\Sigma_R]}) & = & 0.95\% \\ \Pr(|\Sigma_R - \mu| \leqslant 3\cdot\sigma) = \Pr(|\Sigma_R - \mathsf{E}[\Sigma_R]| \leqslant 3\cdot\sqrt{\mathsf{Var}[\Sigma_R]}) & = & 0.997\% \end{array} \quad (5)$$

and because we are looking for the optimal C such that (1) is in force; by the 99.7% Rule we knows that

$$\Pr(|\Sigma_R - \mathsf{E}[\Sigma]| \leqslant 2\sqrt{\mathsf{Var}[\Sigma]}) = 0.95 \implies \Pr(|\Sigma_R - \mathsf{E}[\Sigma]| > 2\sqrt{\mathsf{Var}[\Sigma]}) = 0.05. \quad (6)$$

Notice that although Σ_R is not really symmertic, because it cannot take negative values, since the number of sources are many enough as well as its mean and variance are large, see $\mu = 24\text{Mbps}$, $\sigma = \sqrt{183.8\text{Mbps}}$, than we can safely approximate

$$\Sigma_R \sim \mathcal{N}(24\text{Mbps}, \sqrt{183.8\text{Mbps}})$$

i.e, as normal random variable, and consider the area of right tail equal to 0.05/2, namely,

$$\Pr(\Sigma_R > \mathsf{E}[\Sigma] + 2\sqrt{\mathsf{Var}[\Sigma]}) = \frac{0.05}{2} = 0.025 \leqslant \alpha,$$

the value of C such that (1) is in force can be evaluated by the (6) as the value of C corresponding to $\mathsf{E}[\Sigma_R] + 2\sqrt{\mathsf{Var}[\Sigma_R]}$.

Given that, and because $\sum_{i=1}^{100} R_i$ is also a worst-case upper-estimation of the sufficient capacity to serve all arriving traffic without losses, it results that the C_{opt} must satisfy the next relation

$$\begin{aligned} C_{opt} &= \min[\textstyle\sum_{i=1}^{100} R_i, \mathsf{E}[\Sigma_R] + 2\sqrt{\mathsf{Var}[\Sigma_R]}] \\ &= \min[20 \cdot 10 + 80 \cdot 1, 24 + 2 \cdot \sqrt{183.8}] \\ &= \min[280, 51.1] = 51.1\text{Mbps} \end{aligned}$$

For the sake of completedness note that if α was 0.015% then C_{opt} results to be 64.68Mbps

Solution of Exercise 2. Let $\Sigma_i(t)$ be the traffic generated at time t by each apartament, the total traffic of the residential area can be modelled as a stochastic process $\{\Sigma(t), t \in \mathbb{R}\}$, where $\Sigma(t) = \sum_{i=1}^{4} \Sigma_i(t)$

Moreover it further results that

$$\mu = \mathsf{E}[\Sigma(t)] = \mathsf{E}[\sum_{i=1}^{4} \Sigma_i(t)] = \sum_{i=1}^{4} \mu_{\Sigma_i(t)} = 4 \cdot 15 = 60\text{Mbps}$$

Because of $\Sigma(t)$ is a non-negative random variable, by means of the Markov inequality

$$\Pr(\Sigma \geqslant C) \leqslant \frac{\mu}{C}$$

we can derives that

$$\Pr(\Sigma \geqslant C) \leqslant \frac{60}{200} = 0.3$$

Which means that the available capacity of the channel will be saturated the thirty percent of time.

To determine the maximal tolerated standard deviation σ for the total traffic such that it will exceeds the available capacity no more then the 10% of the time we can invoke the Chebyshev inequality, given below

$$\Pr(|\Sigma - \mu| \geqslant \theta) \leqslant \frac{\sigma^2}{\theta^2} \tag{7}$$

where $\sigma^2 = \mathsf{Var}[\Sigma]$ and $\theta > 1$.

Now, by letting $\theta = \Theta\sigma$, and because nothing is stated about the simmetricity of the traffic distribution, then it can only be stated that

$$\Pr(|\Sigma - \mu| \geqslant \Theta\sigma) \leqslant \frac{1}{\Theta^2} \quad \Longrightarrow \quad \Pr(\Sigma \geqslant \overbrace{\mu + \Theta\sigma}^{C=200\text{Mbps}}) \leqslant \overbrace{\frac{1}{\Theta^2}}^{10\%}$$

Thus, by solving the following linear system

$$\begin{cases} 200\text{Mbps} &= \mu + \Theta\sigma \\ 0.1 &= \frac{1}{\Theta^2} \end{cases} \implies \begin{cases} 200\text{bps} &= 60 + \sqrt{10}\sigma \\ \Theta &= \sqrt{10} \end{cases} \implies \begin{cases} \Theta = \sqrt{10} = 3.16 \\ \sigma = \frac{200-60}{3.16} = 44.3\text{Mbps} \end{cases}$$

Solution of Exercise 3. We want to compute $\mathsf{E}[C]$. Since each minifigure cost 20€, then C is a multiple 20€, namely $C = 20K$ where K is the total number of purchases, each of 20€, to collect them all.

Note that the process of collecting n distinct minifigure is a couting process

$$\{X_k, k \in \mathbb{N}\}$$

where $X_k \in S_X = \{0, 1, 2, 3, 4\}$.

On the other hand, here we are interested in the number of purchases to pass from a collection with 0 minifigures to one with 4 distinct minifigures. This problem could be easily solved by exploting a discrete-time markov chain modelizzation and the concept of hitting-times (see module of DT-MC). However, here we will try to solve the problem as we do not know the existence of MC, and by exploting only consideration on random variables and generic stochastic processes.

Thus let us focus on the stochastic process

$$\{K_i, i = 0, 1, 2, 3\}$$

where $K_i \in S_K = \mathbb{N}$ models the random variable consisting on the number of purchases to obtain a full collection, namely $S_K = \{k_0, k_1, k_2, k_4\}$, where the index set is associated to the number of distinct minifigure we actually have.

Now, by means of simple consideration, one can easily note that each K_i is geometrically distributed with parameter

$$K_i \sim Geo(p_i) \quad \text{with} \quad p_i = \frac{(n-i)}{n}, \quad i = 0, 1, 2, 3$$

where i is the actual number of minifigure we actually have and $n = 4$ is the cardinality of the full collection. To understand this notice that, at the first step, we have $i = 0$ minifigure, and the 100% of chance to find a new minifigure (success) thus

$$\Pr(X_1 = 1|X_0 = 0) = 1 = \Pr(K_1 = k_1) = 1 \implies p_0 = 1,\ K_0 = 1 \text{ purchase of } 20€ \sim Geo(1)$$

To find your second minifigure in one purchase you have now a chance of 3/4, which implies that

$$\Pr(X_1 = 2|X_0 = 1) = \frac{3}{4} = \Pr(K_1 = k_1) = p_1 \quad \implies p_1 = \frac{3}{4},\ K_1 \sim Geo\left(\frac{3}{4}\right).$$

Namely you need to a Similarly it results that a purchase of 20

$$\Pr(X_1 = 3|X_0 = 2) = \frac{2}{4} = \Pr(K_3 = k_3) = p_2 \quad \implies p_2 = \frac{2}{4},\ K_2 \sim Geo\left(\frac{1}{2}\right).$$

and

$$\Pr(X_1 = 4|X_0 = 3) = \frac{1}{4} = \Pr(K_4 = k_4) = p_3 \qquad \Longrightarrow\ p_3 = \frac{1}{4},\ K_3 \sim Geo\left(\frac{1}{4}\right).$$

From that, and because of $\mathsf{E}[Geo(p_i)] = \frac{1}{p_i}$, we can thus conclude that

$$C = 20€ \cdot K = 20€ \cdot \sum_{i=0}^{n-1} K_i \implies \mathsf{E}[C] = 20€ \sum_{i=0}^{3} \overbrace{\mathsf{E}[K_i]}^{\frac{1}{p_i}} = 20€ \overbrace{\left(1 + \frac{4}{3} + 2 + 4\right)}^{\approx 8.333} \approx 166€$$

Let us further note that, for a generic n, it further results that

$$\mathsf{E}[C] = 20 \cdot \sum_{i=0}^{n-1} \frac{1}{p_i} = 20 \cdot \sum_{i=0}^{n-1} \frac{n}{(n-i)} = 20 \cdot n \cdot \sum_{i=0}^{n-1} \frac{1}{n-i}$$

namely

$$\mathsf{E}[C] = 20 \cdot 4 \cdot \sum_{i=0}^{3} \frac{1}{4-i} = 20 \cdot 4 \left(\frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1\right)$$

At hom, try to solve this exercise by exploting a DT-MC modellization and the concept of mean hitting time.

Solution of Exercise 4. Let us first note that the sample space of each random variable X, associated with my stochastic process $\{X_\tau, \tau = 1, 2, \ldots\}$ is discrete and equal to $1, 2, 3, 4, 5, 6$.

Let us now compute the marginal probabilities associated with each value of the sample space at time τ, namely, $\pi_i(\tau) = \Pr(X_\tau = i)$. Since the possibility to roll the dice again depends of what is happenned before, by the "Total Probability Law" it yields that,

$$\begin{aligned}
\pi_1(\tau) &= \Pr(X_{\tau-1} = 1) + \textstyle\sum_{i=2}^{5} \Pr(X_\tau = 1, X_{\tau-1} = i) \\
&= \Pr(X_{\tau-1} = 1) + \textstyle\sum_{i=2}^{5} \Pr(X_\tau = 1|X_{\tau-1} = i) \cdot \Pr(X_{\tau-1} = i) \\
&= \pi_1(\tau-1) + \tfrac{1}{6} \cdot \textstyle\sum_{i=2}^{5} \pi_i(\tau-1) \\
\pi_2(\tau) &= \textstyle\sum_{i=2}^{5} \Pr(X_\tau = 2, X_{\tau-1} = i) \\
&= \textstyle\sum_{i=2}^{5} \Pr(X_\tau = 2|X_{\tau-1} = i) \cdot \Pr(X_{\tau-1} = i) = \\
&= \tfrac{1}{6} \textstyle\sum_{i=2}^{5} \pi_i(\tau-1)
\end{aligned} \qquad \forall\ \tau \geqslant 1 \qquad (8)$$

Moreover, it further results that

$$\begin{gathered}
\pi_3(\tau) = \pi_4(\tau) = \pi_5(\tau) = \pi_2(\tau), \\
\pi_6(\tau) = \pi_1(\tau).
\end{gathered} \qquad (9)$$

Thus, by substituting (9) into (8), it thus results that

$$\begin{aligned}
\pi_i(\tau) &= \tfrac{1}{6} \cdot (4 \cdot \pi_2(\tau-1)) = \tfrac{4}{6} \cdot \pi_2(\tau-1), \qquad i = 2, 3, 4, 5 \\
\pi_i(\tau) &= \pi_1(\tau-1) + \tfrac{1}{6} \cdot (4 \cdot \pi_2(\tau-1)) = \pi_1(\tau-1) + \tfrac{4}{6} \cdot \pi_2(\tau-1), \qquad i = 1, 6
\end{aligned} \qquad (10)$$

Now, let us explicitly calculate the closed solutions of (10), recursively, for all $\tau \geqslant 1$. Let us first note that

$$\pi_i(0) = \Pr(X_0 = i) = \frac{1}{6} \quad , \quad \forall i = 1, 2, 3, 5, 6.$$

Now, let us consider the case of $\pi_2(\tau)$, for $\tau = 1, 2, \ldots, \tau$,

$$\begin{aligned} \pi_2(1) &= \tfrac{4}{6} \cdot \pi_2(0) = \tfrac{4}{6} \cdot \tfrac{1}{6} \\ \pi_2(2) &= \tfrac{4}{6} \left(\tfrac{4}{6} \cdot \tfrac{1}{6} \right) = \tfrac{1}{6} \left(\tfrac{4}{6} \right)^2 \\ \vdots \quad & \vdots \quad \vdots \\ \pi_2(\tau) &= \tfrac{1}{6} \left(\tfrac{4}{6} \right)^\tau \end{aligned} \tag{11}$$

On the other hand for $\pi_1(\tau) = \pi_6(\tau)$ are such that

$$2 \cdot \pi_1(\tau) + 4 \cdot \pi_2(\tau) = 1 \implies \pi_1(\tau) = \frac{1 - 4 \cdot \frac{1}{6} \left(\frac{4}{6} \right)^\tau}{2} = \frac{\left(1 - \left(\frac{4}{6} \right)^{\tau+1} \right)}{2}$$

Let us now evaluate the mean value associated with this stochastic process as the time passes

$$\begin{aligned} \mu_X(\tau) &= \textstyle\sum_{i=1}^{6} i \cdot \pi_i(\tau) = \overbrace{(1+6)}^{7} \cdot \pi_1(\tau) + \overbrace{(2+3+4+5)}^{14} \cdot \pi_2(\tau) \\ &= 7 \cdot \frac{\left(1 - \left(\frac{4}{6} \right)^{\tau+1} \right)}{2} + 14 \cdot \tfrac{1}{6} \left(\tfrac{4}{6} \right)^\tau \\ &= \tfrac{7}{2} - \tfrac{7}{2} \cdot \tfrac{4}{6} \left(\tfrac{4}{6} \right)^\tau + \tfrac{14}{6} \left(\tfrac{4}{6} \right)^\tau \\ &= \tfrac{7}{2} - \tfrac{7}{\not{2}} \cdot \tfrac{\not{2}\not{4}}{\not{6}3} \left(\tfrac{4}{6} \right)^\tau + \tfrac{\not{14}7}{\not{6}3} \left(\tfrac{4}{6} \right)^\tau = \tfrac{7}{2} + \left(\tfrac{4}{6} \right)^\tau \cdot \left(\tfrac{7}{3} - \tfrac{7}{3} \right) = \tfrac{7}{2}, \quad \forall\ \tau \geqslant 0 \end{aligned} \tag{12}$$

The previous computation shows that, despite probabilities are time-varying, the mean associated with this stochastic process is time-invariant with respect to $\tau \geqslant 0$.

This information combined with the fact the stochastic process has a finite sample-space, which implies a bounded second-order moment, allows to conclude this process is weak-sense stationary.

This result further implies that also its long-term mean $\mu_X(\infty) = \mu_X(\tau)$ is time invariant as well. To confirm this, let us note that $\pi_1(\infty) = \pi_6(\infty) = 0.5$ and $\pi_2(\infty) = \pi_3(\infty) = \pi_4(\infty) = \pi_5(\infty) = 0$, thus it yields that

$$\mu_X(\infty) = \lim_{\tau \to \infty} \mu_X(\tau) = \sum_{i=1}^{6} i \cdot \pi_i(\infty) = \lim_{\tau \to \infty} 7 \cdot \frac{\left(1 - \left(\frac{4}{6} \right)^{\tau+1} \right)}{2} + 14 \cdot \frac{1}{6} \left(\frac{4}{6} \right)^\tau = \frac{7}{2}$$

Notice that, mean is finite and time-invariant thus the stochastic process, since it has finite sample-space, thus finite second order moment, is wide-sense stationary.

Let us now provide 3 possible realization of this stochastic process

$$\begin{aligned} R_1 &= \{5,5,6,6,6,6,6,6,6\} \\ R_2 &= \{3,2,1,1,1,1,1,1,1\} \\ R_3 &= \{5,1,1,1,1,1,1,1,1\} \end{aligned}$$

By inspecting the three realizations R_i, it can be noted that the steady value of this stochastic process depends on the given realization. This aspect could be easily noted, also by evaluation the stationary marginal probabilities associated with X_τ, namely,

$$\pi_i(\infty) = \lim_{\tau \to \infty} \pi_i(\tau) = \begin{cases} 0 & i = 2,3,4,5 \\ \frac{1}{2} & i = 1,6 \end{cases} \tag{13}$$

It further follows that this stochastic process is not ergodic since its stationary distribution is not unique, and thus its sampled mean approaches 1 or 6 as the number of observations approach infinity on the the basis of which of the two possible outcomes is occurred for first.

To confirm this, let us not that, if we model this process as a DT-MC (notice that the conditional probability of moving to the next state at time τ depends only to the states at time $\tau - 1$), it has two absorbing components, resp., $\{1\}$ and $\{6\}$, or equivalently its transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 \\ 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 \\ 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 \\ 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

has an eigenvalue equal to 1 with algebraic multiplicity 2.

Finally, note that the Joint probability

$$\begin{aligned} \Pr(X_0 = 2, X_1 = 6, X_2 = 6) &= \Pr(X_2 = 6 | X_1 = 6, X_0 = 2) \cdot \Pr(X_1 = 6, X_0 = 2) && (14) \\ &= \Pr(X_2 = 6, X_1 = 6) \cdot \Pr(X_1 = 6 | X_0 = 2) \cdot \Pr(X_0 = 2) && (15) \\ &= \Pr(X_2 = 6 | X_1 = 6) \cdot \Pr(X_1 = 6 | X_0 = 2) \cdot \Pr(X_0 = 2) && (16) \\ &= \frac{1}{6} \cdot \frac{1}{6} \cdot 1 = \frac{1}{36} && (17) \end{aligned}$$

where $\Pr(X_\tau = i) = \pi_i(\tau)$, $\tau \geqslant 0$, $\forall\ i$. From this result it is also evident that this process is not independent because

$$\Pr(X_0 = 2, X_1 = 6, X_2 = 6) \neq \underbrace{\Pr(X_0 = 2)}_{\frac{1}{6}} \cdot \underbrace{\Pr(X_1 = 6)}_{\frac{1}{6}\left(1+\frac{4}{6}\right)=\frac{5}{18}} \cdot \underbrace{\Pr(X_2 = 6)}_{\frac{5}{18}+\frac{1}{6}\left(\frac{4}{6}\right)^2=\frac{19}{54}} .$$

Similarly we have that

$$\Pr(X_0 = 1, X_1 = 6, X_2 = 6) = \Pr(X_0 = 1) \cdot \Pr(X_1 = 6 | X_0 = 1) \cdot \Pr(X_1 = 6 | X_0 = 6) =$$

$$= \frac{1}{6} \cdot 0 \cdot 1 = 0.$$

Summarizing the process is weak-stationary because it always admits a stationary distribution. However it is neither independent nor ergodic becasue their limiting statiorary distributions are not identical. Finally it results that process is clearly memoryless, and thus markovian.

Solution of Exercise 5. The sample-space of $\{X_n, n = 0, 1, 2, \dots\}$ is

$$S_X = \{0, \pm 1, \pm 2, \dots, \pm\infty\}$$

whereas the sample space of the generic

$$S_{X_n} = \{0, \pm 1, \pm 2, \dots, \pm n\}.$$

Since X_n does not met the conditions of counting process, e.g., it could take negative values, then the random walk $\{X_n, \tau \in \mathbb{N}\}$ is not a counting process.

Let us now provides a possible realization for the random walk, e.g.,

n	0	1	2	3	4	5	6	7	8	9	10
X_n	0	1	0	-1	0	1	2	1	2	3	2

Let us now define $\pi_k(n) = \Pr(X_n = k)$, where k is the actual value of the random walk at the n-th time instant. Then, we can note that

$$\begin{aligned}
X_1 &= X_0 + Z_1 = Z_1 \\
X_2 &= X_0 + Z_1 + Z_2 = X_1 + Z_2 \\
X_3 &= X_0 + Z_1 + Z_2 + Z_3 = X_2 + Z_3 \\
\vdots & \quad \vdots \quad \vdots \\
X_n &= X_{n-1} + Z_n, \quad \forall\ n = 1, 2, \dots
\end{aligned}$$

it results that,

$$\begin{aligned}
\pi_k(n) &= \Pr(Z_n = 1 | X_{n-1} = k-1) \cdot \Pr(Z_{n-1} = k-1) + \\
&\quad \Pr(Z_n = -1 | X_{n-1} = k+1) \cdot \Pr(X_{n-1} = k+1) \\
&= p \cdot \pi_{k-1}(n-1) + (1-p) \cdot \pi_{k+1}(n-1)
\end{aligned}$$

This proves that the random walk is a Markovian process, since the probability law governing the changes at a given time instant n depends only to the state at the previous time $n-1$, and not by whole probability trajectory.

As known, the mean of sum of random variables is always the sum of their means, thus

$$\mathsf{E}[X_n] = \mathsf{E}\left[X_0 + \sum_{k=1}^{n} Z_k\right] = \mathsf{E}[X_0] + \sum_{k=1}^{n} \mathsf{E}[Z_k]$$

Moreover because of all the Z_k are i.i.d it further results that

$$\mathsf{E}[Z_k] = 1 \cdot p + (-1) \cdot (1-p) = p - 1 + p = 2p - 1 = 2 \cdot 0.5 - 1 = 0,$$

then

$$\mathsf{E}[X_n] = \mathsf{E}[X_0] + n \cdot \mathsf{E}[Z_k] = \mathsf{E}[X_0] + n \cdot (2p-1) = \mathsf{E}[X_0]$$

where, because of X_0 is known, it results

$$\mathsf{E}\left[X_0\right] = (0) \cdot \Pr(X_0 = 0) = 0 \cdot 1 = 0$$

Moreover, although in general the variance of sum of random variables is not always equal to the sum of their variances. Since our random variable are independent and indentical, then

$$\mathsf{Var}[X_n] = \mathsf{Var}\left[X_0 + \sum_{i=1}^{n} Z_n\right] = \mathsf{Var}[X_0] + \sum_{i=1}^{n} \mathsf{Var}[Z_n] = \mathsf{Var}[X_0] + n\mathsf{Var}[Z_n]$$

where, obliviously $\mathsf{Var}[X_0] = 0$, because it is known, in fact

$$\mathsf{Var}[X_0] = \mathsf{E}[X_0^2] - \mathsf{E}[X_0]^2 = (0)^2 \cdot 1 - 0^2 = 0$$

whereas

$$\mathsf{Var}[Z_n] = \mathsf{E}[Z_n^2] - \mathsf{E}[Z_n]^2 = \mathsf{E}[Z_n^2] = \mathsf{E}[Z_n^2] = 1^2 \cdot p + (-1)^2 \cdot (1-p) = 1$$

Thus it results that

$$\mathsf{Var}[X_n] = 0 + n \cdot 1 = n.$$

This results could also obtained by invoking the Central Limit Theorem, which states that the sum of n independent and identically distributed random variable with common mean m and variance σ^2 has approximately a Gaussian distribution with mean $n \cdot m$ and variance $n \cdot \sigma^2$ regardless of the distribution of these variables.

Finally, we conclude this exercise by noticing that this process is not wide-sense stationary, because of although its mean is finite, namely

$$\mathsf{E}\left[X_n\right] = \mathsf{E}\left[X_0\right] = x_0 = 0 < \infty \quad \forall\, n$$

its second-order moment and thus also its variance goes to infinity as $n \to \infty$, namely

$$\mathsf{Var}[X_\infty] = 0 + \infty \cdot 1 = +\infty.$$

Notice also that since the process is not weak stationary, it cannot be ergodic, because of the weak stationarity is a prerequiste for the ergodicity.

Let us further note that if the considered random walk would have, in place of an unbounded sample-space, a bounded sample-space such that, for instance, $S_X = \{0, \pm 1, \pm 2 \ldots, \pm k\}$, and where

$$\Pr(X_n = x_n | X_{n-1}) = x_{n-1}) = 0.5 \quad \forall\, x_{n-1} \neq \pm k$$

and

$$\Pr(X_n = k | X_{n-1}) = k) = 0.5 \quad , \quad \Pr(X_n = k-1 | X_{n-1}) = k) = 0.5$$

$$\Pr(X_n = -k | X_{n-1}) = -k) = 0.5 \quad , \quad \Pr(X_n = -(k-1) | X_{n-1}) = -k) = 0.5$$

then the resulting process would be stationary and also ergodic. To understand this fact, try to model this random walk as a finite state DT-MC. Specifically, by means of this analysis one can further note that its unique limiting distribution, is also uniform, and such that

$$\pi_{\ell,i} = \Pr(X_\infty = i) = \frac{1}{2k+1} \quad \forall\ -k \leqslant i \leqslant k$$

whereas

$$\mathsf{Var}[X_\infty] = \lim_{n\to\infty} \mathsf{Var}[X_n] = \lim_{n\to\infty} (\mathsf{E}[X_n^2] - \mathsf{E}[X_n]^2)$$

whereas

$$\mathsf{E}\left[X_\infty\right] = \sum_{i=-k}^{k} i \cdot \frac{1}{2k+1} = 0$$

$$\mathsf{E}[X_n^2] = \sum_{i=-k}^{k} i^2 \cdot \pi_{\ell,i} = \pi_{\ell,i} \cdot \sum_{i=-k}^{k} i^2 = \frac{2}{2k+1} \cdot \sum_{i=1}^{k} i^2 =$$

$$= \frac{2}{2k+1} \cdot \frac{k(k+1)(2k+1)}{6} = \frac{k(k+1)}{3}$$

Solution of Exercise 6. The MatLab code that solves this assignment is the following:

```
clear all, clc, close all;

n=1e6;                          % the length is arbitrary
lambda=40;                      % Inter-even time of a Z~Pois(40 arrivals/sec)
x=rand(n,1);                    % Observation x of X~Uniform(0,1)
mu_x=(1/n)*sum(x);              % Sampled mean of X (approx. E[X]=1/2)
y = -lambda^-1*log(1-x);        % Observation y of Y~Exp(lambda)
mu_y=(1/n)*sum(y);              % Sampled mean of Y (approx. E[Y]=1/lambda)

Pr_y_smaller_then_mu=(1/n)*sum(y<=mu_y);    % clearly equal to 1-exp(-1)
Pr_y_greater_then_mu=(1/n)*sum(y>=mu_y);    % clearly equal to exp(-1)
Pr_y_smaller_then_mu+Pr_y_greater_then_mu;  % clearly equal to 1

%% Point a)
% The y2E~Erlang(2,lambda) is a random variable that it may represent the
% inter-event time needed for having exactly 2 points on a Poisson process.
% Thus, because y~Exp(lambda) is the inter-event time of a Poisson point,
% then yE2 can be generated from
%
%   y = [y(1), y(2), y(3), y(4), y(5),...] = [dt1, dt2, dt3, dt4, dt5,...]
%
%   yE2 = [y(1)+y(2), y(3)+y(4), ...] = [dt1+dt2, dt3+dt4,...]
%
% Thus one has that:
```

```
yE2=zeros(floor(n/2),1);
for i=1:1:floor(n/2)
yE2(i)=sum(y(i*2-1:i*2));
end

% Sampled mean of an Erlang(2,40) distribution (approx. E[y]=2/lambda)
mu_yE2=(1/floor(n/2))*sum(yE2)
figure(1)
histogram(yE2,'Normalization','probability');
ylabel('f_{yE2}')
xlabel('t (sec)')
set(gca,'FontName','times','FontSize',18)

%% Point b)
% Vector
%
%   y = [y(1), y(2), y(3), y(4), y(5),...] = [dt1, dt2, dt3, dt4, dt5,...]
%
% contain a collection of interevent times. From y we can generate
% the time axis for generating a poisson process, as follows
%
% t=[0 y(1) y(1)+y(2), y(1)+y(2)+y(3),...]=[0 dt1 dt1+dt2, dt1+dt2+dt3,...]
%
t=zeros(n,1);
for i=2:n
t(i)=t(i-1)+y(i-1);                    % Time axis
end
% Generates an observation z of Z~Poisson(lambda*dt) every dt, with dt=y
z= poissrnd(lambda*y);

% Let us now sample our distribution every second in order to check its
% mean corresponds to lambda and if its shape looks like a bell, centered
% at k=40.

k=1:floor(t(end));  % Number of samples necessary to sample "t" @1sec
index_t=zeros(k(end),1); % indexes of with t sampled "t" @1sec
z_sampled_at_1sec=[];
for i=1:k(end)
index_t(i)=find(t>k(i),1);
if i==1
z_sampled_at_1sec=[z_sampled_at_1sec; sum(z(1:index_t(i)-1))];
else
z_sampled_at_1sec=[z_sampled_at_1sec; sum(z(index_t(i-1):index_t(i)-1))];
end
```

```
end

% Sampled mean of a poisson distribution (approx. lambda)
mu_z_sampled_at_1sec=(1/k(end))*sum(z_sampled_at_1sec)

figure(2)
title('Probability density functions')
subplot(311)
histogram(x,'Normalization','probability');
ylabel('f_x')
subplot(312)
histogram(y,'Normalization','probability');
ylabel('f_y')
xlabel('t (sec)')
subplot(313)
histogram(z_sampled_at_1sec,'Normalization','probability');
hold on
plot(poisspdf(0:66,lambda),'r')
ylabel('f_z sampled @1sec')
xlabel('k (arrivals)')

figure(3)
title('Poisson stochastic process {Zt, t}')
stairs(t, z,'b')
hold on
stairs(t(index_t), z_sampled_at_1sec,'r')
legend('Poisson process','Poisson process averaged @1sec')
xlim([0, 20])
xlabel('t (sec)')
```

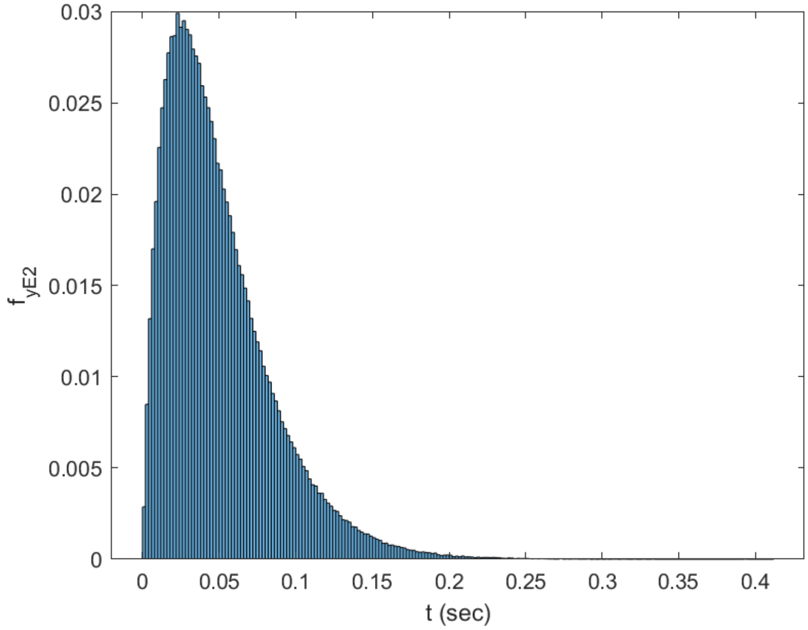

Figure 1: Probability density functions of $Erlang(2, 40)$.

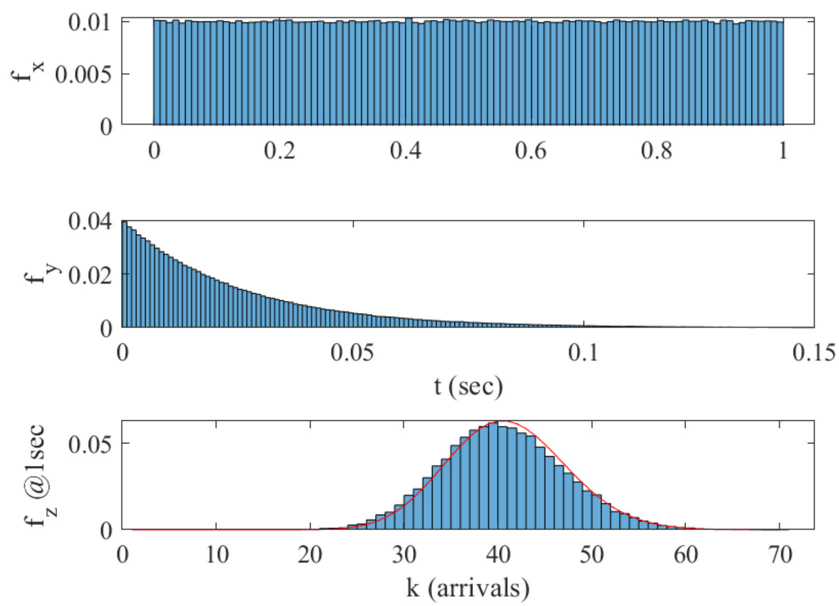

Figure 2: Probability density functions of $X \sim Uniform(0, 1)$ and $Y \sim Exp(\lambda)$, and probability mass function of $Z_\tau = Pois(\lambda)$, where $\tau = 1, 2, 3, \ldots,$ seconds, namely, $Z_\tau = \sum_{\forall t \in [\tau-1, \tau)} X_t$.

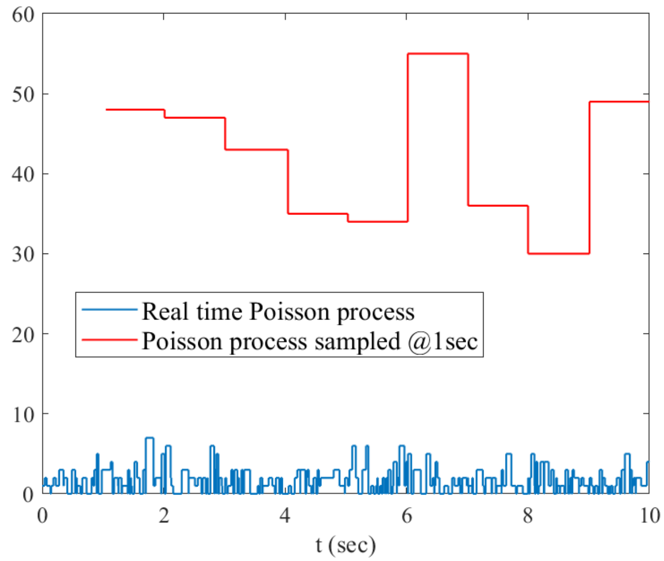

Figure 3: In blue it shown the real-time poisson point process, namely, the Poisson process where the arrivals are scheduled by the $Y \sim Exp(\lambda)$ used for modelling the inter-event times, namely, one has that $Z_t = Pois(\lambda Y)$. In red the same Poisson process, sampled every 1 second, namely, let $\tau = 1, 2, 3, \ldots,$ seconds, $Z_\tau = \sum_{\forall t \in [\tau-1,\tau)} X_t$.