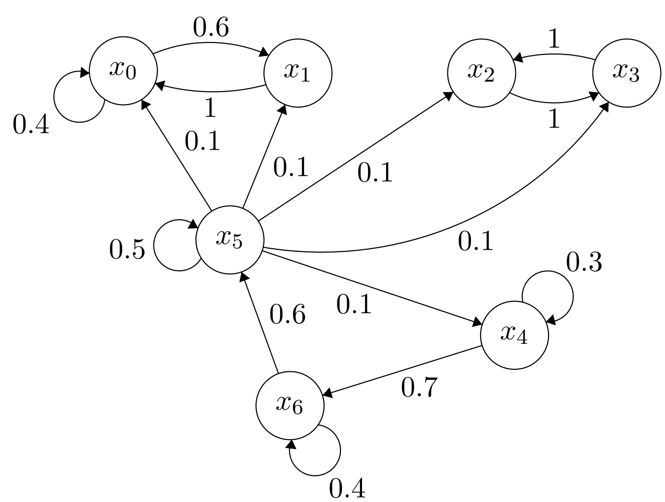
Stochastic Models

A.A. 2024/2025

Assignment 4

Exercise 1. Consider the Discrete-Time Markov Chain (DT-MC) depicted in the next figure.



(a) Evaluate the strongly connected components. Which are ergodics and which transients?

(b) Evaluate the periodicity of the ergodic components.

(c) Evaluate if the given DT-MC is ergodic and explain what this property implies.

(d) For each absorbing component, determine its absorption probabilities.

(e) Evaluate if by changing the some arcs weights (and without adding new arcs), the conclusion drawn at point (c) would change.

(f) If the response to (e) is negative, evaluate the possibility to make the MC graph corresponding to that of an ergodic DT-MC by changing the orientation of a single arc while adjusting the arc weights accordingly.

Exercise 2. The marital status for a man/woman in his 30s is considered using the following Markov process $\{x(\tau), \tau = 30, 31, \ldots\}$. The state space of the process consists of three states: Single ($x(\tau) = x_0$), Cohabiting ($x(\tau) = x_1$), Married ($x(\tau) = x_2$).

The transition matrix from year to year has the following form:

$$\boldsymbol{P} = \begin{pmatrix} 0.6 & 0.35 & 0.05 \\ 0.20 & 0.70 & 0.10 \\ 0.10 & 0.05 & 0.85 \end{pmatrix}$$

Draw the transition graph for this Markov chain and then answer the following questions:

(a) Determine the edge set $E \subset X \times X$ of the graph associated with the given DT-MC.

(b) Calculate the probability that a 35-year-old married man/woman gets divorced next year, and marries again the following year.

(c) Given a 30-year-old man is single this year, calculate the the probability that he will still be single in two years time.

(d) Determine the mean recurrence time of state x_2, namely

$$\mathsf{E}[\min\{k \geqslant 1 \ : \ x(k) = x_2 | x(0) = x_2\}]$$

and the mean recurrence time of state x_2 conditioned to the fact that $x(1) \neq x_2$, i.e.,

$$\mathsf{E}[\min\{k \geqslant 1 \ : \ x(k) = x_2 | x(1) \neq x_2, x(0) = x_2\}]$$

Then, provide an interpretation of the obtained results.

(e) Do you think this is a good model for the marital status of a man? What is good/bad?

Exercise 3. The production line of a manufacturing system consists of single machine with two servers working in parallel. At each time unit T, a workpiece arrives in accordance with a Bernoulli process with parameter $p = 0.8$.

If one server at time T is busy, it will complete its task within the next time interval with probability $q = 0.9$. The machine has no waiting lines, thus if the two servers are busy a new arrival is discharged.

(a) Model the manufacturing system status as a DT-MC. Drawn its transition graph and calculate its transition probability matrix (express probabilities as functions of p and q).

(b) Is the resulting DT-MC ergodic? Specify what the ergodicity property implies in practice.

(c) Calculate the stationary probability distribution.

(d) Calculate the long term probabilities that at the next time step: a) an incoming piece would be discarded (unconditioned by an arrival event); b) an arrived piece is discharged.

Exercise 4. Consider a service station consisting of a robot arm and a pre-load buffer with capacity of 1 piece. A transfer line, at each time cycle $\Delta\tau = 10$ seconds, introduces pieces with probability p. The piece is introduced into the pre-load buffer if only and only if the robot is free, otherwise the piece is discharged. The robot is able to complete its task in a time cycle. This operation may fail with probability q. If thus, it is needed another cycle time to repeat the task. This operation may fail again with the same probability.

(a) Model the occupancy of the service station as a Discrete-Time Markov Chain (DT-MC) and draw its transition graph. Then, calculate its transition probability matrix.

(b) Let $p = 0.9$ and $q = 0.2$, evaluate if this DT-MC is ergodic.

(c) Calculate the marginal stationary distribution of this stochastic process

(d) Determine the mean rate of correctly elaborated pieces at the steady state λ_e in pieces/minute.

(e) Determine the percentage of pieces discharged at the steady state η_d. Then derive the resulting discharging rate λ_d in pieces/minute.

(f) Determine the mean number of pieces in the system in the steady state.

(g) Under the assumption the robot may fail its task at most once. Model the status of the system by exploiting this additional information.

Solutions

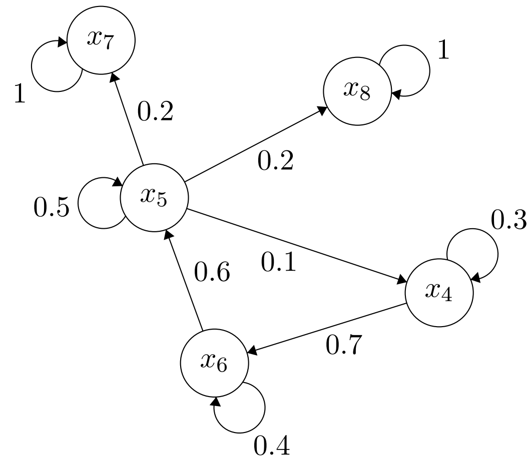


Figure 1: Rearrenged DT-MC for the calculus of the absorption probabilities.

Solution of Exercise 1. Remanding the following defintions:

A directed graphs is said to be strongly connected if every vertex is reachable (i.e., exists a sequence of arcs) from every other vertex.

The Strongly Connected Components (SCCs)of a directed graph forms a partition into subgraphs that are themselves strongly connected.

It follows that the SCCs associated with the given DT-MC are, resp.,

$$\begin{aligned} S_1 &= \{x_0, x_1\} \\ S_2 &= \{x_2, x_3\} \\ S_3 &= \{x_4, x_5, x_6\} \end{aligned}$$

Moreover, since a SCC is said to be "transitory" if, once leaved, it is not possible reach it again by following any path. Whereas it is "ergodic" or "absorbing" if, once reached, there are no paths enabling the possibility to leave it, then S_1 and S_2 are ergodic, whereas S_3 is transient.

Since, S_1 has at least a self it follows that it is aperiodic. On the other hand S_2 is periodic with period $d = 2$.

The given DT-MC is not ergodic because it has more then one ergodic SCCs. The ergodicity is a property of the MC which implies that its limiting distribution is unique, and does not depend on the particular initial distribution $\pi(0) = [\pi_0, \pi_1, \pi_2, \ldots]$. Thus it results that to understand the behaviour of an ergodic MC chain, we can simply study one their realizations. However, this is not our case.

Let us know compute the absorption probabilities of each ergodic component. To do that we follows its standard procedure, thus rearrange the DT-MC by replacing to S_1 and S_2 with two absorbing states as in Figure 1.

Then by letting

$$a_i = \Pr(\exists k \geqslant 0 \ : \ x(k) = x_7 | x(0) = i)$$

then, the next linear system is derived:

$$\begin{cases} a_7 &=& 1 \\ a_8 &=& 0 \\ a_6 &=& 0.6a_5 + 0.4a_6 \\ a_5 &=& 0.5a_5 + 0.2a_7 + 0.2a_8 + 0.1a_4 \\ a_4 &=& 0.3a_4 + 0.7a_6 \end{cases} \rightarrow \begin{cases} a_7 &=& 1 \\ a_8 &=& 0 \\ a_6 &=& a_5 \\ a_5 &=& 0.5a_5 + 0.2 + 0.1a_4 \\ a_4 &=& 0.3a_4 + 0.7a_6 \end{cases} \rightarrow$$

$$\rightarrow \begin{cases} a_7 &=& 1 \\ a_8 &=& 0 \\ a_6 &=& a_5 \\ a_5 &=& 0.5a_5 + 0.2 + 0.1a_5 \\ a_4 &=& a_6 \end{cases} \begin{cases} a_7 &=& 1 \\ a_8 &=& 0 \\ a_6 &=& 0.4 \\ a_5 &=& 0.2/0.4 = 0.5 \\ a_4 &=& a_5 \end{cases} \rightarrow \begin{cases} a_7 &=& 1 \\ a_8 &=& 0 \\ a_6 &=& 0.5 \\ a_5 &=& 0.5 \\ a_4 &=& 0.5 \end{cases}$$

Finally let

$$b_i = \Pr(\exists k \geqslant 0 \ : \ x(k) = x_8 | x(0) = i)$$

and because of the absorption of the SCC x_8 is a mutually exclusive event with respect to the absorption of x_7, then it yields that

$$\begin{cases} b_7 &=& 1 - a_7 = 0 \\ b_8 &=& 1 - a_8 = 1 \\ b_6 &=& 1 - a_6 = 0.5 \\ b_5 &=& 1 - a_5 = 0.5 \\ b_4 &=& 1 - a_4 = 0.5 \end{cases} \tag{1}$$

Note that the fact that $a_5 = b_5 = 0.5$ is fairly reasonable because of to jump to x_7 or x_8, we need to pass through x_5, but there $p_{57} = p_{58} = 0.2$.

Regarding the possibility to make this process ergodic by changing the weight of 1 edge. As known there is no possibility to make the given DT-MC ergodic by the change of the weight of an existing arc. This is because ergodicity does not depend by the MC's transition probabilities arcs, but only to their orientation.

On the other hand, if we change the orientation of either $x_5 \rightarrow x_2$ to $x_2 \rightarrow x_5$ or $x_5 \rightarrow x_3$ to $x_3 \rightarrow x_5$ then, the resulting DT-MC becomes ergodic because it now has a single ergodic, aperiodic component denoted by $S_1 = \{x_0, x_1\}$. Note that the change of the orientation of either $x_5 \rightarrow x_0$ or $x_5 \rightarrow x_1$ do not make the DT-MC ergodic because S_2 is periodic, and thus its stationary distribution would depend on the given initial marginal distribution.

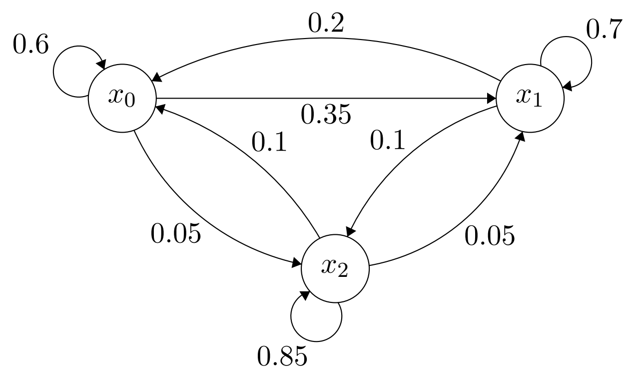


Figure 2: Transition probability graph of the DT-MC in Exercise 2.

Solution of Exercise 2. The transition graph associated with this DT-MC is in Figure 2. Then its edges set is

$$E = \{(x_0, x_0), (x_0, x_1), (x_0, x_3), (x_1, x_0), (x_1, x_1), (x_1, x_2, (x_2, x_0), (x_2, x_1), (x_2, x_2)\}$$

The probability that a man which status is "married" at 35 gets divorsed at 36 and then remarries at 37 years old, is equivalent to the following conditional probability

$$\Pr(X_{37} = x_2, X_{36} \neq x_2 | X_{35} = x_2).$$

By means of Total probability Law, and by inspecting the above transition graph we can note that

$$\begin{aligned} \Pr(X_{37} = x_2, X_{36} \neq x_2 | X_{35} = x_2) &= \\ &= \Pr(X_{37} = x_2 | X_{36} = x_0)\Pr(X_{36} = x_0 | X_{35} = x_2) \\ &\quad +\Pr(X_{37} = x_2 | X_{36} = x_1)\Pr(X_{36} = x_1 | X_{35} = x_2) \\ &= p_{20} \cdot p_{02} + p_{21} \cdot p_{12} = 0.1 \cdot 0.05 + 0.05 \cdot 0.1 = 0.01 \end{aligned}$$

Notice that this probability, differs to the probability that a man which status was "married" at 35 will remain married with same person at 37, that is instead

$$\begin{aligned} \Pr(X_{37} = x_2, X_{36} = x_2 | X_{35} = x_2) &= \Pr(X_{37} = x_2 | X_{36} = x_2)\Pr(X_{36} = x_2 | X_{35} = x_2) \\ &= p_{22} \cdot p_{22} = 0.85 \cdot 0.85 = 0.7225 \end{aligned}$$

Moreover, notice that the above two probabilities differs also to the probability that a man which status is "married" at 35 will be married at 37 that is instead the "2-step return probability" to state x_2 that is

$$p_{22}^{(2)} = \Pr(X_{37} = x_2 \mid X_{35} = x_2) = 0.01 + 0.7225 = 0.7325$$

The probability that a single 30-year old man will still be single in two years, by the total probability law is :

$$
\begin{aligned}
p_{00}^{(2)} = \Pr(X_{32} = x_0 | X_{30} = x_0) &= \Pr(X_{32} = x_0 | X_{31} = x_0)\Pr(X_{31} = x_0 | X_{30} = x_0) + \\
&\quad + \Pr(X_{32} = x_0 | X_{31} = x_1)\Pr(X_{31} = x_1 | X_{30} = x_0) + \\
&\quad + \Pr(X_{32} = x_0 | X_{31} = x_2)\Pr(X_{31} = x_2 | X_{30} = x_0) \\
&= 0.6 \cdot 0.6 + 0.35 \cdot 0.2 + 0.05 \cdot 0.1 = 0.4350
\end{aligned}
$$

This quantity could be easily evaluated by means of the N-steps Transition Probability, thus by exploiting the Chapman Kolmogorov Relation as follows

$$
\boldsymbol{P}^2 = \begin{pmatrix} \Pr(X_{m+2} = x_0 | X_m = x_0) & \Pr(X_{m+2} = x_1 | X_m = x_0) & \Pr(X_{m+2} = x_2 | X_m = x_0) \\ \Pr(X_{m+2} = x_0 | X_m = x_1) & \Pr(X_{m+2} = x_1 | X_m = x_1) & \Pr(X_{m+2} = x_2 | X_m = x_1) \\ \Pr(X_{m+2} = x_0 | X_m = x_2) & \Pr(X_{m+2} = x_1 | X_m = x_2) & \Pr(X_{m+2} = x_2 | X_m = x_2) \end{pmatrix} =
$$

$$
= \begin{pmatrix} 0.4350 & 0.4575 & 0.1075 \\ 0.2700 & 0.5650 & 0.1650 \\ 0.1550 & 0.1125 & 0.7325 \end{pmatrix}
$$

In accordance with its definition, the mean recurrence time of state x_2

$$
t_2^* = \mathsf{E}[\min\{k \geqslant 1 \ : \ x(k) = x_2 | x(0) = x_2\}]
$$

we are interested in the mean number of years $k \geqslant 1$ to be in the state married given that at time $k = 0$ we were in that state. As we know t_2^* can be evaluated by means of the well-known formula of the *recurrence time*, namely,

$$
t_2^* = 1 + \sum_{j \neq 2} p_{ij} \cdot t_j
$$

where

$$
t_j = \mathsf{E}[\min\{k \geqslant 0 \ : \ x(k) = x_2 | x(0) = i\}]
$$

denotes the mean first hitting time to x_2, namely the mean of the minimum time to jump into x_2, given that $X_0 = j$.

Thus, as a first step we need to rearrange our DT-MC in a such a way x_2 becomes an absorbing component as in Figure 3, then solve following linear system, computed in accordance with the procedure used for the calculus of the mean absorbing times of an absorbing state, i.e.,

$$\left\{\begin{array}{lcl} t_2 & = & 0 \\ t_0 & = & 1+0.6t_0+0.35t_1+0.05t_2 \\ t_1 & = & 1+0.2t_0+0.7t_1+0.1t_2 \end{array}\right. \rightarrow \left\{\begin{array}{lcl} t_2 & = & 0 \\ t_0 & = & 1+0.6t_0+0.35t_1 \\ t_1 & = & 1+0.2t_0+0.7t_1 \end{array}\right.$$

$$\rightarrow \left\{\begin{array}{lcl} t_2 & = & 0 \\ 0.4t_0 & = & 1+0.35t_1 \\ 0.3t_1 & = & 1+0.2t_0 \end{array}\right. \rightarrow \left\{\begin{array}{lcl} t_2 & = & 0 \\ t_0 & = & 2.5+0.875t_1 \\ 0.3t_1 & = & 1+0.2t_0 \end{array}\right.$$

$$\rightarrow \left\{\begin{array}{lcl} t_2 & = & 0 \\ t_0 & = & 2.5+0.875t_1 \\ 0.3t_1 & = & 1+0.2(2.5+0.875t_1) \end{array}\right. \rightarrow \left\{\begin{array}{lcl} t_2 & = & 0 \\ t_0 & = & 2.5+0.875t_1 \\ 0.3t_1 & = & 1+0.5+0.175t_1 \end{array}\right.$$

$$\rightarrow \left\{\begin{array}{lcl} t_2 & = & 0 \\ t_0 & = & 2.5+0.875t_1 \\ t_1 & = & \frac{1.5}{0.125}=12 \end{array}\right. \rightarrow \left\{\begin{array}{lcl} t_2 & = & 0 \\ t_0 & = & 13 \\ t_1 & = & 12 \end{array}\right.$$

This result means that on average we need to wait 13 years to get married if we were single at time $k=0$, and 12 if we were co-habitation at time $k=0$.

From that it finally results that

$$t_2^* = 1 + p_{20}t_0 + p_{21}t_1 + p_{22}t_2 = 1 + 0.1 \cdot 13 + 0.05 \cdot 12 + 0.85 \cdot 0 = 2.9\text{years}$$

This means that if we where married at given time, on average 2.9 year later almost three years later will be married as well either because we have same wive or because we get divorced and we found a new wife.

Let us now compute the mean recurrence time of state x_2 conditioned by the fact that we are sure that at time $k=1$ we will have divorced, i.e, $x(1) \neq x_2$. More formally we are interested in

$$t_2^{**} = \mathsf{E}[T] = \mathsf{E}[\overbrace{\min\{k \geqslant 1 \ : \ x(k) = x_2 | x(1) \neq x_2, x(0) = x_2\}}^{T}], \quad T \in \{1,2,3,\dots\}$$

Due to the Markov property it holds that

$$t_2^{**} = 1 + \mathsf{E}[T'] = 1 + \mathsf{E}[\overbrace{\min\{k \geqslant 0 \ : \ x(k) = x_2 | x(1) \neq x_2\}}^{T'}], \quad T' \in \{0,1,2,3,\dots\}$$

Let us further note that such a quantity differs to t_2^* because of in the computation of t_2^* we could stay in x_2 at time $k=1$, whereas in t_2^{**} such event is not allowed.

Now by invoking the definition of expectation, and the Law of Total probability one obtains that

$$
\begin{aligned}
\mathsf{E}[T'] = \sum_{k=0}^{\infty} k \cdot \overbrace{\sum_{x_j \in \{x_0, x_1\}} \Pr(T' = k | X_1 = x_j) \cdot \Pr(X_1 = x_j)}^{\Pr(T'=k)\,:\,\text{Law of Total Probability}} \\
= \overbrace{\sum_{x_j \in \{x_0, x_1\}} \mathsf{E}[T' | X_1 = x_j] \cdot \Pr(X_1 = x_j)}^{\text{Law of Total Expectation}} \\
= \mathsf{E}[T' | x(1) = x_0] \cdot \Pr(x(1) = x_0) + \mathsf{E}[T' | x(1) = x_1] \cdot \Pr(x(1) = x_1) \\
= t_0 \cdot \Pr(x(1) = x_0) + t_1 \cdot \Pr(x(1) = x_0)
\end{aligned}
$$

Let us now compute $\Pr(x(1) = x_j)$ for $j = 0, 1$. This probability is conditioned by the fact that at time $k = 1$ we have left x_2. Thus, on the basis of such prior information, and by means of the Bayes Theorem we can find that

$$
\begin{aligned}
\Pr(x(1) = x_1) = & \Pr(x(1) = x_1 | x(1) \in \{x_0, x_1\}, x(0) = x_2) \\
= & \frac{\Pr(x(1) \in \{x_0, x_1\} | x(1) = x_1) \cdot \Pr(x(1) = x_1 | x(0) = x_2)}{\Pr(x(1) \in \{x_0, x_1\} | x(0) = x_2)} \\
= & \frac{1 \cdot p_{21}}{p_{20} + p_{21}} = \frac{0.05}{0.1 + 0.05} = 0.333
\end{aligned}
$$

$$
\Pr(x(1) = x_0) = 1 - \Pr(x(1) = x_1) = 0.667
$$

From that we straight forwardly derive that

$$
t_2^{**} = 1 + \mathsf{E}[T'] = 1 + 0.333 \cdot t_1 + 0.667 \cdot t_0 = 13.667
$$

Notice that this result, namely $t_2^* \ll t_2^{**}$, is clearly reasonable because of in t_2^*, the term $t_2 = 0$ is weighted 0.85, whereas $t_1 = 12$ and $t_0 = 13$ are weighted with very small probabilities.

Finally, regarding the last point of this exercise, we can note that the use of only three states is dubious. Indeed, the use of only three states implies that, due to the Markov Property that a man who has cohabited with his partner for many years as the same likely to get married than a man who has cohabited for only one year. A better modelizzation would, for instance, consider k more co-habiting states x_{1j}, with $j = 1, 2, \ldots, k$, one for year the man is co-habiting, which transition probabilities into x_2 from x_{1j} such that

$$
\Pr(x(k) = x_2 | x(k-1) = x_{1j}) > \Pr(x(k) = x_2 | x(k-1) = x_{1j-1}) \quad \forall\, j = 1, 2, \ldots, k.
$$

```
%% Simulation for this process
clc, clear all, close all
P=[0.6  0.35  0.05; 0.2 0.7  0.1; 0.1 0.05 0.85]
```

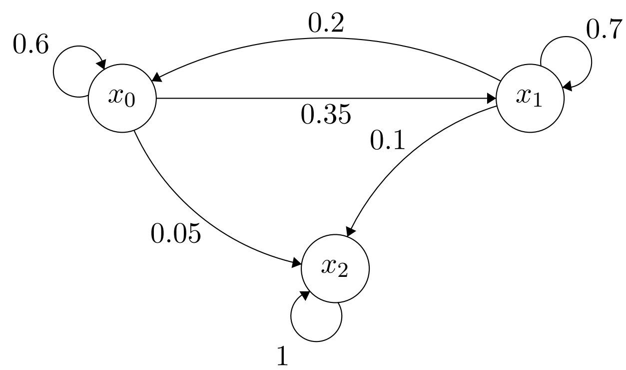


Figure 3: Rearrenged DT-MC for the calculus of the mean first passage of x_2 in Exercise 2. Because of in the calculus of the mean first passage, i.e. $t_i = \mathsf{E}[\min\{k \geqslant 0 \,:\, x(k) = x_2 | x(0) = x_i\}]$ with do not care where the chain will move after passing through x_2, then we can simply set to 0 the outgoing probabilities of x_2 with a selfloop with of probability 1.

```
mc=dtmc(P)
ht = hittime(mc,3)
pi0=[1 0 0]
% Limiting distribution
n=10000
pi0*P^n
figure
graphplot(mc,'ColorNodes',true);

% Simulation of 9 realization of the process
X=simulate(mc,n,'X0', 3*ones(1,3))
prX1=sum((X(:,:)==1))/n
prX2=sum((X(:,:)==2))/n
prX3=sum((X(:,:)==3))/n
```

Solution of Exercise 3. Since the manufacturing process can operate at most two pieces at time and there is no waiting line, the stochastic process $\{X_\tau, \tau = T, 2T, \ldots\}$ describing the functioning of this process can be modelled by a three state DT-MC which takes into account the number of pieces in the system at the same time, namely, $X_\tau \in X = \{0, 1, 2\}$.

Because of both a new arrival and the service process are modelled as a Bernoulli trials with parameters p and q, and because of independence, the probability that the piece under works does not terminate its process while a new piece is arrived corresponds to

$$p_{12} = \Pr(X_\tau = 2 | X_{\tau-1} = 1) = p \cdot (1 - q).$$

On the other hand we have that

$$p_{10} = \Pr(X_\tau = 0 | X_{\tau-1} = 1) = q \cdot (1 - p)$$

which corresponds to the probability that there were only a piece in the process at time τ, its processing is terminated correctly while no further pieces are arrived.

Finally, by exclusion it results that

$$p_{11} = \Pr(X_\tau = x_1 | X_{\tau-1} = x_1) = 1 - p(1 - q) - q(1 - p) = pq + (1 - q)(1 - p),$$

which correspond to the probability that the current process is terminated correctly a new piece is arrived, or alternatively, the process is not terminated within T and no further pieces are arrived.

Similarly, it results that

$$p_{20} = \Pr(X_\tau = 0 | X_{\tau-1} = 2) = q^2(1 - p)$$

which corresponds to the probability that there were two pieces in the process at time τ, both the processes are terminated correctly while a no further pieces are arrived.

On the other hand we have that

$$p_{21} = \Pr(X_\tau = 1 | X_{\tau-1} = 2) = 2 \cdot (q \cdot (1 - q)(1 - p) \cdot q^2) + q^2 \cdot p$$

which corresponds to the probability that:

- there were two pieces in the process at time τ, only one process terminates correctly, the other not, and a no further pieces are arrived. This event is counted twice because it may correspond on either the first machine terminates correctly and the other not, or vice-versa
- there were two pieces in the process at time τ, both the processes terminates correctly within τ and a piece is arrived.

Similarly, it results that

$$p_{22} = \Pr(X_\tau = 2 | X_{\tau-1} = 2) = p(1-q)^2 + (1-p)(1-q)^2 + 2 \cdot pq(1-q) = (1-q)^2 + 2 \cdot pq(1-q),$$

which corresponds to the probability that both the processes do not terminate correctly independently on the fact that a new piece is arrived, or one of the two process is terminated correctly and a new piece is arrived, counted twice.

The transition graph associated with this DT-MC is drawn in Figure 4.

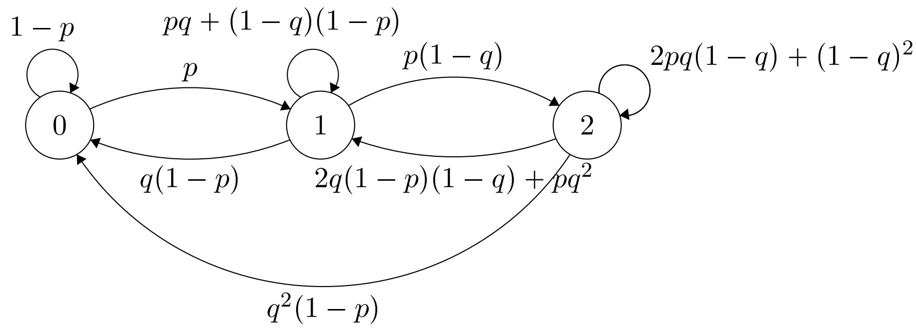


Figure 4: Transition probability graph of the DT-MC of Exercise 3.

It can be noted that the resulting graph is strongly connected because each state is reachable from another by an oriented path. It results that the DT-MC is irriducible. Moreover, because there are self-loops we can immediatly assert that the strongly connected component associated with this DT-MC is also aperiodic, and thus the DT-MC is ergodic. It follows that, for this process, we can estimate the long term marginal unconditional probability distributions, and other performance metrics, independently to the system's initial probability distribution.

Let us now evaluate the corresponding limiting probability distribution Π_ℓ. Let us first define the transition matrix of the DT-MC as follows

$$\boldsymbol{P} = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.18 & 0.74 & 0.08 \\ 0.162 & 0.684 & 0.154 \end{pmatrix}$$

Then, since the DT-MC is ergodic then Π_ℓ correspond to its stationary distribution Π_s. Thus, by solving the following linear system

$$\begin{cases} \Pi_s = \Pi_s \cdot \boldsymbol{P} \\ \Pi_s \cdot \mathbf{1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,0} = 0.2\pi_{s,0} + 0.18\pi_{s,1} + 0.162\pi_{s,2} \\ \pi_{s,1} = 0.8\pi_{s,0} + 0.74\pi_{s,1} + 0.684\pi_{s,2} \\ \pi_{s,2} = 0.08\pi_{s,1} + 0.154\pi_{s,2} \\ \pi_{s,0} + \pi_{s,1} + \pi_{s,2} = 1 \end{cases} \tag{2}$$

it results that

$$\Pi_\ell \equiv \Pi_s \approx (0.1824, 0.7470, 0.0706)\,.$$

Let us now calculate the probability that an incoming piece would be discarded. This probability is independent on the fact the would be arrived of not, thus it follows that

$$\Pr(\text{a piece would be discharged}) = (1-q)^2 \cdot \pi_2(\infty) = (1-q)^2 \cdot \pi_{2,\ell} = 0.1^2 \cdot 0.0706 = 0.0706.$$

On the other hand, the probability a piece arrives and then it is discharged is

$$\Pr(\text{an arrived piece is discharge}) = p(1-q)^2\pi_2(\infty) = 0.0706 \cdot 0.8 = 0.05648$$

Finally note that, most of the previous analysis can be done on MatLab. Here a list of instruction that can help the analysis.

```
clc, clear all, close all
% Define the transtion probability matrix
P=[0.2 0.8 0;0.18 0.74 0.08; 0.162 0.684 0.154]
% Cardinality of the sample space of the DT-MC
n=length(P)
% Evaluate the eigenvalues of P
eig(P)
% Simple algorithm to evaluate if the DT-MC is ergodic
sum( abs(eig(P))>1 )>1
% To evaluate the stationary distribution Pi_s numerically we can simply
% use the Chapman-Kolmogorov equation. In particular, let k_end be a
% suffienctly big number of iteration
k_end=100;
% and let the initial probability distribution of DT-MC be, for instance,
Pi0=1/n*ones(1,n)
% then
Pi_s=Pi0*P^k_end

% Moreover its also possible simulate the behaviour of a DT-MC as follow:
k_end=10;                   % Simulation lenght
pi_0=[1 0 0];               % Initial marginal probability
vec_pi=zeros(k_end+1,n);    % DT-MC evolution vector
vec_pi(1,:)=pi_0;           % Initialization of vec_pi @ k=0
for k=2:1:k_end+1
    vec_pi(k,:)=vec_pi(k-1,:)*P % DT-MC state update
end

% Notice that
round(vec_pi(k_end,:),5)==round(pi_0*P^(k_end),5)

figure(1)
subplot(311)
```

```
stairs(0:k_end,vec_pi(:,1))
ylim([0 1])
ylabel('$\pi_1(k)$','fontsize',14,'interpreter','latex')
set(gca,'FontSize',18,'FontName','times')
subplot(312)
stairs(0:k_end,vec_pi(:,2))
ylim([0 1])
ylabel('$\pi_2(k)$','fontsize',14,'interpreter','latex')
set(gca,'FontSize',18,'FontName','times')
subplot(313)
stairs(0:k_end,vec_pi(:,3))
ylim([0 1])
xlabel('# Iteration')
ylabel('$\pi_3(k)$','fontsize',14,'interpreter','latex')
set(gca,'FontSize',18,'FontName','times')
```

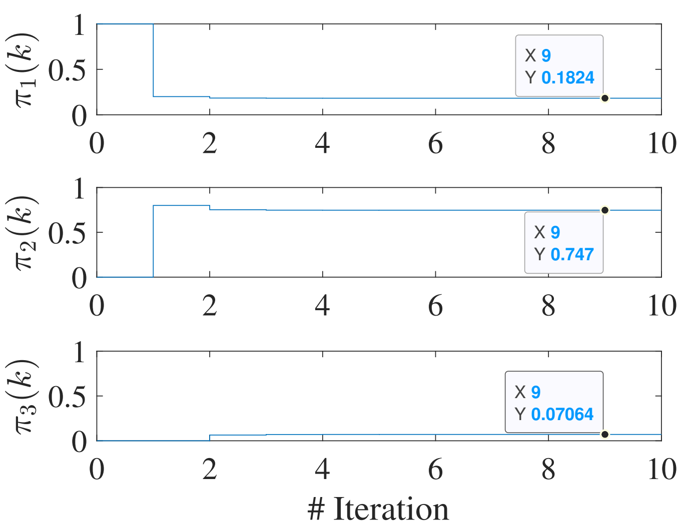


Figure 5: Output figure generated by the above MatLab script which show the temporal evolution of the marginal probabilies of the DT-MC. It is evident that, after only 10 iterations the system has reached its steady limiting distribution, namely $\Pi(\infty) \equiv \Pi_s = \Pi_\ell = (0.1824, 0.7470, 0.0706)$.

Let us further note that on MatLab exists also a dedicated tool-box which helps the analysis of DT-MC. Hereinafter some of their commands are listed.

```
clc, clear all, close all
% Define the transtion probability matrix
P=[0.2 0.8 0;0.18 0.74 0.08; 0.162 0.684 0.154]
% Cardinality of the sample space of the DT-MC
```

```
n=length(P)
mc=dtmc(P)
% Evaluate if it is "erogodic" or not
tf = isergodic(mc)
% Evaluate if it is "reducible" or not
tf = isreducible(mc)
% Draw its transition graph, while classifying the number and type of
% trongly connected components. The edge probabiliteis color correspond
% to the right legend
figure(1)
graphplot(mc,'ColorNodes',true,'ColorEdges',true);
% Draw the eigenvalues of P within the unit circle
figure(2)
eigplot(mc)
% By means of the command "simulate" we can generate n random walk of lenght
=1000
% each staring by one of the n states.
X = simulate(mc,k_end,'X0',ones(1,n))

estimate_prX1=sum((X(:,:)==1))/k_end
estimate_prX2=sum((X(:,:)==2))/k_end
estimate_prX3=sum((X(:,:)==3))/k_end
pi0=[1 0 0];
% limiting distribution by means of the chapman-kolgomorov equation
pi0*P^k_end
```

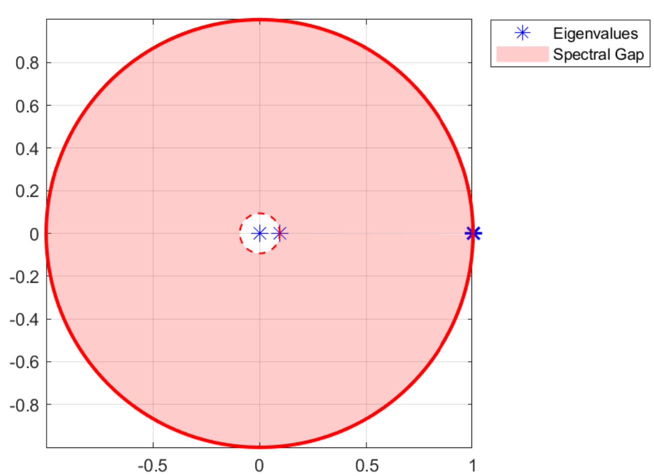


Figure 6: Output of the command “eigplot(mc)”.

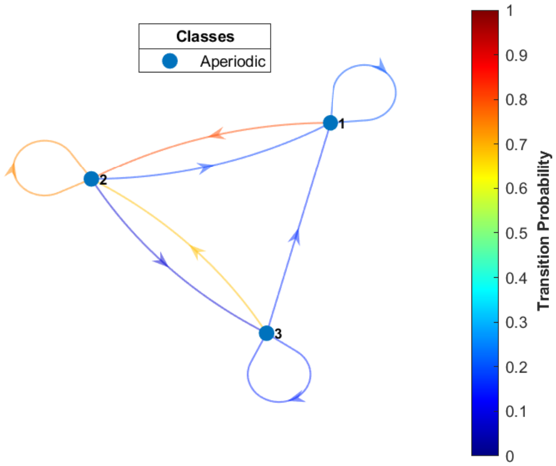
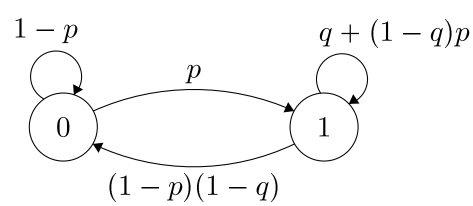


Figure 7: Output of the command “graphplot(mc,’ColorNodes’,true,’ColorEdges’,true)”.

Solution of Exercise 4. Since an incoming piece can be introduced into the buffer if and only if the robot is free, and that no more than a piece may stay within the system. It follows that this stochastic process $\{X_\tau, \tau = \Delta\tau, 2\Delta\tau, \dots\}$ associated with the buffer state may assume only two values, either 0, or 1, representing the number of pieces in the station, i.e., $X = \{0, 1\}$. Let us further note that the event of having an incoming piece from the transfer line, that has probability equal to p, and the event that the robot fails its task, that has probability q, are clearly independent. Let us further note events are scheduled at discrete-time instants of $T = 10$ seconds, and that the future events depends only by the actual state of the system. It follows that the buffer state can be modelled as a DT-MC.

The transition graph of the system is provided below whereas its transition probability matrix



is

$$\boldsymbol{P} = \begin{pmatrix} 1-p & p \\ (1-p)(1-q) & q+p(1-q) \end{pmatrix} = \begin{pmatrix} 0.1 & 0.9 \\ 0.08 & 0.92 \end{pmatrix}$$

Let us now evaluate the ergodicity of this DT-MC. For instance, by means of the Eigenvalue criteria we may found that

$$\det(s\boldsymbol{I} - \boldsymbol{P}) = \det \begin{pmatrix} s-0.1 & -0.9 \\ -0.08 & s-0.92 \end{pmatrix} = s^2 - \frac{50}{51}s + \frac{1}{50} = 0 \to \{s_1 = 1, s_2 = 0.02\}.$$

Since there is only one eigenvalue sastifying $|\lambda_i| = 1$, then the DT-MC is ergodic. Thus, the limiting distribution can be evaluated by solving the following linear system

$$\begin{cases} \Pi_s = \Pi_s \cdot \boldsymbol{P} \\ \Pi_s \cdot \mathbf{1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,0} = 0.1\pi_{s,0} + 0.08\pi_{s,1} \\ \pi_{s,1} = 0.9\pi_{s,0} + 0.92\pi_{s,1} \\ \pi_{s,0} + \pi_{s,1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,0} = \frac{0.08}{1-0.1}\pi_{s,1} \\ \pi_{s,1} = \frac{1}{1+\frac{0.08}{1-0.1}} \end{cases} \tag{3}$$

from which, it results that

$$\Pi(\infty) = \Pi_\ell = [\pi_{\ell,0}, \pi_{\ell,1}] \equiv \Pi_s \approx (0.0816, 0.9184)\,.$$

Because of pieces arrives every $\Delta\tau = 10$ seconds, the rate of correctly elaborated pieces at the steady state, denoted here by λ_e can expressed as the averaged number of correctly elaborated pieces, with respect to the time unit $\Delta\tau$. Since in a time cycle no more than 1 piece can be correctly processed within the system, thus the number of correctly elaborated pieces can be described by random variable Z which may takes only two values 0, or 1. The long term probability a piece is correctly elaborated is

$$\Pr(Z(\infty) = 1) = (1-q)\cdot\pi_{\ell,1},$$

namely, the probability of being in 1 and the process terminates correctly. On the other hand

$$\Pr(Z(\infty) = 0) = 1 - \Pr(Z(\infty) = 1)$$

It follows that

$$\mathsf{E}[\text{correctly elaborated pieces}] = \mathsf{E}[Z] = 1\cdot(1-q)\cdot\pi_{\ell,1} = 0.8\cdot 0.9181 = 0.73472 \text{ pieces}$$

and thus

$$\lambda_e = \left.\frac{\mathsf{E}[\text{correctly elaborated pieces}]}{\Delta\tau}\right|_{\Delta\tau=10sec} \cdot \frac{6\Delta\tau}{\text{minute}} = 0.73472\cdot 6 \approx 4.4 \left[\frac{\text{pieces}}{\text{min}}\right]$$

Let us not calculates the percentage of discharged pieces η_d at steady state, that is the ratio between the probability the DT-MC has its buffer busy and the robot fails its task, with respect to the probability that a piece is introduced in the system, that is the sum between the probability the buffer is empty and a piece arrive, plus the probability the robot terminates correctly its task, while in the same time interval a piece arrives. It follow that

$$\eta_d = \frac{\Pr(\text{an arrived piece is discharge})}{\Pr(\text{a piece is introduced in the system})} = \frac{pq\cdot\pi_{\ell,1}}{p\cdot\pi_{\ell,0} + p(1-q)\pi_{\ell,1}} = 0.0225.$$

That means that every $\Delta\tau$, a proportion of 0.0225 pieces is discharged, on average (i.e. 22.5%). It further follows that the discharging rate in pieces/minutes is

$$\lambda_d\left[\frac{\text{pieces}}{\text{min}}\right] = \eta_d\left[\frac{\text{pieces}}{\Delta\tau}\right]_{\Delta\tau=10sec} \cdot \frac{6\Delta\tau}{\text{minute}} = 0.135\left[\frac{\text{pieces}}{\text{min}}\right].$$

The mean number of pieces in the buffer at the the steady state, can be computed as next

$$\mu_X(\infty) = \lim_{\tau\to\infty} \mathsf{E}[X(\tau)] = \sum_{i=0}^{1} i \cdot \Pr(X(\infty) = i) = 0 \cdot \pi_{0,\ell} + 1 \cdot \pi_{1,\ell} = 0.9184 \text{ pieces.}$$

Finally, under the assumption that the robot may fail its task only once,it results that the now it is needed a way to count how many times the process is failed. To do that we can simply modify the DT-MC sample space to account the following events, resp.:

- "0": the system is empty in idle mode;
- "1_P": the piece is in the system is under its first elaboration;
- "1_R": the piece in the system is under its second elaboration.

Thus, by analogous considerations of that made at point (a) the DT-MC that describes this process can modified in accordance with the following transition graph