# Stochastic Models (IA/0134/EN)

# Sample of candidate Theoretical Questions

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**Abstract**

*The aim of this document is to help the student prepare for the exam by providing an overview of the theoretical concepts and aspects of the course they should pay particular attention to. The document is organized as follows: Section 1 briefly outlines the exam's structure. Section 2 emphasizes the importance of theory in understanding the concepts. Finally, Section 3 provides a list of potential theory questions for each module.*

## 1 Exam's Organization

The final examination for the Stochastic Models course consists of an oral discussion.

The exam begins by asking the candidate to talk (in English) about a topic of their choice from the main topics of the course. The chosen topic is flexible and can encompass multiple modules covered during the course. Examples of possible topics are:

- From the definition of a random variable to stochastic processes: An overview of the main notions and concepts related to these topics.
- The dimensioning problem of a shared communication link: An overview of the main notions and concepts related to the formalization of this problem.
- From the Markov property to Markov chains: A discussion of the main notions and concepts related to these topics.
- From birth-death processes to queuing theory: An overview of the main notions and concepts related to these topics.
- From queuing theory to queuing networks: An overview of the main notions and concepts related to these topics.
- From rumor spreading to consensus: An overview of the main concepts and notions.
- Distributed optimization: An overview of the main concepts and notions.

Generally, this part takes approximately 30 minutes. No slides can be used; it should be an informal discussion where you need to demonstrate your understanding of the concepts you are presenting at the blackboard.

Based on the quality of the presentation, some details about it may be questioned.

Then, the exam continues with the candidate being asked to solve/sketch the solutions of two exercises.

Those exercises are similar to those covered in class or are generally easier. Note that the focus of these exercises is on how you approach solving the given problem. Finally, related theoretical questions may also be asked.

To pass the exam the student must demonstrate an appropriate and correct knowledge of the tools and methods of analysis and design seen during the course, while showing, during the solution of simple exercises, autonomy in making judgments regarding its design choices.

To help students prepare for the exam, a list of potential theory questions is provided below. The aim of these questions is to help students understand which aspects they need to focus on during their study.

# 2 What is Theory?

One of the major purposes of "Theory", regardless of the particular field of study, is to provide an answer to the question "Why?"

Asking "Why?" to deepen your knowledge of a subject allows you to realign your thoughts and opinions on a topic, and it is an essential skill for anyone who wants to learn and develop.

It is also worth noting that developing theories is humanity's response to their innate nature to ask "Why?" from childhood.

- Why do you wash my blankie?
- Why is snow cold?
- Why is the sky blue?

Although such questions can be endless, they reflect a child's first attempts to understand the world. By continually asking "Why?", a child is fueling their learning.

Although there are no hard and fast rules, modern theories are usually developed through a series of steps. Typically, this process starts with the "observation" of a given phenomenon, which is generally the "effect", not the "cause" we are seeking. This process involves several phases, such as the description of the "phenomenon", the development of "hypotheses", the modification of these hypotheses, and ultimately the acceptance of the theory by a wider community.

Moreover, it is important to note that to support all these phases, the use of a common (and "international") language is clearly needed. That language is mathematics", which, through its formalism, rules, and operators, allows us to formulate problems unambiguously, state theorems, and characterize the validity ranges of the developed results.

# 3 List of potential theory question per topic

Let us now list some potential theory questions, per topic of the course, which will help in your exam preparation, self-assessment, and in reinforcing the fundamental concepts of the course.

It should also be noted that the ability to answer the following questions is essential for properly approaching every numerical assignment.

## 3.1 Module of "Probability theory"

- Describe the meaning of experiment, trial, outcome, sample space and event, and that of probability function.

- By using the Set's Theory, provide an interpretation of the following probability rules and laws: Addition's Rule, Conditional probability law, Law of total probability, Bayes' theorem.
- Explain the difference between independent and mutually exclusive events.
- Provided the definition of random variable in either the discrete or continuous case, then describe the differences.
- Clarify the difference and/or provide an interpretation for probability mass function, probability density function and cumulative probability function.
- On the basis of their formal definition, provide an interpretation for the concept of "Mean" and that of "Variance".
- Provides an example and/or an interpretation for the selected probability distributions seen in the course (students are enabled to consult the table of known probability distributions to pick-up their related formulas).
- Let $X$ be a random variable defined as the sum of many indipendent random variable. Discuss how its mean and variance can be computed.
- Consider two exponential random variables $X_1$ and $X_2$, where $f_{X_i}(t) = \lambda_i \cdot e^{-\lambda_i \cdot t}$ with $\lambda_i > 0$ and $t \geqslant 0$. Consider the competition between the two, such that $X_1 < X_2$. Compute $\Pr(X_1 < X_2)$, then provide an interpretation of this result.

### 3.2 Module of "Link Dimensioning"

- Provided an interpretation and explain the differences between the Markov an the Chebyshev inequalities.
- Discuss the meaning of the Law of Large numbers.
- Provide an interpretation for the Central limit theorem. Then explain the difference to the Lyapunov's central limit theorem (no formulas are needed).
- Discuss the project of Link Dimensioning in the case of homogeneous individual sources. Then, generalize the discussion for non-homogeneous sources.

### 3.3 Module of "Stochastic processes"

- Provide the definition of stochastic process, then discuss its meaning.
- Discuss the importance of the Joint probability function in the context of stochastic processes.
- Explain the differences between a strictly stationary process and, a weak sense stationary process.
- Discuss the reasons why the ergodicity property is often desired in practice.
- Provide the definition of Markov process than discuss what the Markov property implies.
- Discuss the properties a Poisson process, then show its connection with the Exponential distribution
- Discuss the differences between a random and regular splitting of a Poisson process.
- Illustrate the concept of superposition of Poisson Processes.

### 3.4 Module of "Discrete-time Markov chains"

- Provides the definition of discrete-time Markov chain, then by means of a simple example discuss its key properties.
- Discuss the differences between a transient, a recurrent periodic, and a recurrent aperiodic state of a DT-MC.
- Discuss the differences between a reducible and a irreducible DT-MC.
- Draw the graph of a reducible periodic Markov chain. Is this DT-MC ergodic?
- Discuss the differences between an ergodic DT-MC, and an irreducible periodic DT-MC and an irreducible aperiodic DT-MC
- Provide an interpretation highlighting the differences between the stationary and the limiting distribution of a DT-MC. Then, under which conditions is a limiting distribution unique?
- Discuss how the ergodicity in a DT-MC can be evaluated.
- Draw a DT-MC with two ergodic strongly connected absorbing components. For this chain explain the concept of absorption probabilities.
- What is a discrete-time birth-death processes.
- Draw a simple DT-MC, then provide an interpretation of the mean hitting time concept and the mean recurrence time concept for a given state.

### 3.5 Module of "Continuous-time Markov chains"

- Provides the definition of CT Markov chain, then by means of some example discuss its key properties.
- Discuss the connections between the transition rate matrix and the transition probabilities of a CT-MC. Then, under the hypothesis that the process is time-homogenous, discuss how the system analysis becomes easier.
- Provide the definitions and then discuss the differences between the stationary and the limiting distribution of a CT-MC. Then, under which condition a limiting distribution is unique.
- Provide an interpretation of the mean hitting time concept for a CT-MC. What is the major difference compared to the DT-MC formulation?
- Discuss how to evaluates the ergodicity in a CT-MC.
- Discuss the main characteristics of a time-homogenous continuous-time birth-death processes and its connection with a M/M/1 resource
- Discuss the ergodicity of a CT-BDP process than derive its limiting distribution
- Provide an interpretation for the mean hitting time and for the mean recurrence time a state in a CT-MC

### 3.6 Module of "Queueing theory"

- Discuss the functioning of a queue and its characterization by means of the Kendall Notation.
- Consider the Table of the Queues systems' Quantity of interest. Explain the meaing of each quantity. e.g. for idle probability, for utilization factor, for throughput, for steady-state probability.
- Provide the definition and an interpretation of the Little's Law.

- Discuss the functioning of a deterministic queue and the conditions such that it behaves as an ergodic process.
- Discuss the concept and provide an interpretation of traffic intensity for a M/M/1 and a M/M/$\infty$
- Let $\{X_t, t \geqslant 0\}$ be the counting process of arrival to a queue system where

$$\Pr(X_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

  Let $\Delta t = t_{k+1} - t_k$ be the inter-event time is Exponentially distributed with the same rate. Then, provide an interpretation why $\mathsf{E}[\Delta t] = 1/\lambda$.
- Provide an interpretation for the PASTA property
- Discuss the advantages in terms of $\rho$, $\bar{x}$ abd $th\bar{e}ta$ of the stochastic multiplexing access method with respect to the TDMA.
- Discuss the ergodicity for a M/M/m/m queue
- Provide the definition and an interpretation for blocking probability and abandonement rate in a markovina queue.
- Model a M/M/3 queue as a CT-BDP, the discuss under which condition this stochastic process is ergodic.
- Model a M/M/3/4 queue as a CT-BDP, the discuss under which condition this stochastic process is ergodic.

### 3.7 Module of "Queueing networks"

- Discuss the differences between open and closed queueing networks
- Which is the meaing of the routing probability matrix?
- In the context of a queuing network, provide an interpretation to the following Joint probabilities

$$\Pr(X_1 = x_1, X_2 = x_2, \ldots, X_v = x_n) \quad , \quad \Pr(X_1 \geqslant 1, X_2 = 1)$$

- Discuss ergodicity conditions for an open and a closed queuing network
- Draw the transition graph associated with the CT-MC modellization of a tandem queuing network consisting of two M/M/1 queues
- Model a cyclic queuing network consisting of two M/M/3 nodes as a CT-MC, and a population of $n = 3$ costumers

### 3.8 Module of "Rumor spreading"

- Describe the functioning of the so-called push-model
- Discuss modellization of the rumor spreading problem as a CT-MC. Then, discuss if the resulting MC is ergodic.
- Discuss, in words, which are the major differences between the rumor spreading problem over a complete communication graph with respect to the case graph is arbitrary.

### 3.9 Module of “Consensus problem”

- What do we mean for a multi-agents system?
- Illustrate the determistic CT consensus problem
- Discuss the existing connections between the Laplacian matrix associated with the communication graph, and the transitions rate matrix of a CT-MC
- Illustrate how a DT average consensus problems can be formulated starting from its corresponding CT counterpart
- Illustrate the agents' update rule in the randomized gossip-based consensus problem
- Discuss, in words, the existing relationship between matrix $P$ in the DT consensus problem, and matrix $\mathsf{E}[W]$ of the gossip-based algorithm

### 3.10 Module of “Distributed optimization”

- Discuss the difference between a convex, a strictly convex, a strong convex, and a not convex function.
- Present the conceptual ideas behind the so-called gradient descent solver algorithm.
- Discuss how to approach the numerical solution of a convex optimization with equality constraints.
- Illustrate the main ideas behind the so-called Lyapunov stability theorem.
- Illustrate the main ideas behind the so-called LaSalle Invariant Principle.
- Illustrate the main ideas behind the problem of solving a convex optimization with equality constraints and solving the so-called unconstrained Lagrangian dual problem.
- Illustrate the similarities behind the problem of solving a convex optimization with equality and solve an uncostrained optimization in a distributed way over a multi-agent system infrastructure.
- Consider a distributed optimization problem. Which are the advantages of considering the so-called augmented Lagrangian dual problem within the solver design?
- Provide the main ideas, or the algorithm structure, of the so-called gossip-based distributed optimization algorithm. Discuss, in words, which steps would involve its converge proof.

### 3.11 Module of “Hidden Markov Models”

- Provide the definition of Hidden Markov Model and that of DT-MC chain, then discuss the conceptual differences between the two.
- Illustrate, in words, the three main problems that can be solved by means of a Hidden Markov Model.
- Which are the advantages of the Viterbi algorithm with respect to the Backward-Forward algorithm.