Stochastic Models

A.A. 2024/2025

Assignment 2

Exercise 1. It's 16:05. You are in smart working and the today's meeting should have already finished since 5 minutes. In general your meetings have no more then 15 minutes of delay. Today you have also agreed on a delivery by the courier between 4pm and 5pm.

What is the joint probability the meeting will finish in the next 10 minutes while the courier will pass between the 16:05 and 16:15?

Exercise 2. Let us consider a Poisson random variable X with parameter λ:

$$p_X(k) = \Pr(X = k) = \frac{e^{-\lambda}\,\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

(a) Recalling the exponential Taylor expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ show $\sum_{i=0}^{\infty} \Pr(X = i) = 1$.

(b) Calculate $\Pr(X > 2)$ with $\lambda = 4$.

Exercise 3. Let x_i be the number of request received by a web-server after i seconds. Assume requests independent and let $x_0 = 0$. Two case studies are now considered:

(a) requests have a regular rate of 60 requests/min.

(b) requests are random with an average 60 requests/min.

Let X_i be the random variable associated with this process, determine for the two cases the sample of X_i, and its probability function, then calculate

$$\Pr(X_{10} = 10) \quad , \quad \Pr(X_{10} \leqslant 9) \quad , \quad \Pr(X_9 \geqslant 9 | X_{10} \leqslant 15)$$

Exercise 4. Let X_1 and X_2 be two independent exponential random variables with rate λ. Let $Y = X_1 + X_2$ be a new random variable. Provide the probability density function of Y.

Exercise 5. Consider a binary communication channel between a transmitter and a receiver where b_n is the value of the n-th bit at the receiver. Assume the probability that a bit is erroneous follows a Bernoulli distribution with parameter $p = 0.01$.

(a) Calculate the probability of having more than one error on a bit-stream of 10 bits.

(b) Discuss under which condition can a Poisson distribution approximate this process? Under this assumption, repeat the previous calculus.

Exercise 6. The random variable X counts the number of heads when we flip 3 coins. Calculate:

(a) the sample space of X

(b) the probability mass function $p_X(x)$

(c) the expected value $\mathsf{E}[X]$ and its variance $\mathsf{Var}[X]$.

Exercise 7. Let X be a continuous random variable which probability density function follows

$$f_X(x) = \begin{cases} kx & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $k \in \mathbb{R}$ is a constant. Calculate

(a) the value of k in order to get a valid probability then evaluate its probability density function $f_X(x)$.

(b) the corresponding cumulative distribution function $F_X(x) = \Pr(X \leqslant x)$.

(c) the $\Pr(1/4 < X \leqslant 2)$.

(d) the expected value $\mathsf{E}[X]$.

(e) the variance $\mathsf{Var}[X] = \mathsf{E}[(X - \mathsf{E}[X])^2$.

Exercise 8. Let $X \sim Uni_c(0, 1)$ be an uniformly distributed random variable.

Let $Y = \sqrt{1 - X^2}$ calculates it probability density function. Finally, calculate its expectation by taking advantage of the web-service `https://www.wolframalpha.com/`.

Solutions

Solution of Exercise 1. Here we are looking for the joint probability the meeting will finish in the next 10 minutes while the courier will pass between the 16:05 and 16:15, conditioned to the fact that the actual time is 16:05. Well, because of the two events are independent, and for sure our meeting will end in the next 10 minutes (i.e. with probability equal to 1), then this probability depends only to the probability the courier will pass between the 16:05 and 16:15 given that in equal to 10/55.

Let us now see hot formally prove this result. Let us first define two random variable, one is T which describes the ending time of your meeting, the other is C which describes the arrival time of the courier. Clearly, both are uniform and also independent, such that

$$f_T(t) = Uni_c(0, 15) \quad , \quad f_C(c) = Uni_c(0, 60) \quad , \quad f_{C,T}(c, t) = f_C(c) \cdot f_T(t)$$

The key point of this exercize is the knowledge of the actual time, namely that it's 16:05. Let us now compute the Joint probability the meeting will finish in the next 10 minutes while the courier will pass between the 16:05 and 16:15. To do that we have to define the event $\mathcal{C} = 5 \leqslant C \leqslant 15 | C \geqslant 5$ and $\mathcal{T} = 5 \leqslant T \leqslant 15 | T \geqslant 5$.

Then because of the independence one has

$$\Pr(\mathcal{C}, \mathcal{T}) = \Pr(\mathcal{C}) \cdot \Pr(\mathcal{T}) \tag{1}$$

Let us now compute the two probabilities. Let us first note that

$$\begin{aligned} \Pr(\mathcal{T}) =& \Pr(5 \leqslant T \leqslant 15 | T \geqslant 5) && (2) \\ =& \frac{\Pr(5 \leqslant T \leqslant 15 \cap T \geqslant 5)}{\Pr(T \geqslant 5)} = \frac{\Pr(T \geqslant 5)}{\Pr(T \geqslant 5)} = 1, && (3) \end{aligned}$$

since the event $5 \leqslant T \leqslant 15$ and $T \geqslant 5$ denotes the same event.

On the other hand, one has that

$$\begin{aligned} \Pr(\mathcal{C}) =& \Pr(5 \leqslant C \leqslant 15 | C \geqslant 5) = \frac{\Pr(5 \leqslant C \leqslant 15 \cap C \geqslant 5)}{\Pr(C \geqslant 5)} && (4) \\ =& \frac{\Pr(10 \leqslant C \leqslant 15)}{\Pr(C \geqslant 5)} = \frac{\frac{10}{60}}{\frac{55}{60}} = \frac{10}{55} && (5) \end{aligned}$$

We can thus conclude that

$$\Pr(\mathcal{C}, \mathcal{D}) = \Pr(\mathcal{C}) \cdot \Pr(\mathcal{D}) = \frac{10}{55} \approx 0.1818. \tag{6}$$

Solution of Exercise 2. Point (a): To prove that $\sum_{i=0}^{\infty} \Pr(X = i) = 1$ let us manipulate the previous relation as follows

$$\sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \cdot \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

Point (b): Let $\lambda = 4$, $\Pr(X > 2)$ can instead be calculated as

$$\Pr(X > 2) = 1 - \Pr(X \leqslant 2) = 1 - \left(\frac{e^{-4} \cdot 4^0}{0!} + \frac{e^{-4} \cdot 4^1}{1!} + \frac{e^{-4} \cdot 4^2}{2!} \right) = 1 - e^{-4}(1+4+8) \approx 0.762$$

Solution of Exercise 3. Point (a): Because of requests arrive with a deterministic policy with a fixed rate $\lambda = 60$ arrivals/min or equivalently 1 arrivals/sec, one has that the number of request at time i correspond to i, thus arrivals can be described by the following random variable:

$$\text{case (a):} \quad X_i(S_i, p_{X_i}) \ : \ S_i = \{i\}, \quad p_{X_i}(i) = 1, \ \forall \ i \in \mathbb{N}_{\geqslant 0}$$

From that one straightforwardly obtain

$$\begin{aligned} \Pr(X_{10} = 10) &= p_{X_{10}}(10) = 1 \\ \Pr(X_{10} \leqslant 9) &= P_{X_{10}}(9) = 0 \\ \Pr(X_9 \geqslant 9 | X_{10} \leqslant 15) &= \frac{\Pr(X_{10} \leqslant 15 | X_9 \geqslant 9)\Pr(X_9 \geqslant 9)}{\Pr(X_{10} \leqslant 15)} \\ &= \frac{\left(\sum_{k=0}^{6} p_{X_1}(k)\right)(1 - P_{X_9}(8))}{P_{X_{10}}(15)} = \frac{p_{X_1}(1)(1-0)}{p_{X_{10}}(10)} = \frac{1 \cdot 1}{1} \end{aligned}$$

Point (b): Here requests are random and independent with an average arrival rate, then we can model $X_i \sim Pois(\lambda\Delta)$, where Δ is playing as a scale factor for λ, namely

$$\text{case (b):} \quad X_i(S_i, p_{X_i}) \ : \ S_i = \{0, 1, 2, \dots\}, \quad p_{X_i}(k) = \frac{(\lambda\Delta)^k}{k!} e^{-\lambda\Delta}, \quad \Delta = \frac{i}{60}, \quad \forall \ k, i \in \mathbb{N}_{\geqslant 0}$$

From that one straightforwardly obtain

$$\Pr(X_{10} = 10) = p_{X_{10}}(10) = \frac{(10)^{10}}{10!} e^{-10} = 0.1251 \qquad (\texttt{poisspdf(10,10)})$$

$$\Pr(X_{10} \leqslant 9) = P_{X_{10}}(9) = \sum_{i=0}^{9} p_{X_i}(k) = 0.4579 \qquad (\texttt{poisscdf(9,10)})$$

$$\begin{aligned} \Pr(X_9 \geqslant 9 | X_{10} \leqslant 15) &= \frac{\Pr(X_{10} \leqslant 15 | X_9 \geqslant 9)\Pr(X_9 \geqslant 9)}{\Pr(X_{10} \leqslant 15)} = \frac{\left(\sum_{k=1}^{6} p_{X_1}(k)\right)(1 - P_{X_9}(8))}{P_{X_{10}}(15)} \\ &= \frac{0.9999 \cdot 0.5443}{0.9513} = 0.5722 \end{aligned}$$

(`sum(poisspdf(0:6,1))*(1-poisscdf(8,9))/poisscdf(15,10)`)

Solution of Exercise 4. As known by the probability theory, the probability density function (pdf) of the sum of k independent and identically distributed (IID) exponential random variables is a known to be $Erlang(k, \lambda)$. In particular one has that, let $Y = \sum_{i=1}^{k} X_i$, with $X_i \sim Exp(\lambda)\ \forall\ i = 1, 2, \dots, k$ then

$$Y \sim f_Y(y) = Erlang(k, \lambda) = \frac{\lambda^k \cdot y^{k-1} \mathrm{e}^{-\lambda y}}{(k-1)!}$$

Thus in our case, one has that, let $Y = X_1 + X_2$, it follows that

$$Y \sim f_Y(y) = Erlang(2, \lambda) = \lambda^2 \cdot y \cdot \mathrm{e}^{-\lambda y}.$$

Another way to solve this exercise was, by exploiting the concept of convolution. In fact by the theory of random variables, it is known that the pdf of the sum of IID random variables (not necessarily exponential), can be evaluated directly by solving the next continuous-time convolution integral

$$\begin{aligned} f_Y(y) =& f_{X_1+X_2}(y) = f_{X_1}(y) * f_{X_2}(y) = \int_0^y f_{X_2}(y - x_1) \cdot f_{X_1}(x_1) dx_1 \\ =& \int_0^y \lambda \mathrm{e}^{-\lambda(y-x_1)} \cdot \lambda \mathrm{e}^{-\lambda x_1} dx_1 = \lambda^2 \mathrm{e}^{-\lambda y} \int_0^y dx_1 = \lambda^2 \cdot y \cdot \mathrm{e}^{-\lambda y}. \end{aligned}$$

As expected it results that $Y \sim Erlang(2, \lambda)$.

However, supposed that none of the above mentioned results was known to us. Then, the only way to find a pdf of arbitrary random variables, is to evaluated its cumulative distribution function (cdf) then differentiate it.

So, let us find the cdf of $Y = X_1 + X_2$. Notice that the pdf of $X_1 + X_2$ is not the sum of the two pdf. This is obvious because otherwise we obtain

$$\int_{-\infty}^{\infty} (f_{X_1}(y) + f_{X_2}(y) dy) = 2$$

that is not a probability! Said that another techniques to compute the pdf of Y is by computing

$$F_Y(z) = \Pr(Y \leqslant y) = \int_{-\infty}^{y} f_Y(z) dz$$

by means of the Law of total probability. As an example, we can solve

$$F_Y(y) = \Pr(Y \leqslant y) = \int_0^y \Pr(X_1 + X_2 \leqslant y | X_1 = x_1) \cdot f_{X_1}(x_1)\ dx_1 \tag{7}$$

and then diffentiate it w.r.t. y, so that $f_Y(y) = dF_Y(y)/dy$.

Namely, by recalling the concept of Joint PDF

$$f_{Y,X}(y, x) = f_{Y|X}(y|x) f_X(x)$$

and by marginalizing out with respect to x_1 as follows

$$\begin{aligned}
F_Y(y) = \Pr(X_1 + X_2 \leqslant y) =& \Pr(X_2 \leqslant y - x_1 | X_1 \leqslant y) \\
=& \int_0^{y-x_1} \int_0^{y} f_{X_2|X_1}(x_2|x_1)\ f_{X_1}(x_1)\ dx_1\ dx_2 && \text{(Total probability Law)} \\
=& \int_0^{y-x_1} \int_0^{y} f_{X_1,X_2}(x_1, x_2)\ dx_1\ dx_2 && \text{(Joint probability definition)} \\
=& \int_0^{y} \int_0^{y-x_1} f_{X_1}(x_1)\ f_{X_2}(x_2)\ dx_1\ dx_2 && \text{(independence)} \\
=& \int_0^{y} \left(\int_0^{y-x_1} f_{X_2}(x_2)\ dx_2 \right)\ f_{X_1}(x_1)\ dx_1 && \text{(cdf definition)} \\
=& \int_0^{y} \Pr(X_2 \leqslant y - x_1)\ f_{X_1}(x_1)\ dx_1
\end{aligned}$$

where, since the next constraint hold $x_1 + x_2 \leqslant y$, then the integral with respect to x_1 is between the interval $[0, y] \subset S_1 = [0, \infty)$, whereas that on x_2 is within $[0, y - x_1] \subset S_2 = [0, \infty)$. Now, note that

$$\Pr(X_2 \leqslant y - x_1) = F_{X_2}(y - x_1) = \left(1 - \mathrm{e}^{-\lambda(y-x_1)}\right)$$

Thus, by substituting the above relation, one obtain

$$\begin{aligned}
F_Y(y) =& \int_0^y F_{X_2}(y - x_1) \cdot f_{X_1}(x_1)\ dx_1 = \int_0^y \left(1 - \mathrm{e}^{-\lambda(y-x_1)}\right) \cdot \lambda \mathrm{e}^{-\lambda x_1} dx_1 \\
=& \lambda \int_0^y \left(\mathrm{e}^{-\lambda x_1} - \mathrm{e}^{-\lambda y}\right) dx_1 = \lambda \int_0^y \mathrm{e}^{-\lambda x_1} dx_1 - \lambda \mathrm{e}^{-\lambda y} \int_0^y\ dx_1 \\
=& \lambda \left. \frac{\mathrm{e}^{-\lambda x_1}}{-\lambda} \right|_0^y - \lambda \mathrm{e}^{-\lambda y}\ x_1|_0^y = 1 - \mathrm{e}^{-\lambda y} - (\lambda y) \cdot \mathrm{e}^{-\lambda y}
\end{aligned}$$

Now, let's differentiate the cdf to get the pdf.

$$f_Y(y) = \frac{dF_Y(y)}{dy} = 0 + \lambda \mathrm{e}^{-\lambda y} - \lambda \left(\mathrm{e}^{-\lambda y} + y \cdot \left(-\lambda \mathrm{e}^{-\lambda y}\right)\right) = \lambda^2 \cdot y \cdot \mathrm{e}^{-\lambda y}$$

It thus results $Y \sim Erlang(2, \lambda)$.

Thus, reminding that in a Poisson point process with rate λ inter-event times are exponential distributed, then the sum of two exponential IID random variable can be interpreted as the sum of two inter-event times. It follows that an $Erlang(k, \lambda)$ can be thought as the random variable associated to the time to have exactly k Poisson points (or Poisson events) over the considered time window of interest.

Solution of Exercise 5. Let $b_n \in \{0,1\}$ be the n-th bit at the receiver, and $Z_n \in \{0,1\}$ be the state of the bit at the receiver, where $Z_n = 0$ means that b_n is received correct, whereas, $Z_n = 1$ means that b_n have been received wrong.

Then $Z = \sum_{n=1}^{10} Z_n$ represents the discrete random variable counting the errors on a bit-stream of 10 bits. Then remanding that a binomial distribution count the number of success over n Bernoulli trials, it is straightforward note that $Z \sim Bin(n,p)$. Thus it results that

$$\begin{aligned}\Pr(Z > 1) &= 1 - \Pr(Z \leqslant 1) = 1 - \Pr(Z = 0) - \Pr(Z = 1)\\ &= 1 - \binom{10}{0} p^0 (1-p)^10 - \binom{10}{1} p^1 (1-p)^9\\ &= 1 - (1-0.01)^{10} - 10 \cdot 0.01(1-0.01)^9 = 0.0043.\end{aligned}$$

A Binomial r.v. in order to be well-approximated by a Poisson distribution have to satisfy that $np \leqslant 10$, where np takes the meaning of rate $\lambda = np$, whereas n must satisfy $n \geqslant 100$. In this case we have instead $\lambda = np = 10 \cdot 0.01 = 0.1$ and $n = 10$. One may thing that the condition is not satisfied because $n = 10$, nonetheless since p is sufficiently small such that $np = 0.1$, that is 10 times smaller then the requirement, this lack will be automatically compensated. To confirm this note that $Z \sim Pois(0.1)$ is such that.

$$\Pr(Z \geqslant 1) = 1 - \Pr(Z = 0) - \Pr(Z = 1) = 1 - e^{-0.1} \cdot (0.1^0 + 0.1^1) = 0.0047.$$

From the previous result we can conclude that the Poisson distribution can be used to approximate a Binomial distribution if n is sufficiently big and p is sufficiently small.

Solution of Exercise 6. Since we are tossing three coins, and each coin can assume two values T, H, the number of possible outcomes is equal to the number of simple permutations $2^3 = 8$. In particular, we can have

$$\begin{Bmatrix} TTT, & TTH, & THT, & THH \\ HTT, & HTH, & HHT, & HHH \end{Bmatrix}$$

which thus implies that the sample space of $X \in 0, 1, 2, 3$. Let us further note that each outcome of X is not equally likely. In particular we have

x	$p_X(x) = \Pr(X = x)$
0	1/8=0.125
1	3/8=0.375
2	3/8=0.375
3	1/8=0.125

Let us now evaluate $\mathsf{E}[X]$ and $\mathsf{Var}[X]$. By means of the "Moment generating function" it results that

$$\Pi_X(z) = \sum_{0}^{\infty} z^{-i} \cdot p_X(i) = 0.125 z^{-0} + 0.375 z^{-1} + 0.375 z^{-2} + 0.125 z^{-3}.$$

Then, the expected value and variance of X can be computed as follows

$$\mathsf{E}[X] = -\left.\frac{d\Pi_X(z)}{dz}\right|_{z=1} \quad , \quad \mathsf{Var}[X] = \left.\frac{d^2\Pi_X(z)}{dz^2}\right|_{z=1} - \mathsf{E}[X] - \mathsf{E}[X]^2$$

Thus, by differentiating $\Pi_X(z)$ it results

$$\frac{d\Pi_X(z)}{dz} = 0 - 0.375 \cdot z^{-2} - 0.375 \cdot 2z^{-3} - 0.125 \cdot 3z^{-4}$$

$$\frac{d^2\Pi_X(z)}{dz^2} = 0.375 \cdot 2 \cdot z^{-3} + 0.375 \cdot 2 \cdot 3z^{-4} + 0.125 \cdot 3 \cdot 4z^{-5}$$

It finally results that

$$\mathsf{E}[X] = -(-1.5) = 1.5$$

which corresponds to its value computed by means of its standard definition

$$\mathsf{E}[X] = \sum_{i=0}^{3} i \cdot p_X(i) = 0.125 \cdot 0 + 0.375 \cdot 1 + 0.375 \cdot 2 + 0.125 \cdot 3 = 1.5$$

Similarly we have that

$$\mathsf{Var}[X] = 4.5 - 1.5 - 1.5^2 = 0.75$$

which corresponds to its value computed by means of its standard definition

$$\mathsf{Var}[X] = \sum_{i=0}^{3} (i - \mathsf{E}[X])^2 \cdot p_X(i)$$

$$= 0.125 \cdot (0 - 1.5)^2 + 0.375 \cdot (1 - 1.5)^2 + 0.375 \cdot (2 - 1.5)^2 + 0.125 \cdot (3 - 1.5)^2 = 0.75.$$

Solution of Exercise 7. Let us first prove which value of k makes it a valid probability density function by solving w.r.t. k the following definite integral

$$\int_0^1 k \cdot x \; dx = 1 \Rightarrow k \left.\frac{x^2}{2}\right|_0^1 = \frac{k}{2} = 1 \Rightarrow k = 2.$$

It follows that

$$f_X(x) = \begin{cases} 2 \cdot x & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Let us now compute its cumulative distribution function by integrating it as follows

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^1 2 \cdot x \; dx & 0 \leqslant x < 1 \\ 1 & x \leqslant 1 \end{cases}$$

which implies

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leqslant x < 1 \\ 1 & x \leqslant 1 \end{cases}$$

Now, compute

$$\Pr(1/4 \leqslant X \leqslant 2) = \int_{1/4}^2 f_X(x) \; dx = \int_{1/4}^1 2x \; dx = x^2\big|_{1/4}^1 = 1 - \frac{1}{16} = \frac{15}{16}.$$

Then

$$\mathsf{E}[X] = \int_0^1 x \cdot f_X(x) \; dx = 2 \int_0^1 x^2 \; dx = \left.\frac{2}{3}\right|_0^1 = \frac{2}{3}.$$

Finally, by trivial manipulations we have that

$$\mathsf{Var}[X] = \mathsf{E}[(X - \mathsf{E})^2] = \mathsf{E}(X^2 - 2X\mathsf{E} + \mathsf{E}^2) = \mathsf{E}[X^2] - 2\mathsf{E}[X]^2 + \mathsf{E} = \mathsf{E}[X^2] - \mathsf{E}[X]^2. \quad (8)$$

Thus, we have that

$$\mathsf{E}[X^2] = \int_0^1 x^2 f_X(x) \; dx = \int_0^1 x^2 \cdot 2x \; dx = 2\left.\frac{x^4}{4}\right|_0^1 = \frac{1}{2},$$

then by substituting the two partial results into (8), we have

$$\mathsf{Var}[X] = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

Solution of Exercise 8. Let us first note that since $x \in [0,1] \Rightarrow y \in [0,1]$. Let us now compute $f_Y(y)$ through the definition of its cumulative distribution function

$$F_Y(y) = \Pr(Y \leqslant y) = \int_0^y f_Y(y)\ dy.$$

and that

$$F_Y(y) = \Pr(Y \leqslant y) = \Pr(\sqrt{1 - X^2} \leqslant y) = \Pr(X \geqslant \sqrt{1-y^2}) = 1 - \Pr(X \leqslant \sqrt{1-y^2}),$$

which thus implies that, because of $X \sim U(0,1)$ it follows that

$$F_Y(y) = 1 - F_X(\sqrt{1-y^2}) = 1 - \sqrt{1-y^2}$$

Then, by differentiating it with respect to y, it yields that

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[1 - \sqrt{1-y^2}\right] = -\frac{d}{dy}\left[\sqrt{1-y^2}\right] = \frac{y}{\sqrt{1-y^2}}.$$

Further note that

$$\mathsf{E}[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y) dY = \int_0^1 y \cdot \frac{y}{\sqrt{1-y^2}}\ dy$$

`https://www.wolframalpha.com/input?i=int_0%5E1+y%5E2%2F%28sqrt%281-y%5E2%29%29+dy`

then, by integrating by substitution with $x = \sqrt{1-y^2}$, or alternatively by invoking the property provided in the text of the assignment, one has that

$$\mathsf{E}[Y] \quad = \quad \int_0^1 \frac{y^2}{\sqrt{1-y^2}}\ dy = \frac{1}{2}\left(\operatorname{asin}(y) - y \cdot \sqrt{1-y^2}\right)\Big|_0^1 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

From the above results one concludes that, by generating a sample of n observations of the random variable $Y = \sqrt{1 - X}$ as n increase its mean converges to $\pi/4$, thus it is possible estimates

$$\pi = 3.14159265358979323846264338327950288419716939937510582097494459 23\ldots$$

as next $4 \cdot \mathsf{E}[Y] \approx \pi$. This is a stochastic approach for estimating π.

As an example run the following Matlab script

```
clear all
N = 1e5;
u = rand(N,1);
Y = sqrt(1 - u.^2);
piEst = 4*mean(Y)
error = abs(pi - piEst)
```