

Stochastic Models

A.A. 2024/2025

Assignment 1

Exercise 1. Consider an experiment which universe of samples is S . Let $A \subset S$ and $B \subset S$ be two events such that $\Pr(A) = 0.9$, $\Pr(B) = 0.8$ e $\Pr(A \cap B) = 0.75$. Calculate:

$$(a) \Pr(A \cup B); \quad (b) \Pr(A \cap \bar{B}); \quad (c) \Pr(\bar{A} \cap \bar{B}).$$

where $\bar{A} = S \setminus A$ and $\bar{B} = S \setminus B$. Then, reply the next questions

- (d) Are A and B mutually exclusive?
- (e) Are A and B exhaustive?
- (f) Are A and B independent?

Exercise 2. In a lot of 100 microchips, 20 of them are defective. Two microchips are randomly selected. Calculate the probabilities of:

- (a) the first extracted microchip is defective.
- (b) the second extracted microchip is defective given that the first was defective as well.
- (c) both the extracted microchips were defective.

Exercise 3. In a multiple choice exam, there are 4 answers to a question. A student knows the right answer with probability

$$(a) \ 0.8; \ (b) \ 0.2; \ (c) \ 0.5;$$

If the student doesn't know the answer s/he always guesses with probability of success 0.25. Given that the student marked the right answer, what is the probability s/he knew it.

Exercise 4. Consider an experiment in which two marbles are randomly picked from a bag containing four marbles, each labeled with an integer from 1 to 4.

- (a) In the case when, before picking the second marble, the first extracted is put back in the bag, calculate the sample space S_a and its cardinality $|S_a|$;
- (b) In the case without the replacement of the first ball, calculate the sample space S_b and its cardinality $|S_b|$;
- (c) In the case “without the replacement” of the first ball and by neglecting the extraction order, as it the lotto game, calculate the sample space S_c and its cardinality $|S_c|$.

Exercise 5. Consider the experiment consisting of picking 1 random marble at each time $k \in \mathbb{N}$ from a bag containing four marbles. Each marble is labeled from 1 to 4. Two policies are considered: (a) with replacement, and (b) without replacement.

- (i) Let $E_k(S_k, p_k)$ be the random variable associated with the time index k . Determine the sample-spaces S_k^a and S_k^b , respectively for each $k \in \mathbb{N}$, namely $k = 1, 2, 3, 4, 5, \dots$
- (ii) For the two cases, determine the following joint probability $p_{E_1, E_2, E_3, E_4}(1, 2, 3, 4)$.

Exercise 6. Consider an experiment counting the number of success over n Bernoulli trials with parameter π . Determine which random variable correctly describe this process.

- (a) Which is its sample-space?

Then calculate the following probabilities

- (b) we get at least 1 success over n trial;
- (c) we get n failures over n trial;
- (d) we get n success over n trial.

Exercise 7. Let the random variable Y be defined as a function of a continuous random variable X , such that $Y = F_Y^{-1}(X)$.

- (a) Prove that $X \sim Uni_c(0, 1)$.
- (b) Let $X \sim Uni_c(0, 1)$ and $Y \sim Exp(\lambda)$ be exponential with parameter λ . Determine which is the function enabling the generation of Y from X .

Exercise 8. The (normalized) temperature $X \in [0, 1]$ of an engine is a random variable whose probability density function is:

$$f_X(x) = \begin{cases} n \cdot (1 - x)^{n-1} & \text{with } 0 \leq x \leq 1, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Show for which values of $n > 0$, the function $f_X(x)$ is a probability density function.
- (b) Calculate its expected value of X , namely $E(X)$.

Solutions

Solution of Exercise 1. Point (a): By invoking the *Addition's Rule* one obtains

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 0.9 + 0.8 - 0.75 = 0.95 < 1. \quad (1)$$

Please refer to Figure 1 for a graphical interpretation of A and B with respect to S .

Point (b): By exploiting that $\bar{B} = S \setminus B$, and by means of the distributive property of sets, one derives

$$A \cap \bar{B} = A \cap (S \setminus B) = (A \cap S) \setminus (A \cap B) = A \setminus (A \cap B).$$

Thus one has

$$\Pr(A \cap \bar{B}) = \Pr(A) - \Pr(A \cap B) = 0.9 - 0.75 = 0.15.$$

Point (c): By means of the distributive property, after straightforward manipulations one has

$$\begin{aligned} \bar{A} \cap \bar{B} &= (S \setminus A) \cap (S \setminus B) \\ &= [(S \setminus A) \cap S] \setminus [(S \setminus A) \cap B] \\ &= [(S \cap S) \setminus (S \cap A)] \setminus [(S \cap B) \setminus (A \cap B)] \\ &= [S \setminus A] \setminus [B \setminus (A \cap B)] \\ &= S \setminus [A + B \setminus (A \cap B)] \\ &= S \setminus (A \cup B) \end{aligned}$$

Notice that the above derivation proves one of the two De Morgan's Law, which states that

$$\bar{A} \cap \bar{B} \equiv \overline{A \cup B} = S \setminus (A \cup B)$$

From that one further derives that

$$\Pr(\bar{A} \cap \bar{B}) = \Pr(S) - \Pr(A \cup B) = 1 - 0.95 = 0.05. \quad (2)$$

Point (d): Because of $\Pr(A \cap B) \neq \emptyset$ then A and B are not mutually exclusive.

Point (e): Because of $\Pr(A \cup B) = 0.95 < 1$, then A and B are not exhaustive, namely we can define a set $C = S \setminus (A \cup B)$, such that $\Pr(C) = 1 - 0.95 = 0.05$. It straightforwardly results that A , B and C are exhaustive.

Point (f): The sets A and B are not independent because $\Pr(A \cap B) \neq \Pr(A) \cdot \Pr(B)$. Indeed, $\Pr(A \cap B) = 0.75$, whereas $\Pr(A) \cdot \Pr(B) = 0.72$.

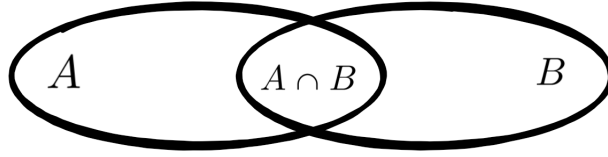


Figure 1: Venn diagram of set A and B in Exercise 1.

Solution of Exercise 2. Point (a): Let D_1 and D_2 be the events that, respectively, the first and the second extracted chips are defective, then one has that

$$\Pr(D_1) = \frac{20}{100} = 0.2.$$

Point (b): The probability that second extracted microchip would be defective, given that the first is defective is

$$\Pr(D_2|D_1) = \frac{19}{99} = 0.\widehat{19}.$$

This probability can be easily derived by noticing that we now have only 19 defective chips over 99.

Point (c): Finally, the “a-priory” probability that we may extract two defective microchips by means of two consecutive extractions is instead the joint probability that both the first and second extracted microchip would be defective, that is

$$\Pr(D_1 \cap D_2) = \Pr(D_2|D_1) \cdot \Pr(D_1) = \frac{19}{99} \cdot \frac{20}{100} = 0.0\widehat{38}.$$

Another way to compute $\Pr(D_2|D_1)$ is by means of the conditional probability formula, namely,

$$\Pr(D_2|D_1) = \frac{\Pr(D_1 \cap D_2)}{\Pr(D_1)} = \frac{\frac{19}{99} \frac{20}{100}}{\frac{20}{100}} = \frac{19}{99} = 0.\widehat{19}.$$

It is also worth noting that $\Pr(D_2|D_1)$ and $\Pr(D_1 \cap D_2)$ differ, conceptually, from the available information at moment of the question. Finally let us further note that D_1 and D_2 are dependent, in fact

$$\Pr(D_2|D_1) = \frac{19}{99} \neq \Pr(D_2|\bar{D}_1) = \frac{20}{99} \neq \Pr(D_2) = \frac{19}{99} \frac{20}{100} + \frac{20}{99} \frac{80}{100} = \frac{20}{100}$$

In the first case we are sure that D_1 is occurred, and in fact the chance that D_2 would occur given that D_1 is occurred is much higher than the a-priory probability that both were defective, namely, $\Pr(D_2|D_1) \gg \Pr(D_1 \cap D_2)$.

Solution of Exercise 3. Let us first define the following events

- A : Student knows the answer
- \bar{A} : Student does not know the answer
- B : Student marks correctly
- \bar{B} : Student marks incorrectly.

Let us now note that this assignment ask the calculation of a posterior, thus Bayesian probability $\Pr(A|B)$, from a prior probability $\Pr(A)$, given the evidence of B . Thus it results that

$$\Pr(A|B) = \frac{\Pr(B|A) \cdot \Pr(A)}{\Pr(B)}. \quad (3)$$

Let us first note that, for the first case $\Pr(A) = 0.8$, $\Pr(\bar{A}) = 0.2$, $\Pr(B|A) = 1$, and $\Pr(B|\bar{A}) = 0.25$.

By the “Law of Total Probability” we have that

$$\Pr(B) = \Pr(B|A)\Pr(A) + \Pr(B|\bar{A})\Pr(\bar{A}) = 1 \cdot 0.8 + 0.25 \cdot 0.2 = 0.85$$

Then by replacing into (3) it results that

$$\Pr(A|B) = \frac{\Pr(B|A) \cdot \Pr(A)}{\Pr(B)} = \frac{1 \cdot 0.8}{0.85} = 0.94117647058.$$

For case (b), we have that

$$\Pr(A|B) = \frac{\Pr(B|A) \cdot \Pr(A)}{\Pr(B|A)\Pr(A) + \Pr(B|\bar{A})\Pr(\bar{A})} = \frac{1 \cdot 0.2}{1 \cdot 0.2 + 0.25 \cdot 0.8} = \frac{0.2}{0.4} = 0.5.$$

For case (c), we have that

$$\Pr(A|B) = \frac{\Pr(B|A) \cdot \Pr(A)}{\Pr(B|A)\Pr(A) + \Pr(B|\bar{A})\Pr(\bar{A})} = \frac{1 \cdot 0.5}{1 \cdot 0.5 + 0.25 \cdot 0.5} = \frac{0.5}{0.625} = 0.8.$$

Solution of Exercise 4. Point (a): The sample space S_a is:

$$S_a = \left\{ \begin{array}{cccccccc} (1,1), & (1,2), & (1,3), & (1,4), & (2,1), & (2,2), & (2,3), & (2,3), \\ (3,1), & (3,2), & (3,3), & (3,4), & (4,1), & (4,2), & (4,3), & (4,4) \end{array} \right\} \quad (4)$$

and its cardinality $|S_a| = 16$. It would be noted that, due to the “replacement discipline” repetitions like (i, i) or (j, j) are allowed, thus the cardinality of this set could be also computed by the well-known formula of simple permutation¹ n^k , where n is the number of things to choose from, and k is the number of available choices, follows that $4^2 = 16$.

Point (b): The sample space S_b is:

$$S_b = \left\{ \begin{array}{cccccc} (1,2), & (1,3), & (1,4), & (2,1), & (2,3), & (2,4), \\ (3,1), & (3,2), & (3,4), & (4,1), & (4,2), & (4,3) \end{array} \right\} \quad (5)$$

and its cardinality $|S_b| = 12$. It would be noted that, since we are considering a discipline “without replacement” repetitions are not allowed. Moreover after the first extraction the possible number of combination w.r.t. the previous case is reduced. Thus the cardinality of this set could be also computed by the well-known formula of *permutation without repetition*

$$\frac{n!}{(n-k)!} = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot 2!}{2!} = 4 \cdot 3 = 12$$

Further note that this is not the formula of the binomial coefficient.

Point (c): The sample space S_c is:

$$S_c = \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \} \quad (6)$$

and its cardinality $|S_c| = 6$. It would be noted that, since we are considering a discipline “without replacement” repetitions are not allowed. Moreover since we are not considering the order of extraction, repetitions like (i, j) , or (j, i) for all $i, j = 1, 2, 3, 4$, are counted once. Notice that the cardinality of this set could be also computed by the well-known “binomial coefficient”

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad , \quad \text{with } n \geq k \geq 0 \quad (7)$$

which count the number of combination without repetition.

It thus results that

$$|S_3| = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2!}{2!2!} = 2 \cdot 3 = 6$$

Notice that the count of permutation differs from that of combination because in the count of permutation the order matter.

¹When the order doesn't matter, it is a Combination. When the order does matter it is a Permutation.

Solution of Exercise 5. Point (i.(a)): Let us now consider the random variable E_k , that models the outcome of the k -th extraction for $k \geq 1$, and which may take values $e_k \in S_k$. For the policy “with replacement” (a), one observes that

$$(a) \quad S_1 = S_2 = \dots = S_k = S = \{1, 2, 3, 4\}, \quad \forall k \geq 1$$

Point (i.(b)): On the other hand in the case “without replacement” (b), since we cannot know a priori which marble is extracted at step k , we can simply say that, at the first iteration one has that

$$S_1 = S = \{1, 2, 3, 4\}$$

then

$$\begin{aligned} S_k &= S_{k-1} \setminus \{e_{k-1}\} \subset S \quad \forall 2 \leq k \leq 4 \\ S_k &= \emptyset \quad \forall k \geq 5 \end{aligned}$$

Let us further note that, because of the considered scenarios, both (a) and (b), accounts the repetition of consecutive trials of the same experiment, scheduled at different countable time instances $k = 1, 2, \dots$, then we can call this discrete-time stochastic process, which is in general denoted with the following notation $\{E_k, k = 1, 2, 3, \dots\}$, where each discrete random variable $E_k(S_k, p_k(e_k))$ has its own sample-space $S_k \subset S$, and its own probability mass function $p_k(e_k)$.

Point (ii.(a)): Let us now consider the calculation of $p_{E_1, E_2, E_3, E_4}(1, 2, 3, 4)$. Before that, let us remind the definition of joint probability function for discrete random variables, that is

$$(a) \quad p_{E_1, E_2, E_3, E_4}(e_1, e_2, e_3, e_4) = \Pr(E_1 = e_1, E_2 = e_2, E_3 = e_3, E_4 = e_4) \in [0, 1]$$

where

$$\{e_1, e_2, e_3, e_4\} \in S_1 \times S_2 \times S_3 \times S_4 \subset S \times S \times S \times S,$$

where $\text{card}\{S \times S \times S \times S\} = 4^4 = 256$.

Said that, in the case with replacement (a), because of the random variables E_k are IID, one obtains that

$$\begin{aligned} p_{E_1, E_2, E_3, E_4}(1, 2, 3, 4) &= \Pr(E_1 = 1, E_2 = 2, E_3 = 3, E_4 = 4) \\ &= \prod_{k=1}^4 p_k(k) = \left(\frac{1}{4}\right)^4 \approx 0.0039 \equiv 0.4\%. \end{aligned}$$

where $p_k(1) = p_k(2) = p_k(3) = p_k(4) = 1/4, \forall k \geq 1$.

Point (ii.(b)): On the other hand, in the case without replacement, because of the dependence

of events, by applying the definition of joint probability recursively, one obtains

$$\begin{aligned} \text{(b)} \quad p_{E_1, E_2, E_3, E_4}(1, 2, 3, 4) &= \Pr(E_1 = 1, E_2 = 2, E_3 = 3, E_4 = 4) \\ &= \Pr(E_4 = 4 | E_3 = 3, E_2 = 2, E_1 = 1) \times p_{E_1, E_2, E_3}(1, 2, 3) \\ &= 1 \times p_{E_1, E_2, E_3}(1, 2, 3) \\ &= 1 \times \Pr(E_3 = 3 | E_2 = 2, E_1 = 1) \times p_{E_1, E_2}(1, 2) \\ &= 1 \times \frac{1}{2} \times \Pr(E_2 = 2, E_1 = 1) \\ &= 1 \times \frac{1}{2} \times \Pr(E_2 = 2 | E_1 = 1) \times p_{E_1}(1) \\ &= 1 \times \frac{1}{2} \times \frac{1}{3} \times \Pr(E_1 = 1) \\ &= 1 \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \approx 0.041 \equiv 4\% \end{aligned}$$

Solution of Exercise 6. Point (a): Let X_i be a random variable associated with the results of the i -th trial, such that $S_i = \{0, 1\}$. Let us first remind that the distribution which describes the probability to obtain k successes over n independent experiments (trial) is the so-called “Binomial distribution”, which refer to a newer random variable counting the successes k over n trials, namely the Binomial random variable is defined as sum of Bernoulli random variable as next

$$X = \sum_{i=1}^n X_i$$

Thus, its sample space is $S = 0, 1, 2, \dots, n$, whereas its probability mass function is given by the so-called “Binomial distribution”, namely

$$p_X(k) = \Pr(X = k) = \binom{n}{k} \pi^k \cdot (1 - \pi)^{n-k} \quad (8)$$

Point (b): Said that, it results

$$\Pr(X \geq 1) = \Pr(X > 0) = \bar{P}_X(0) = 1 - P_X(0) = 1 - \binom{n}{0} p^0 \cdot (1 - \pi)^n = 1 - (1 - \pi)^n.$$

Point (c): Since Bernoulli trials are, by definition, stochastically independent, it is obvious the probability of getting $k = n$ failures over n trials is

$$P_X(0) = \Pr(X = 0) \equiv \Pr(X_1 = 0) \cdot \Pr(X_2 = 0) \cdots \Pr(X_n = 0) = (1 - \pi)^n. \quad (9)$$

Point (d): Finally, the probability of getting exactly $k = n$ success over n trials is instead

$$\Pr(X = n) = \frac{n!}{n!(n-n)!} p^n \cdot (1 - \pi)^{n-n} \equiv \prod_{i=0}^n p_{X_i}(1) = \pi^n.$$

Solution of Exercise 7. Point (a): Let us first prove that $X \sim Uni_c(0, 1)$. To do that, we need to prove that

$$F_X(x) = \Pr(X \leq x) = x.$$

Notice that since $Y = F_Y^{-1}(X)$ holds, this implies F_Y is bijective. Thus, we can write $X = F_Y(Y)$. Then, by simple manipulations, it results

$$F_X(x) = \Pr(F_Y(Y) \leq x) = \Pr(Y \leq F_Y^{-1}(x)) = F_Y(F_Y^{-1}(x)) = x \implies X \sim Uni_c(0, 1)$$

Now, let us derive how a exponential random variable can be generated, starting from an uniformly distributed random variable.

Point (b): Let us first compute the cumulative distribution on of an exponential distribution starting from its probability density function

$$f_Y(y) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

with parameter λ , that is

$$F_Y(y) = \int_{-\infty}^y f_Y(\tau) d\tau = \int_0^y \lambda \cdot e^{-\lambda y} dy = \lambda \cdot \frac{e^{-\lambda y}}{-\lambda} \Big|_0^y = 1 - e^{-\lambda y}$$

Now, by exploiting the previous results it yields that

$$X = F_Y(Y) = 1 - e^{-\lambda Y} \Rightarrow X - 1 = -e^{-\lambda Y} \Rightarrow Y = \frac{\ln(1 - X)}{-\lambda}$$

It follows that, whenever we want to generate a realization of length n of an exponential distribution with parameter λ , we simply generate a sequence of outcome of the process X , namely x_1, x_2, x_3, \dots . From that our sequence of exponentially distribute outcomes can be derived as next $y_i = \ln(1 - x_i)/\lambda$. Then next set of MatLab instructions verifies that

```
clear all, clc, close all;
s=rng(0);
n=1e4; % the length is arbitrary
lambda=40; % rate of the Exp r.v.
x=rand(n,1); % x~Uniform(0,1)
y = -lambda^-1*log(1-x); % y~Exp(lambda)
mu_y=(1/n)*sum(y); % Sampled mean of X (approx. E[X]=1/lambda)

Pr_y_smaller_than_mu=(1/n)*sum(y<=mu_y); % clearly equal to 1-exp(-1)
Pr_y_greater_than_mu=(1/n)*sum(y>=mu_y); % clearly equal to exp(-1)
Pr_y_smaller_than_mu+Pr_y_greater_than_mu; % clearly equal to 1

% From y one can easily generate generate a time line of the
% arrivals in Poisson process
t=zeros(n+1,1);

for i=1:1:n
t(i+1)=t(i)+y(i);
end
% check it out!
[x(1:10) y(1:10) t(1:10)]
```

This method of generating random outcome (called also *deviates*) from any distribution starting from a uniform distribution is known by the following names: inverse transform sampling, inverse transformation method, inverse probability integral transform, and Smirnov transform, and it is used whenever we are condong on devices which instruction set accounts only the `rand()` function.

Notice that from a realization of exponential random variables, as it is y in the above script, one can easily generate the time line corresponding to each arrival of a Poisson process.

Solution of Exercise 8. Point (a): It's enough to check for which values of $n > 0$ the integral of the PDF equals 1, namely

$$n : \int_0^1 f_X(x) dx = 1.$$

Here we have that

$$\int_0^1 n(1-x)^{n-1} dx.$$

Thus by integrating for substitution, with $t = 1 - x$, which further implies $dx = -dt$, we have

$$\int_0^1 n(1-x)^{n-1} dx = n \int_{1-0}^{1-1} -t^{n-1} dt = n \cdot \frac{t^n}{n} \Big|_0^1 = 1.$$

Thus our $f_X(x)$ is a valid PDF for $\forall n > 0$.

Point (b): Now, by applying the definition of expected value it further results that

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{+\infty} x \cdot f_X(x) dx = \int_0^1 x \cdot n(1-x)^{n-1} dx \\ &= n \cdot \left(\underbrace{x \cdot \frac{-(1-x)^n}{n}}_0 \Big|_0^1 - \int_0^1 \frac{-(1-x)^n}{n} dx \right) = \int_0^1 (1-x)^n dx \\ &= - \int_{1-0}^{1-1} t dt = \int_0^1 t^n dt = \frac{t^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1} \end{aligned}$$

Namely, as n increases, then expected value of temperature get closer to the origin.