UNIVERSITY OF CAGLIARI

DIEE - Department of Electrical and Electronic Engineering

STOCHASTIC MODELS

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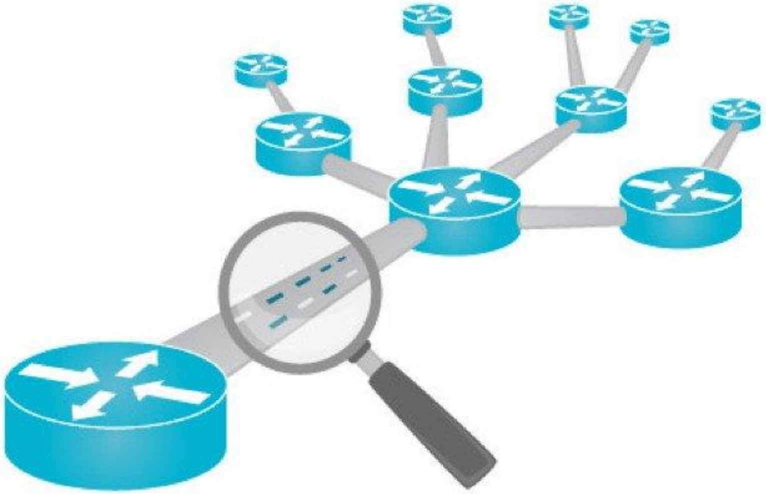
Link Dimensioning



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Random variables

- A random variable X is a variable which value is unknown a priory, and it may assume only the values $x \in S_X$, where S_X is its sample-space
- To each value $x \in S_X$ is associated a function depending of S_X

If S_X is countable (discrete case) *we have prob. mass function* (pmf) $p(x)$

Interpretation: The pmf tell us the *probability that* $X = x$, i.e., $p(x) = \Pr(X = x)$

If S_X is uncountable (continuous case) *we have a prob. density function* (pdf) $f(x)$

Interpretation: If the pdf around x is large, it means the X is more likely to be close to x, namely $X = x$, is more frequent

- All random variables have a *cumulative distribution function* (cdf), which meaning is that of a “probability”

Discrete case: $$P(x) = \Pr(X \leq x) = \sum_{\forall\ z \leq x} p(z)$$

Continuous case: $$F(x) = \Pr(X \leq x) = \int_{-\infty}^{x} f(z)dz$$

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Mean and Variance

The **mean $\boldsymbol{\mu}$** corresponds to the "most expected" value X may takes

- It is defined as the Expectation of X, or First-order moment
- It may or may not belong to S_X, in fact it corresponds to the "center of mass" of its pmf (discrete case), or pdf (continuous case), namely:

Discrete case:

$$\mu = \mathsf{E}[X] = \sum_{\forall x \in S_X} x \cdot p_X(x)$$

Continuous case:

$$\mu = \mathsf{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

The **variance $\boldsymbol{\sigma^2}$** gives info about how data are expected to be scattered around their expected value. It is defined as the Second-order central moment of X

$$\sigma^2 = \mathsf{Var}[X] = \mathsf{E}\left[(X-\mu)^2\right] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2$$

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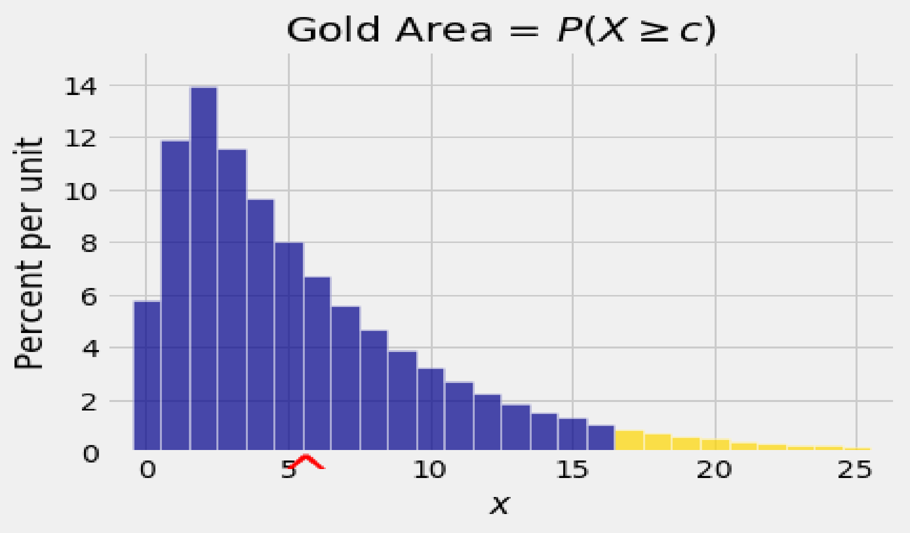
Markov Inequality

- Let X be a non-negative, arbitrarily distributed, random variable.

- To understand the accuracy of estimates, it helps examining the chance that X exceed a given value $c > E[X]$.

- Let Markov Inequality

$$\boxed{\Pr(X \geq c) \leq \frac{\mathrm{E}[X]}{c}}$$



- It provides an upper-bound on the tail probability (or complementary cdf)

- An upper-bound is just a ceiling, not an approximation, namely, $\Pr(X \geq c)$ may be also much smaller than $E[X]/c$, but for sure $\Pr(X \geq c) \leq E[X]/c$

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Markov Inequality (Proof discrete case)

- Let X be a non-negative random variable

$$
\begin{aligned}
\mathsf{E}[X] &= \sum_{x \in S_X} x \cdot \Pr(X = x) \\
&= \sum_{x < c} x \cdot \Pr(X = x) + \sum_{x \ge c} x \cdot \Pr(X = x)
\end{aligned}
$$

$$\mathsf{E}[X] \ge \sum_{x \ge c} x \cdot \Pr(X = x)$$

(obvious result)

$$
\begin{aligned}
\mathsf{E}[X] &\ge \sum_{x \ge c} x \cdot \Pr(X = x) \\
&\ge c \cdot \sum_{x \ge c} \Pr(X = x) \\
&= c \cdot \Pr(X \ge c)
\end{aligned}
$$

(replace x with c)

$$\Rightarrow \quad \Pr(X \ge c) \le \frac{\mathsf{E}[X]}{c}$$

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Markov Inequality (Proof continuous case)

- Proof: Let X be any non-negative random variables such that

$$\begin{aligned}\mathsf{E}[X] &= \int_{-\infty}^{+\infty} x \cdot f_X(t)dt \\ &= \int_{0}^{+\infty} x \cdot f_X(t)dt \quad \text{(since } X \text{ is positive-valued)} \\ &\geq \int_{c}^{+\infty} x \cdot f_X(t)dt \quad \text{(for any } c > 0\text{)} \\ &\geq \int_{c}^{+\infty} c \cdot f_X(t)dt \quad \text{(since } X \text{ is positive-valued)} \\ &= c \int_{c}^{+\infty} f_X(t)dt \\ &= c \cdot \Pr(X \geq c).\end{aligned}$$

$$\mathsf{E}[X] \geq c \cdot \Pr(X \geq c) \implies c \cdot \Pr(X \geq c) \leq \mathsf{E}[X] \implies \boxed{\Pr(X \geq c) \leq \frac{\mathsf{E}[X]}{c}}$$

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Probability Inequalities and their applications

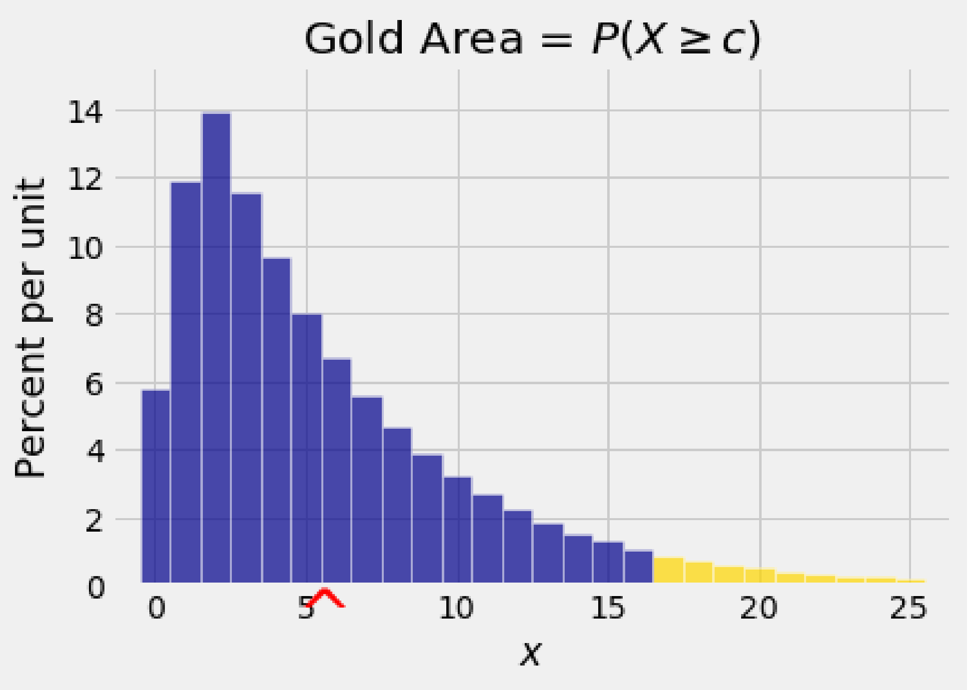
- Let X be a non-negative random variable representing the traffic demand of a communication channel in terms of #packets (or its intensity in #packets/sec).
- Let $C > 0$ be a scalar representing the link capacity (or its bit rate).

- We are now interested in the probability that

$$\Pr(X > C)$$

- Independently by the distribution of X, the traffic probability can be upper-bounded by the Markov Inequality as:

$$\Pr(X \geq C) \leq \frac{\mathrm{E}[X]}{C}$$



Chebyshev inequality

- Let $X \geq 0$ and its mean $\mu = E[X]$ and variance $\sigma^2 = Var[X]$ be known.
- Define a new r.v. $Y = (X - E[X])^2$. Since X and Y are non-negative, then

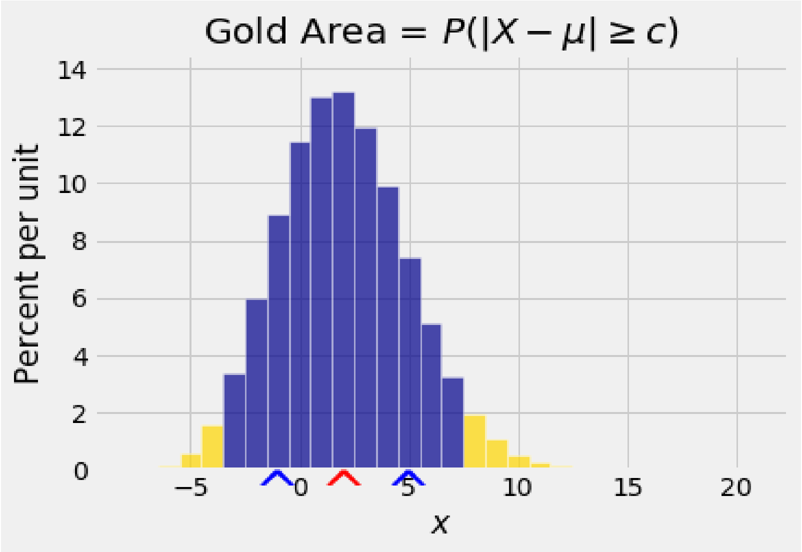
$$\mathrm{E}[Y] = \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{Var}[X] \quad \Rightarrow \quad \Pr(Y \geq C^2) \leq \frac{\mathrm{E}[Y]}{C^2} = \frac{\mathrm{Var}[X]}{C^2}$$

- By root squaring the arguments

$$\Pr\left(Y \geq C^2\right) = \Pr\left((X - \mathrm{E}[X])^2 \geq C^2\right) = \Pr(|X - \mathrm{E}[X]| \geq C).$$

Chebyshev inequality

$$\Pr(|X - \mathrm{E}[X]| > C) \leq \frac{\mathrm{Var}[X]}{C^2}$$



- If the pdf of X is known to be symmetric to μ

$$\Pr(X > \mu + C\sigma) = \Pr(X < \mu - C\sigma) \leq \frac{1}{2C^2}$$

- A refinement of the Chebyshev inequality is the Kolmogorov inequality, which holds for a sum of k independent r.v.

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Kolmogorov inequality (or maximal inequality)

- It extend the Chebyshev inequality to the sum of independent r.v.
- Let $X_1, X_2, \ldots, X_k$ be a family of mutually independent, not necessarily identical, random variables with a common sample-space S_X
- Let

$$S_k = \sum_{i=1}^{k} X_i = X_1 + X_2 + \cdots + X_k$$

- Let the process be such that $Var[S_k] < +\infty$
- Then for each $\theta > 0$, and $\forall\, n \geq k$, one has that

$$\boxed{\Pr\left(\max_{1\leq k\leq n} |S_k - \mathrm{E}[S_k]| \geq \theta\right) \leq \frac{1}{\theta^2} \cdot \mathrm{Var}[S_n] = \frac{1}{\theta^2} \cdot \sum_{i=1}^{n} \mathrm{Var}[X_i]}$$

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Kolmogorov inequality (cont’d)

- It allows to bound the worst-case deviation of the sum of r.v. w.r.t its mean by using info only on S_n

- Example: Let $X_1, X_2, \dots, X_k \sim Ber(\pi)$ and they may takes values 1 or -1

$$\mathrm{E}[X_i] = 1 \cdot \pi + (-1) \cdot (1 - \pi) = 2\pi - 1|_{\pi=0.5} = 0$$

$$\mathrm{Var}[X_i] = 1^2 \cdot \pi + (-1)^2 \cdot (1 - \pi) = 1$$

- Let us further define the random walk as

$$S_n = \sum_{k=1}^{n} X_k, \qquad n = 1, 2, \dots$$

- Then

$$\Pr\left(\max_{1 \le k \le n} |S_k - \mathrm{E}[S_k]| \ge \theta\right) \le \frac{1}{\theta^2} \cdot \mathrm{Var}[S_n] = \frac{1}{\theta^2} \cdot \sum_{i=1}^{n} \mathrm{Var}[X_i] = \frac{n \cdot \mathrm{Var}[X_i]}{\theta^2} = \frac{n \cdot 1}{\theta^2}$$

- This is an upper-bound of the probability to be far from the mean of a quantity θ at any step $k \le n$ of this random walk

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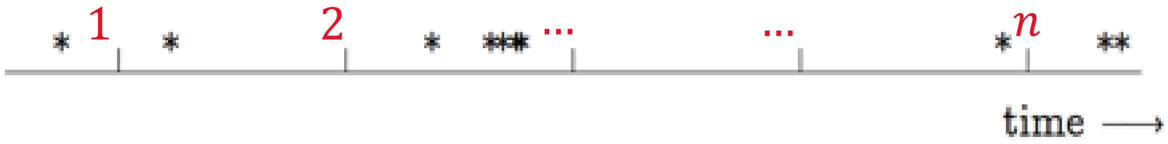
Teletraffic implications

- Assume the number of packets that arrive during different time periods "k" of finite length Δ are mutually independent, $\forall\, \Delta > 0$
- Let: 1) The time axis be divided into "n" consecutive intervals



- Let: 2) Let X_i be the number of packets arrive during the i-th subsequent intervals and let

$$S_k = X_1 + X_2 + \cdots + X_{k-1} + X_k \implies S_k = S_{k-1} + X_k \le S_n$$

- Then:

$$\Pr\left(\max_{1\le k\le n} |S_k - \mathsf{E}[S_k]| \ge \bar{\theta}\sigma(S_n) \right) \le \frac{1}{(\bar{\theta}\sigma(S_n))^2} \cdot \sigma(S_n)^2 \le \frac{1}{\bar{\theta}^2}$$

- Let for instance $\theta \gg 1$ it says, it is rare the maximal traffic at any point in time k is significantly far from the average

- More precisely such probability decrease by a factor that is $\bar{\theta}^2$

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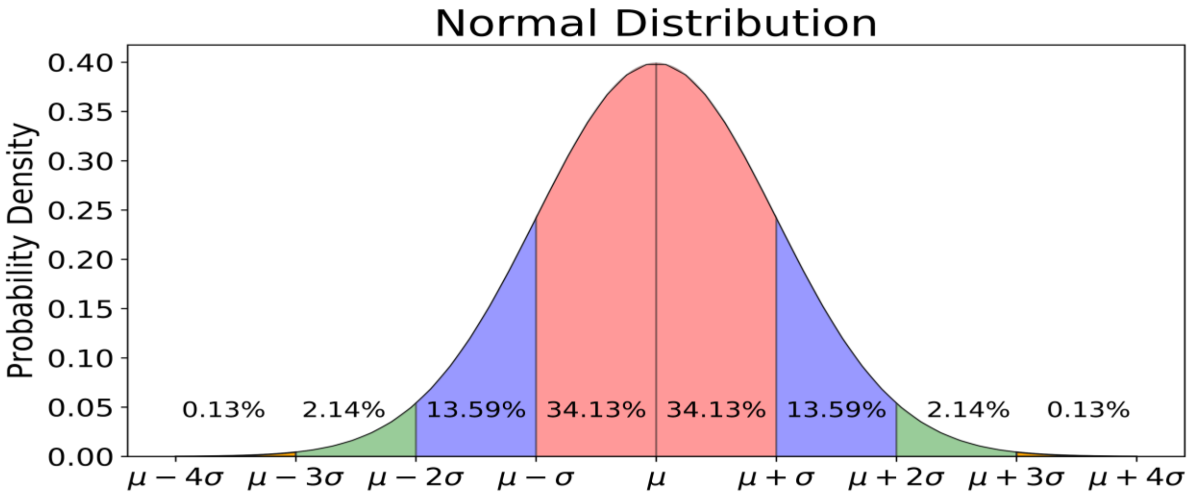
Example

- Verify the 68-95-99.7 Rule by means of the Kolgomorov Inequality for $n = k = 1$



- Let $X \sim N(0,1)$ be a standard normal random variable.
- By letting $n = k = 1$ the Kolmogorov Inequality becomes the Chebychev inequality, then it results that

$$\Pr(|X-\mu| < 2\sigma) \geq 1 - \left(\frac{1}{2\sigma}\right)^2_{\sigma=1} = 1 - \frac{1}{4} = 0.75 = 75\% \ll 95\%$$

- The Kolmogorov Inequality is more conservative, but more general than the 68-95-99.7 rule which holds only for normal distributions.

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Weak law of large numbers (Weak LLN)

Sampled-mean of a large number of i.i.d. r.v converges to its expectation

- Let $X_1, X_2, \dots, X_k$ be a sequence of IID with mean μ and variance σ^2.
- Let $S_k = \sum_{i=1}^{k} X_k$ and $\bar{S}_k = S_k/k$, namely $\bar{S}_k$ is defined as the sample mean of S_k
- Since $X_1, X_2, \dots, X_k$ are IID then $\mathrm{E}[S_k] = k\mu$ and $\mathrm{Var}[S_k] = k\sigma^2$, moreover

$$\mathrm{E}[\bar{S}_k] = \mathrm{E}\left[\frac{\sum_{i=1}^{k} X_i}{k}\right] = \frac{\sum_{i=1}^{k} \mathrm{E}[X_i]}{k} = \frac{k\mu}{k} = \mu$$

$$\mathrm{Var}\left[\bar{S}_k\right] = \mathrm{E}\left[\left(\frac{\sum_{i=1}^{k} X_i}{k}\right)^2\right] - \mathrm{E}[\bar{S}_k]^2 = \frac{\mathrm{E}\left[\left(\sum_{i=1}^{k} X_i\right)^2\right] - k^2\mu^2}{k^2} = \frac{\mathrm{Var}\left[S_k\right]}{k^2} = \frac{k \cdot \sigma^2}{k^2} = \frac{\sigma^2}{k}$$

- By the Chebyshev inequality, one has that $\forall\ \varepsilon > 0$, as k increases, namely the sample-space of S_k increases as well, then $Var[\bar{S}_k] \to 0$, and thus $\bar{S}_k \to \mu$

$$\Pr(|\bar{S}_k - \mu| \geq \varepsilon) \leq \frac{\mathrm{Var}[\bar{S}_k]}{\varepsilon^2} = \frac{\sigma^2}{k \cdot \varepsilon^2} \quad \Rightarrow \quad \lim_{k\to\infty} \Pr\left(\left|\frac{X_1 + X_2 + \cdots + X_k}{k} - \mu\right| \geq \varepsilon\right) = 0$$

- Interpretation: Despite ε can be small, as $k \to \infty$, $\bar{S}_k$ become closer to μ with a probability tending to 1. This is called "convergence in probability".
- Strong LLN states the same but in the "almost surely" sense: $\Pr\left(\lim_{k\to\infty} \bar{S}_k = \mu\right) = 1$

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In other words the Weak LLN states the sampled mean $\overline{S_k}$ of a large number k of observation of a random variable X with mean $\mu = E[X]$ will converge to μ

Example: Rolling a fair dice

```
% Possible outcomes when you flip a dice
x=[1,2,3,4,5,6]'
% If the dice is fair, its pmf is X~Uni(1,6), namely
pmf_of_X=unidpdf(1:6,6)
% Mean and Variance
Mean=sum(pmf_of_X*x)
Variance=sum(pmf_of_X*x.^2)-Mean^2

% Let us prove that with an experiment
experiments=unidrnd(6,10^7,1); % Realization of 10^7 samples
% Sample Mean and Sample Variance
Sample_mean=sum(experiments)/10^7
Sample_variance=sum( (experiments-Sample_mean).^2)/10^7
```

Central limit theorem

The sum of many i.i.d. r.v. is always gaussian regardless their original distributions

- Let $X_1, X_2, \dots, X_k$ be k IID r.v. with common mean μ and variance σ^2.
- Let S_k and Y_k be two new random variables such that

$$S_k = \sum_{i=1}^{k} X_i \ : \ \begin{cases} \mathsf{E}[S_k] = k\cdot\mu \\ \mathsf{Var}[S_k] = k\cdot\sigma^2 \end{cases} \qquad\qquad Y_k = \frac{S_k - \mathsf{E}[S_k]}{\sqrt{\mathsf{Var}[S_k]}} = \frac{\sum_{i=1}^{k} X_i - k\cdot\mu}{\sigma\sqrt{k}}$$

- Then (given without proof) the cdf of Y_k satisfies

$$\lim_{t\to\infty} \Pr(Y_k \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp^{-\frac{x^2}{2}} dx \implies Y_k \sim N(0,1), \ \text{namely} \ \mathsf{E}[Y_k] = 0, \quad \mathsf{Var}[Y_k] = 1$$

- Follows that the sum of k IID random variables with common mean μ and variance σ^2 satisfies that $S_k =\sim N(k\cdot\mu, k\cdot\sigma^2)$, i.e.,

$$S_k = \left(\sigma\sqrt{k}\cdot Y_k\right) + k\mu \ : \ \begin{cases} \mathsf{E}[S_k] = \mathsf{E}\left[\sigma\sqrt{k}\cdot Y_k + k\mu\right] = \sigma\sqrt{k}\cdot\mathsf{E}[Y_k] + k\mu = k\mu \\ \\ \mathsf{Var}[S_k] = \mathsf{E}\left[\left(\sigma\sqrt{k}\cdot Y_k + \cancel{k}\mu - \cancel{\mathsf{E}}[S_k]\right)^2\right] = k\cdot\sigma^2\mathsf{E}[Y_k^2] = k\cdot\sigma^2 \end{cases}$$

- regardless their original distribution.

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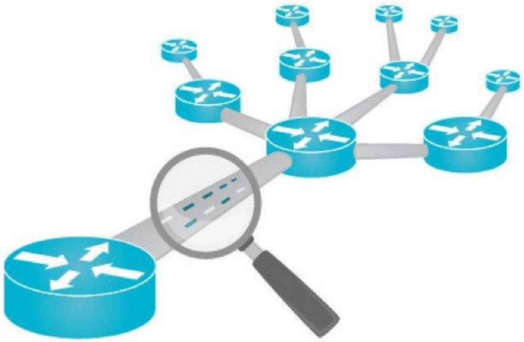
Link Dimensioning

- Let us now show how the probability concepts seen before may finds application to problems of link dimensioning

- Several scenarios where sources (individuals or families) share a communication link with a given capacity C will be considered

- Each source has certain requirements for capacity and the common link must be dimensioned to satisfy some QoS specifications, while minimizing the provider’s costs



- The link is assumed to be used in either directions, namely, when we say a source “transmits”, it means that it may either “transmits” or “downloads”.

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Case 1: Homogeneous Individual Sources

- Consider N independent sources S_i (end-terminals), sharing a transmission link of capacity C [Mb/s]

- Each source alternates two states: ON-OFF.

- During the ON state each source transmits, without loss of generality, at a rate

$$R_i = R \ = 1\ Mb/s$$

- During the OFF state the source is idle, $R_i = 0$

- The proportion of time the source is in state ON and OFF allows to estimates the following probabilities

$$\Pr(X_i = ON) = \frac{T_{ON}}{T_{ON} + T_{OFF}} = \pi \quad , \quad \Pr(X_i = OFF) = 1 - \pi$$

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Case 1: Homogeneous Individual Sources (cont'd)

- Since $R_i = R = 1\ Mb/s$ the total traffic demand at any time can be modeled as next

$$\Sigma = \sum_{i=1}^{N} R_i \quad \Rightarrow \quad \text{thus} \quad \Sigma \sim Bin(N, \pi)$$

- Question: How much capacity C [Mb/s] should the link have so it can serve the N sources, such that

$$\Pr\left(\sum_{i=1}^{N} R_i > C\right) \leq \alpha$$

- where α is a pre-assigned QoS measure.
- In particular, under the zero buffer approximation, α represents the proportion of time that the demand exceeds the supply

$\Downarrow$

Clearly during that period some traffic will be lost

Case 1: Homogeneous Individual Sources (cont'd)

- Realizing that the demand generated by a single source is Bernoulli distributed

$$\Pr(X_i = ON) = \pi \quad , \quad \Pr(X_i = OFF) = 1 - \pi$$

- Then, the demand generated by all N sources has Binomial distribution with parameters π and N.

$$\Pr(\Sigma = i) = \binom{N}{i} \pi^i (1-\pi)^{N-i}$$

Probability that, over N sources, "i" are transmitting at the same time

- Thus, since $R = 1[\mathrm{Mb/s}]$, then

$$\Pr\left(\sum_{i=1}^{N} R_i > C\right) = \sum_{i=C+1}^{N} \binom{N}{i} \pi^i (1-\pi)^{N-i}$$

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Case 1: Homogeneous Individual Sources (cont'd)

- Finding the desired capacity is reduced to finding the smallest C such that

$$\min C \;:\; \Pr\left(\sum_{i=1}^{N} R_i > C\right) = \sum_{i=C+1}^{N} \frac{N!}{i!(N-i)!}\pi^i(1-\pi)^{N-i} \le \alpha$$

- For a given C, the complementary cdf, namely the left-hand side increases as N decrease, and it is zero if $C = N$.

$$\bar{P}_{\Sigma}(N) \quad = \Pr(\Sigma > N) = 1 - \Pr(\Sigma \le N) = 1 - 1 = 0$$

- The optimal C comes out by the computation of the previous inequality for all

$$C = N-1, N-2, N-3, \ldots$$

- until the inequality is violated.

The desired capacity, is that value increased by 1.

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Case 1: Homogeneous Individual Sources (cont'd)

- If N is large, thanks to the Central Limit Theorem, the Binomial distribution can be approximated by a Gaussian distribution

$$\Sigma = \sum_{i=1}^{N} R_i \sim N(\mu, \sigma)$$

- where the demand random variable has, resp.,

$$\mu = \mathrm{E}[Bin(N, p)] = N \cdot \pi \quad , \quad \sigma^2 = \mathrm{Var}[Bin(N, \pi)] = N \cdot \pi(1 - \pi)$$

- The task is to find C such that

$$C \; : \; \Pr(N(\mu, \sigma) > C) \leq \alpha$$

- Since now the demand is approximated by a Gaussian Random Variable we can simply invoke the

68-95-99.7% Rule

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Case 1: Homogeneous Individual Sources (cont'd)

68-95-99.7% Rule

By this rule we have an information on the area below the bell

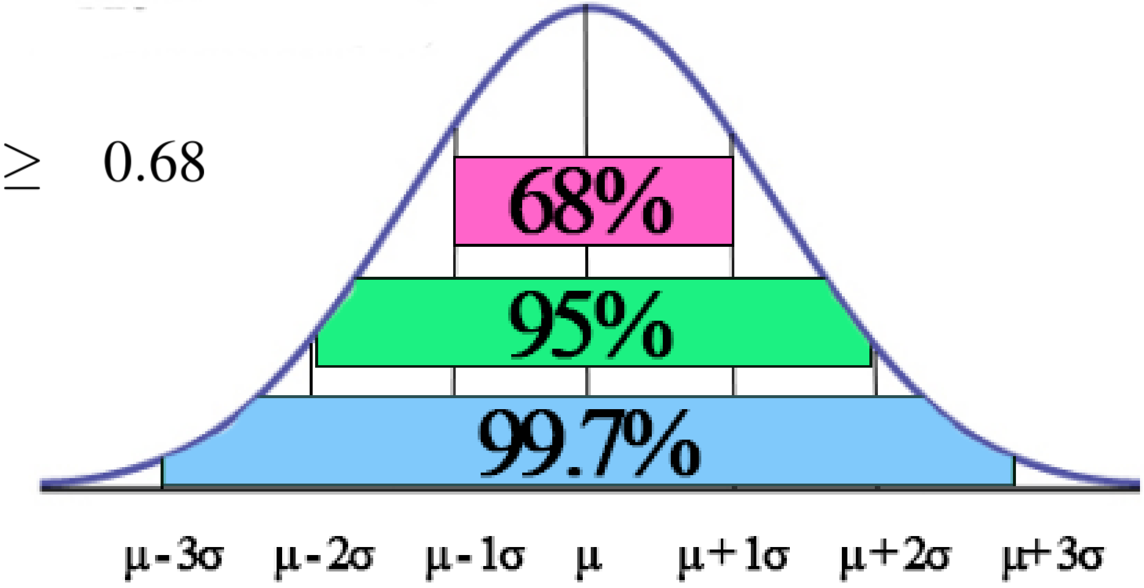
$$\Pr(|\Sigma-\mu| \le \theta \cdot \sigma) = \Pr(|\Sigma| \le \mu+\theta \cdot \sigma)$$

Thus, it further results that

$$\Pr(|\Sigma| \le \mu+\sigma) \ge 0.68$$

$$\Pr(|\Sigma| \le \mu+2\sigma) \ge 0.95$$

$$\Pr(|\Sigma| \le \mu+3\sigma) \ge 0.997$$



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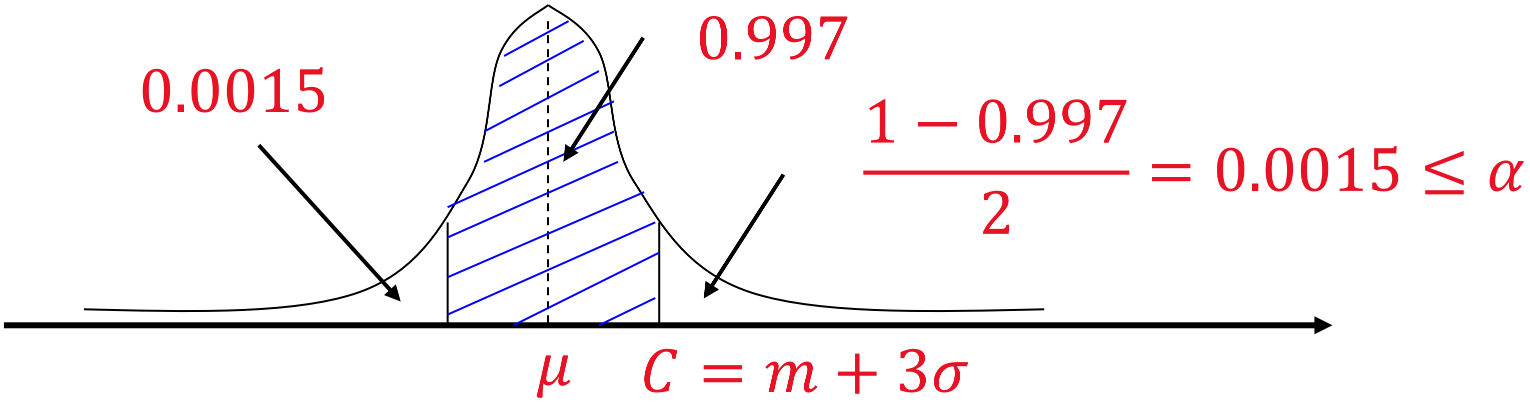
Example

- Let, for instance, $\alpha = 0.0015$ and because

$$\mu = N \cdot \pi \quad , \quad \sigma = \sqrt{N \cdot \pi(1-\pi)}$$

- It results that, if $\theta = 3$, then $\Pr(|\Sigma - \mu| \geq 3\sigma) \quad \leq \quad 1 - 0.997 = 0.003$



- Then C should be 3 times the standard deviations σ above the mean μ total traffic, namely

$$C > N\pi + 3\sqrt{N\pi(1-\pi)}$$

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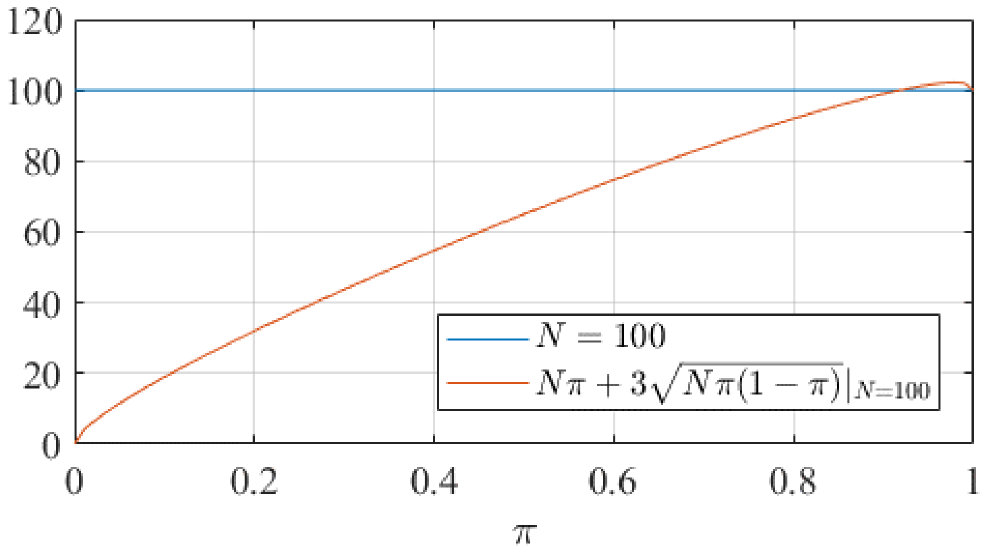
Example (cont'd)

- Recall that:

$C = N$ always guarantees that there is sufficient capacity to serve all traffic without losses.

- Therefore, our dimensioning rule for the optimal C value is as follows:

$$C_{opt} = \min\left\{N, N\pi + 3\sqrt{N\pi(1-\pi)}\right\}$$



This approximation error is because of the real phenomenon is not really Gaussian since the true total traffic is bounded in practice

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Case 2: Non-homogeneous Individual Sources

- Consider N sources, each transmitting at a rate R_i.

- Let X_i be the nonnegative r.v. associated with the i-th source state, we have

$$\Pr(X_i = ON) = \pi_i \quad , \quad \Pr(X_i = OFF) = 1 - \pi_i$$

$$\forall i = 1, 2, \ldots, N$$

- Now the sources are non-homogeneous ($\pi_i \neq \pi_j$), thus we must invoke a generalization of the central limit theorem that held for independent but NOT identically Distributed random variables:

"Lyapunov's central limit theorem"

- Question: How much C should the link have so it can serve all N sources?
- Namely, such that:

$$\Pr\left(\sum_{i=1}^{N} R_i > C\right) \leq \alpha$$

Case 2: Non-homogeneous Individual Sources (cont'd)

- The state of each source is Bernoulli distributed, as it is its generated traffic
- Let R_{X_i} be the r.v. associated with the generated traffic, one has

$$\mathrm{E}[R_{X_i}] = 0 \cdot (1-\pi_i) + R_i \cdot \pi_i \quad , \quad \mathrm{Var}[R_{X_i}] = R_i^2 \pi_i - (R_i \pi_i)^2 = R_i^2 \pi_i (1-\pi_i)$$

- Let us define a new random variable representing the whole traffic

$$\Sigma = \sum_{i=1}^{N} R_{X_i}$$

- Now Σ is not binomially distributed, however, if N is large, by the "Lyapunov's central limit theorem", the total traffic is Gaussian as well, which mean and variance are as next

$$\mathrm{E}[\Sigma] = \sum_{i=1}^{N} \mathrm{E}[R_{X_i}] = \sum_{i=1}^{N} R_i \cdot \pi_i \qquad \mathrm{Var}[\Sigma] = \sum_{i=1}^{N} \mathrm{Var}[R_{X_i}] = \sum_{i=1}^{N} R_i^2 \cdot \pi_i (1-\pi_i)$$

- Thus, by applying the 68-95-99.7% Rule, we can derive the optimal link size.

Example:

- Consider N non-homogeneous individual sources,
- Each transmits at a rate R_i with probability π_i.
- Let $\alpha = 0.0015$ be the preassigned QoS measure
- Thus, similarly with the homogeneous case, and by invoking the 68-95-99.7% Rule, we can derive the optimal link size as

$$C_{opt} = \min\left\{ \sum_{i=1}^{N} R_i, \sum_{i=1}^{N} \mathsf{E}[R_{X_i}] + 3\sqrt{\sum_{i=1}^{N} \mathsf{Var}[R_{X_i}]} \right\}$$

- Then, one has

$$C_{opt} = \min\left\{ \sum_{i=1}^{N} R_i, \sum_{i=1}^{N} R_i \cdot \pi_i + 3\sqrt{\sum_{i=1}^{N} R_i^2 \pi_i (1-\pi_i)} \right\}$$

Case 3: Capacity Dimensioning for a Community

- Assume each source "i" consists of a cluster of sub-sources $R_{i,j}$
- For instance each source may model the traffic generated by a family member

✓ $R_{i,0} = 0$ all the members in the family are in idle.

✓ $R_{i,1}$: the rate associated with one individual family member browsing the web;

✓ $R_{i,2}$: the rate associated with one individual family member using Voice over IP;

✓ $R_{i,3}$: the rate associated with one individual family member watching video;

✓ $R_{i,4}$: the rate associated with one individual family member watching video and another browsing the web;

✓ …

✓ R_{i,J_i} denotes the peak rate of source "i"

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Case 3: Capacity Dimensioning for a Community Community (cont'd)

- A family "i" may consist of n_i independent sub-sources (members), each generating a traffic of R_k^i Mbps with probability $\pi_k^i \in [0,1]$, $k = 1,2,\ldots,n_i$
- At any point in time, one or more of members may access the link
- Thus J_i-many combinations of traffic scenarios can be generated by the n_i members

$$J_i|_{n_i=3} = \sum_{k=0}^{n_i} \binom{n_i}{k}\Bigg|_{n_i=3} = 1+3+3+1 = 8$$

$$\begin{aligned} R_{i,j} \in \{0, R_1^i, R_2^i, R_3^i,& \\ R_1^i + R_2^i, R_1^i + R_3^i, R_2^i + R_3^i,& \\ R_1^i + R_2^i + R_3^i\}& \end{aligned}$$

$$0 = R_{i,0} \le \cdots \le R_{i,j} \le \cdots \le R_{i,J_i} \; [Mb/sec] \quad \forall\, j = 0,1,2,\ldots,J_i$$

Bin. Coefficient gives #combination of k objects chosen among n_i objects

- Because of independence, each rate is associated with a probability π_{ij} where $j \le J_i$

$$n_i = 3 \;:\; \pi_{i0} = \prod_{k=0}^{3}(1-\pi_k^i), \quad \pi_{i1} = \pi_1^i(1-\pi_2^i)(1-\pi_3^i), \ldots, \quad \pi_{i7} = (1-\pi_1^i)\pi_2^i\pi_3^i, \quad \pi_{i8} = \pi_1^i\pi_2^i\pi_3^i$$

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Case 3: Capacity Dimensioning for a Community

- Let R_{X_i} be a non-negative r.v. representing the rate transmitted by source "i"
- The mean traffic generated by source "i" is

$$\mathsf{E}[R_{X_i}] = \sum_{j=0}^{J_i} R_{i,j} \cdot \pi_{ij}$$

- Its traffic variance is instead

$$\mathsf{Var}[R_{X_i}] = \mathsf{E}\left[R_{X_i}^2\right] - (\mathsf{E}[R_{X_i}])^2 = \sum_{j=0}^{J_i} R_{i,j}^2 \cdot \pi_{ij} - \left(\sum_{j=0}^{J_i} R_{i,j} \cdot \pi_{ij}\right)^2$$

- Again, by defining, the total traffic as a random variable, one has

$$\Sigma = \sum_{i=1}^{N} R_{X_i}$$

- Then, we are interested to

$$\Pr\left(\sum_{i=1}^{N} R_{X_i} > C\right) \leq \alpha$$

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Case 3: Capacity Dimensioning for a Community (cont'd)

- If either the number of sources, or that of members in each source, is large, then, total traffic can modelled by a Gaussian random variable

$$\Sigma = \sum_{i=1}^{N} R_{X_i} = \sum_{i=1}^{N} \sum_{j=1}^{J_i} R_{i,j} \sim Norm(\mu, \sigma)$$

$$\mu = \sum_{i=1}^{N} R_{X_i} = \sum_{i=1}^{N} \left[\sum_{j=0}^{J_i} R_{i,j} \cdot \pi_{ij} \right] \qquad \sigma^2 = \sum_{i=1}^{N} \mathsf{Var}[R_{X_i}] = \sum_{i=1}^{N} \left[\sum_{j=0}^{J_i} R_{i,j}^2 \cdot \pi_{ij} - (\mathsf{E}[R_{X_i}])^2 \right]$$

- Thus, let as in previous case $\alpha = 0.015$, then the optimal channel capacity is

$$C_{opt} = \min \left\{ \sum_{i=1}^{N} \sum_{j=1}^{J_i} R_{X_{ij}}, \sum_{i=1}^{N} \mathsf{E}[R_{X_i}] + 3 \sqrt{\sum_{i=1}^{N} \mathsf{Var}[R_{X_i}]} \right\}$$

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