ESAME 13/06/2024

1) $\iint_D \frac{y^2 x}{x^2+y^2}\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1,\ x\ge 0\}$

$\begin{cases} x=\rho\cos\theta \\ y=\rho\,sen\theta \end{cases} \qquad \begin{matrix} 0\le\rho\le 1 \\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2} \end{matrix}$

$$\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho^3 sen^2\theta\cos\theta}{\rho^2\cos^2\theta+\rho^2 sen^2\theta}\cdot\rho\,d\theta\,d\rho$$

$$=\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho^{\cancel{4}\,2}}{\cancel{\rho^2}}\,sen^2\theta\cos\theta\,d\rho\,d\theta=\left[\frac{\rho^3}{3}\right]_0^1\left[\frac{sen^3\theta}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=$$

$$=\frac{1}{3}\cdot(1+1)=\frac{2}{3}$$

2) $f(x,y)=x^2+4y^3-2xy+4$

$f_x=2x-2y \qquad f_y=12y^2-2x$

$\nabla f=0 \iff \begin{cases} 2x-2y=0 \\ 12y^2-2x=0 \end{cases} \quad \begin{cases} x=y \\ \cancel{12}^{\,6}y^2-\cancel{2}y=0 \end{cases} \quad \begin{cases} x=y \\ y(6y-1)=0 \end{cases} \begin{matrix} y=0 \\ x=0 \\ \\ y=\frac{1}{6} \\ x=\frac{1}{6} \end{matrix}$

$(0,0)\ ,\ \left(\frac{1}{6},\frac{1}{6}\right)$

$f_{xx}=2 \qquad f_{xy}=-2 \qquad f_{yy}=24y$

$H^2 f(0,0)=\left|\begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}\right|=-4<0$ punto di sella

$H^2 f\left(\frac{1}{6},\frac{1}{6}\right)=\begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix}=8-4=4 \quad \det>0,\ f_{xx}\left(\frac{1}{6},\frac{1}{6}\right)>0$

matrice definita positiva,
punto di minimo relativo.

3) $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^2+y^2}}$$

$x=0 \quad \lim_{y\to 0}\frac{0}{\sqrt{y^2}}=0 \qquad y=0$ IDEM

$x=y \quad \lim_{x\to 0}\frac{x^3}{\sqrt{2x^2}}=0$

$x=y \quad \lim_{x\to 0} \frac{x^3}{\sqrt{2x^2}} = 0$

$$\left|\frac{xy^2}{\sqrt{x^2+y^2}} - 0\right| = |x| \cdot \left|\frac{y^2}{\sqrt{x^2+y^2}}\right| \leq |x| \left|\frac{y^2+x^2}{\sqrt{x^2+y^2}}\right| = |x|\sqrt{x^2+y^2} \to 0$$

oppure: $\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \end{cases}$

$$\left|\frac{\rho^3 \cos\theta\,\text{sen}^2\theta}{\rho}\right| = |\rho^2|\,\underbrace{|\cos\theta\,\text{sen}^2\theta|}_{\leq 1} \xrightarrow[\rho\to 0]{} 0$$

4) $F(x,y) = (e^{xy} + xye^{xy},\ x^2 e^{xy})$

- irrotazionale:

$$\frac{\partial F_1}{\partial y} = xe^{xy} + xe^{xy} + x^2ye^{xy} = 2xe^{xy} + x^2ye^{xy}$$

$$\frac{\partial F_2}{\partial x} = 2xe^{xy} + x^2ye^{xy}$$

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- irrotazionale su $\mathbb{R}^2$ semplicemente connesso $\Rightarrow$ conservativo.
- potenziale:

$F_1 = f_x,\ F_2 = f_y \qquad \int F_2\,dy = \int f_y\,dy \Rightarrow$

$f = \int x^2e^{xy}\,dy = x\int xe^{xy}\,dy = x\,e^{xy} + \phi(x). \Rightarrow$

$f_x = e^{xy} + xye^{xy} + \phi'(x)$. Ma $F_1 = e^{xy} + xye^{xy}$.

quindi $\phi'(x) = 0 \Rightarrow \phi = c$.

$f(x,y) = xe^{xy} + c$.

- lavoro su una curva chiusa $= 0$

5) $f(x,y,z) = 2xy \qquad S: \gamma(x,y) = \left(x, y, 1 - \frac{x^2}{2} - \frac{y^2}{2}\right)$,

$(x,y) \in [0,1]\times[0,1]$

$\iint_S f\,d\sigma = \int_0^1\int_0^1 2xy \cdot \sqrt{\ldots}\ \ldots$

$$\iint_S f\, d\sigma = \int_0^1 \int_0^1 2xy \cdot \sqrt{1+x^2+y^2}\, dx dy$$

$(x,y) \in [0,1] \times [0,1]$

$d\sigma = \sqrt{1+\gamma_x^2+\gamma_y^2}\, dx dy$

$= \sqrt{1+x^2+y^2}\, dx dy$

$$= \int_0^1 x \left(\int_0^1 2y \sqrt{1+x^2+y^2}\, dy \right) dx =$$

$$= \int_0^1 x \left[\frac{(1+x^2+y^2)^{3/2}}{3/2} \right]_0^1 dx =$$

$$= \int_0^1 \frac{2x}{3} \left((2+x^2)^{3/2} - (1+x^2)^{3/2} \right) dx =$$

$$= \frac{1}{3} \left[\frac{(2+x^2)^{5/2}}{5/2} - \frac{(1+x^2)^{5/2}}{5/2} \right]_0^1 =$$

$$= \frac{1}{3} \left(\frac{2}{5} \cdot 3^{5/2} - \frac{2}{5} \cdot 2^{5/2} - \frac{2}{5} \cdot 2^{5/2} + \frac{2}{5} \right) = \ldots$$

6) • $y'' - y' - 2y = 0$

$\lambda^2 - \lambda - 2 = 0 \qquad (\lambda - 2)(\lambda + 1) = 0$ ⟨ $\lambda = 2$, $\lambda = -1$

$y_0(x) = k_1 e^{2x} + k_2 e^{-x}$

• $y'' - y' - 2y = 3x - 2 \qquad y_S(x) = ax + b \qquad y_S' = a \qquad y_S'' = 0$

$y_S'' - y_S' - 2y_S = 3x - 2 \Rightarrow 0 - a - 2(ax+b) = 3x - 2$

$-a - 2ax - 2b = 3x - 2$

$$\begin{cases} -2a = 3 \\ -a - 2b = -2 \end{cases} \quad \begin{cases} a = -\frac{3}{2} \\ -2b = -2 - \frac{3}{2} \end{cases} \quad \begin{cases} a = -\frac{3}{2} \\ -2b = -\frac{7}{2} \end{cases} \quad \begin{cases} a = -\frac{3}{2} \\ b = \frac{7}{4} \end{cases}$$

$y_S(x) = -\frac{3}{2}x + \frac{7}{4}$

$$y(x) = k_1 e^{2x} + k_2 e^{-x} - \frac{3}{2}x + \frac{7}{4}$$