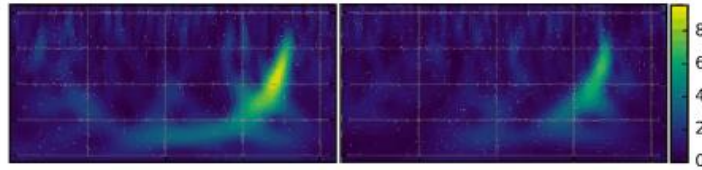


SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Brief Lecture Notes on General Relativity



Lecturer

*Leonardo Modesto*¹

Arranged and typed by TAs serving in 2018 and 2019

Jerry J.-Y ZHANG²

Jacquelyn H.-Y ZHU³

February 26, 2020

¹Department of Physics. 120, Research Building 2. lmodesto@sustc.edu.cn

²skycattmll@gmail.com

³jacquelynzhy@gmail.com

DISCLAIMER

This note doesn't cover all the information in Leonardo's lectures. You should refer to origin note, text book and Leonardo more often than this note. There might be mistakes due to our ignorance and carelessness, we will appreciate if you tell us after debugging.

Contents

I Preliminaries and Special Relativity	8
1 A Classical Perspective	9
1.1 Classical Mechanics	9
1.2 Galileo's Relativity Principle	9
1.2.1 In Inertial Frame	9
1.2.2 In Non-inertial Frame	11
1.2.3 Electromagnetic Field	12
2 When things move fast — Special Relativity	13
2.1 Special Relativity Principle	13
2.2 Event and Interval	13
2.3 Lorentz Transformation	16
2.3.1 How the time change	17
2.3.2 Space/Time-Like Interval and Light Cone	17
2.3.3 Lorentz Matrix and Hyperbolic Function	18
2.3.4 How the length change	19
2.4 Relativistic Mechanics	20
2.4.1 4-Velocity	20
2.4.2 Action and Lagrangian for a Free Relativistic Particle	20
2.4.3 Momentum and Energy	21
2.4.4 Force and Acceleration	21
2.4.5 4-Momentum	22
2.4.6 Least Action Principle	22
3 Relativistic particle in Electromagnetic Field*	23
3.1 Action	23
3.2 Particle Hamiltonian	23
3.3 From Action to EOM	24
3.3.1 Particle EOM	24
3.3.2 Field EOM	25
3.4 Electromagnetic Tensor	25

II	General Relativity	27
4	Fundamental	28
4.1	2 Axioms of General Relativity	28
4.1.1	Manifold	28
4.1.2	Equivalence Principle (EP)	29
4.1.3	So what is gravity?	30
4.2	See some definition in a nutshell from a new metric	31
4.2.1	Light Geodesic	31
4.2.2	Red Shift	31
4.2.3	Eliminating The Singularity	33
4.2.4	Horizon	33
4.3	Rotating Coordinates	34
4.4	Distance Measuring under General Relativity	34
5	Mathematical Formulation	36
5.1	Curvature demystified	36
5.2	Covariant Differentiation	38
5.3	Christoffel Symbol	39
5.3.1	Properties of Christoffel Symbol	39
5.4	Geodesics in General Relativity	40
5.4.1	Killing Vectors	40
5.5	Curvature Tensors	41
5.5.1	Sufficient and necessary condition of Flat Spacetime	42
5.5.2	Properties of Riemann Curvature Tensor	43
5.5.3	Contraction to Ricci Tensor and Scalar	43
5.5.4	Another form of Bianchi Identity and Einstein Tensor	44
5.6	Gravitational Field Equation	44
5.6.1	κ	47
5.6.2	Sufficient and necessary condition of Vacuum Spacetime	47
5.7	Energy-Momentum Tensor*	47
5.7.1	Conservation Law of Energy-Momentum Tensor	48
5.7.2	Examples of Energy-Momentum Tensor	48
6	Solutions to Einstein Equation	50
6.1	Vacuum Solutions	50
6.2	Blackholes	50
6.3	Swarzchild Metric (Spherical symmetry $SO(3)$ and time-invariant)	51
6.3.1	Solve the Metric out	51
6.3.2	Singularity	54
6.3.3	Collapse and Event Horizon	55
6.3.4	Time and Gravitational RedShift	57
6.3.5	Penrose Diagram	58
6.3.6	How Massive Particle Moves	61
6.3.7	How Light is Bend	61
6.4	Kerr Metric	64

<i>CONTENTS</i>	5
6.4.1 Singularity and horizons	65
6.4.2 Frame-dragging	66
6.4.3 Energy Extraction (Penrose Process)	66
6.4.4 Closed Timelike Curve (CTC)	67
6.4.5 Penrose Diagram	67
6.5 Reissner-Nordstrom Metric	67
6.5.1 Penrose Diagram	68
7 The Spacetime's Perturbation — the Gravitational Waves	70
7.1 Perturbation on Vacuum	70
7.1.1 TT Gauge	72
7.2 GW with Sources	72
7.2.1 GW from 2 Black holes	72
8 Cosmology	73
8.1 Perfect Fluid	73
8.1.1 Friedman Equations	73
8.2 Cosmological constant	74
9 Introduction to Nonlocal Gravity*	75
III Appendix	76
10 Notations*	77
10.1 Einstein Sum Convention	77
10.2 Ricci calculus	77
10.3 Scalar, Vector and Tensor	78
10.4 Dual Space	78
10.5 Covariant and Controvariant	78
11 Differential Geometry*	79
11.1 Topological Space	79
11.1.1 Some Basic Definition	79
11.1.2 Continuity	80
11.1.3 Connectness	80
11.1.4 Homeomorphic	81
11.1.5 Homotopy	81
11.2 Differential Manifold	82
11.2.1 Differentiable Manifold and Differentiable Map	82
11.2.2 Tangent Space	84
11.2.3 Cotangent Space	85
11.3 Riemann Geometry	85
11.3.1 (Pseudo) Riemannian Manifold	86
11.3.2 Covariant Derivative, Parellel Transport and Geodesic	87
11.3.3 Affine Connection	88

11.3.4	Metric Connection	89
11.4	From Equivalence Principle Directly to General Relativity	89
11.4.1	Geodesic Equation From the Equivalence Principle	89
11.4.2	Gravitational Action From Equivalence Principle	91
11.4.3	Newtonian Limit	91

Introduction

General Relativity first of all deals with Relativity, namely it is about the symmetries of Nature or mathematically the symmetries of the equations of motion (EoM) that describe the laws of nature. You are already familiar with: space and time translation invariance, rotation invariance, and Galilean relativity. Indeed, the laws of Newtonian mechanics (NM) are based on the latter symmetries. However, when the speed of system is closed to the speed of light NM is not correct anymore. We need a new relativity principle, namely: special relativity.

Let me introduce step by step the reasons that brought Einstein to introduce a new relativity principle.

Part I

**Preliminaries and Special
Relativity**

Chapter 1

A Classical Perspective¹

Since you are going to learn General Relativity, what is Relativity? Please Wiki it!

1.1 Classical Mechanics

As you saw in Analytical Mechanics, a point particle's **Lagrangian** is $L(q, \dot{q}, t)$, and the **Action** is $S = \int_{t_1}^{t_2} L dt$. By the Least Action Principle: $\delta S = 0$, we get the Euler-Lagrange Equation for a system whose degree of freedom is s^2 :

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0}, \quad i = 1, 2, \dots, s. \quad (1.1)$$

The Lagrangian $L = T - U = \sum_i \frac{1}{2} m v_i^2 - U(r_1, r_2, \dots, r_i)$, by Euler-Lagrange Equation, we can get **EOM**(Equation Of Motion):

$$m_i \frac{dv_i}{dt} = -\frac{\partial U}{\partial r_i} = F_i. \quad (1.2)$$

1.2 Galileo's Relativity Principle

1.2.1 In Inertial Frame

If the physics of a reference system K have no causes external to the reference system (simply no external forces is applied on K , the explanation could be found in the next subsection 1.2.2), we call K is **Inertial**. If there's another reference systems K' , and K' has a **constant** velocity V respect to K .

¹Our main reference book is Landau and Lifshitz's *The Classical Theory of Fields*.

²The derivation of Euler-Lagrangian Equation and its properties could be found in Landau and Lifshitz's *Mechanics*.

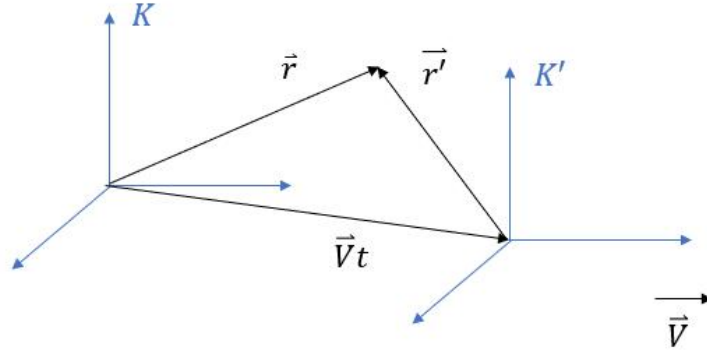


Figure 1.1: Motion in inertial frames

Galileo's Relativity Principle tells us the Law of Mechanics doesn't change under the following transformation:

$$\begin{cases} \vec{r} = \vec{r}' + \vec{V}t \\ t = t'. \end{cases} \quad (1.3)$$

Proof. Consider a free particle $L(v^2) = \frac{1}{2}mv^2$ in K . Take the derivation the first equation of equation 1.3 we get $\dot{\vec{r}} = \dot{\vec{r}}' + \vec{V}$, it is $\vec{v} = \vec{v}' + \vec{V}$. In K' , because $\frac{d\vec{v}'}{dt} = 0$, we get

$$\begin{aligned} L(v^2) &= \frac{1}{2}m(v^2 + V^2 - 2\vec{v}' \cdot \vec{V}) \\ &= \frac{1}{2}mv^2 + \underbrace{\frac{d}{dt}\left(\frac{1}{2}mV^2t\right) - \frac{d}{dt}(m\vec{r}' \cdot \vec{V})}_{\text{total derivative of a function}} \\ &\simeq \frac{1}{2}mv^2 \\ &= L(v^2). \end{aligned} \quad (1.4)$$

which means the **invariance** of the mechanics law, or K' is inertial too. Because a total derivative of a function can be neglected in a Lagrangian. \square

Checking the EOMs derived from the Euler-Lagrangian Equation in K and K' :

$$m \frac{d\vec{v}}{dt} = m \frac{d\vec{v}'}{dt} = 0, \quad (1.5)$$

which indicate the particle is doing a uniform motion in both frames.

A conclusion can be drawn that every reference frame has a constant velocity respect to an inertial frame is inertial.

Galilean invariance or Galilean relativity states that the laws of motion are the same in all **inertial frames**. Galileo Galilei first described this principle in 1632.

Our mother nature doesn't have the prejudice on a special coordinates system. In fact, the coordinates is created by our humans. So the nature of physics law should not be influenced by the coordinates choosing too.

1.2.2 In Non-inertial Frame

If the reference systems K' has a **non-constant** velocity $V(t)$ respect to K , in other words, K' is accelerating respect to K . We have $\vec{v} = \vec{v}' + \vec{V}(t)$.

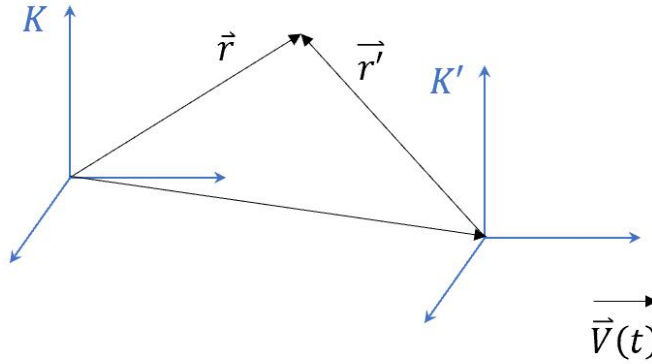


Figure 1.2: Motion in Non-inertial frame

Now in K , without the loss of generality, we add a potential U in particle's Lagrangian, which means that the particle is influenced by an internal conservative force \vec{F} : $L(v^2) = \frac{1}{2}mv^2 - U$. Generally, the potential is not the function of velocity \vec{v} . In K' , the Lagrangian is

$$\begin{aligned}
 L(v'^2) &= \frac{1}{2}m(v^2 + V(t)^2 - 2\vec{v} \cdot \vec{V}(t)) - U \\
 &= \frac{1}{2}mv^2 + \underbrace{\frac{1}{2}mV(t)^2}_{\text{total derivative}} - \left[\underbrace{\frac{d}{dt}(m\vec{r} \cdot \vec{V})}_{\text{total derivative}} - m \frac{d\vec{V}(t)}{dt} \cdot \vec{r} \right] - U \\
 &\approx \frac{1}{2}mv^2 + m \frac{d\vec{V}(t)}{dt} \cdot \vec{r} - U \\
 &= L(v^2) + m \frac{d\vec{V}(t)}{dt} \cdot \vec{r} - U.
 \end{aligned} \tag{1.6}$$

Using the Euler-Lagrangian Equation we obtain the EOM:

$$m \frac{d\vec{v}}{dt} = \underbrace{-\frac{\partial U}{\partial \vec{r}}}_{\vec{F}} + m \underbrace{\frac{d\vec{V}(t)}{dt}}_{\vec{F}_{K'}}. \quad (1.7)$$

We see there is a force $\vec{F}_{K'}$ imposed on the particle at any point in the non-inertial frame K' .

1.2.3 Electromagnetic Field

The discussion above is based on Newtonian Mechanics. You may have learned that Electromagnetism breaks the Galileos Relativity Principle. Try to apply Galileos Transformation on the Maxwell Equations and see the results.

In the end of 19th century, the correctness of classical electromagnetism theory was verified by many experiments. People tried to create 'Aether' to explain the incompatibility but was proved wrong by Michelson-Morley Experiment. In the same time, **vacuum speed of light** c is tested as a constant, which strongly subvert the Galileos Relativity Principle.

We should set a new physical system.

Chapter 2

When things move fast ——— Special Relativity

2.1 Special Relativity Principle

Just like Galileo's Relativity Principle, we have constraint condition in Special Relativity Principle, two simple assumptions constructed based on experimental evidence, although they are counter-intuitive:

- $c = \text{const}$, in any **inertial** reference frame;
- All physical laws are the same in any **inertial** reference frame.

Refer to any reference book if you want to know why Einstein proposed this. Let's find out something interesting by using this principle.

2.2 Event and Interval

There are 2 Events E_1 and E_2 in 2 reference frames K and K' . In K , E_1 is the event that the light signal γ (photon) departs from x_1, y_1, z_1 at time t_1 , and E_2 is the event that the light signal γ gets to x_2, y_2, z_2 at time t_2 .

Now, in K , we have the relationship:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = c^2(t_2 - t_1)^2. \quad (2.1)$$

And in K' , according to the assumption, we know the speed of light is also c ,

$$(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 = c^2(t'_2 - t'_1)^2. \quad (2.2)$$

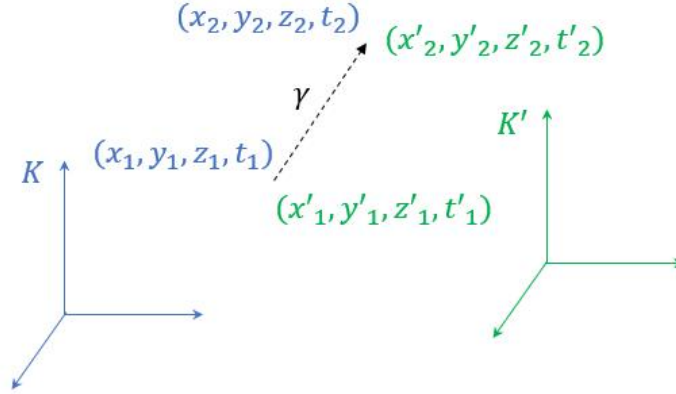


Figure 2.1: Events in 2 frames

Remark 2.1. You can write it simply:

$$\Delta \vec{x}_i \delta_{ij} \Delta \vec{x}_j = c^2 \Delta t^2. \tag{2.3}$$

And in matrix form, the square of Euclidean Distance is

$$(\vec{x}_1 - \vec{x}_2)(\vec{x}_1 - \vec{x}_2) = \begin{pmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}. \tag{2.4}$$

You will use matrix form often later. The matrix in between is called Metric. Here is

3-D Euclidean Metric $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Now we generally define the Interval:

Definition 2.2. The **Interval** of any two Events is¹:

$$s_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2}. \tag{2.5}$$

Obviously, the Interval of \$E_1\$ and \$E_2\$ in 2 reference frames \$K\$ and \$K'\$ are both 0.

We can draw the conclusion that if the Interval is 0 in one frame that it will also be 0 in any other frame because of the invariance of the light speed. And for the propagating of light, the Interval is always 0.

The Infinitesimal Interval is

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{2.6}$$

¹the signs of 4 items in Interval(+, +, +, -) are totally opposed to the description in Landau's (-, -, -, +), but it doesn't matter.

More precisely, this is the Infinitesimal Interval in flat space-time (the Minkowski space) which will be frequently mentioned in Part III. The signature is $(-, +, +, +)$ in this note. The opposite signature $(+, -, -, -)$ is assumed by some people (e.g. Landau), which does not influence the physical fact but you need to be careful during some calculation processes.

Remark 2.3. And in **matrix form** and using Einstein notation²,

$$\begin{aligned} ds^2 &= \begin{pmatrix} cdt & dy & dz & dx \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dy \\ dz \\ dx \end{pmatrix} \\ &= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= dx_\nu dx^\nu. \end{aligned} \quad (2.7)$$

The vector dx^ν is called 4-D vector. The matrix $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is so-called Minkowski

Metric $\eta_{\mu\nu}$. $\eta_{\mu\nu}$ is diagonal.

Any non-singular (non-degenerate) matrix has an inverse matrix, so does the Minkowski Metric matrix $\eta_{\mu\nu}$. The inverse $\eta^{\mu\nu}$ which could be obtained by the reciprocal relationship

$$\eta_{\mu\nu} \eta^{\nu\rho} = \delta_\mu^\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.8)$$

we find that they are actually the same:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.9)$$

As you see in Equation 2.7, the Minkowski Metric can ‘pull down’ and change the indices of a vector. In the contrary, its reciprocal can ‘pull up’ and change the indices³:

$$\eta^{\mu\nu} A_\nu = A^\mu; \quad \eta_{\mu\nu} A^\nu = A_\mu. \quad (2.10)$$

Because ds is a infinitesimal of the same order as ds' , which means

$$ds^2 = ad s'^2. \quad (2.11)$$

²Without using Einstein notation, the front vector should be written transposed, $ds^2 = \vec{x}^T \eta \vec{x}$.

³The reason is about covariant and contravariant vectors that you only need to be familiar with the operation so far. $A_0 = -A^0$ and $A_i = A^i$, $i = 1, 2, 3$. You can also refer to the Appendix and Landau’s *The Classical Theory of Fields* §6 Equation (6.2) ~ (6.6).

We can proof that $a = 1$. Please note that a does not related to t, x, y, z because the spatial uniformity, then a is only related to the relative velocity of two coordinate. For example, there are three coordinate K_1, K_2 and K . The relative velocity of K_1, K_2 to K is \vec{v}_1 and \vec{v}_2 , and the relative velocity between K_1 and K_2 is v_{12} . Then we can say

$$\begin{aligned} ds^2 &= a(\vec{v}_1)ds_1^2 \\ ds^2 &= a(\vec{v}_2)ds_2^2 \\ ds_1^2 &= a(v_{12})ds_2^2. \end{aligned} \quad (2.12)$$

Then it is not difficult to get the relationship

$$\frac{a(\vec{v}_1)}{a(\vec{v}_2)} = a(v_{12}). \quad (2.13)$$

Since v_{12} is related to both the amplitude of \vec{v}_1 and \vec{v}_2 , and the angle between \vec{v}_1 and \vec{v}_2 . The only way to make the equation always correct is that a is also not related to the relative velocity between the coordinate, and it is a constant. From the equation above we can get that

$$a(\vec{v}_1) = a(\vec{v}_2) = a(v_{12}) = 1. \quad (2.14)$$

That is

$$\boxed{ds^2 = ds'^2}. \quad (2.15)$$

It is also the mathematical representation of the immutability of the speed of light.

2.3 Lorentz Transformation

Lorentz Transformation is an isometry which preserves the interval. Here comes the Lorentz transformation of a 4-D vector $x'^\alpha = \Lambda_\beta^\alpha x^\beta$ and because Λ_β^α has constant components, the infinitesimal quantities transform like this: $dx'^\alpha = \Lambda_\beta^\alpha dx^\beta$. The components will be shown in the subsection 2.3.3.

The Interval has property of invariance:

$$\begin{aligned} ds'^2 &= \eta_{\gamma\delta} dx'^\gamma dx'^\delta \\ &= \eta_{\gamma\delta} (\Lambda_\mu^\gamma dx^\mu) (\Lambda_\nu^\delta dx^\nu) \\ &= \eta_{\gamma\delta} \Lambda_\mu^\gamma \Lambda_\nu^\delta dx^\mu dx^\nu \\ &= \eta_{\mu\nu} dx^\mu dx^\nu = ds^2. \end{aligned} \quad (2.16)$$

$\eta_{\gamma\delta}$ has the same form as $\eta_{\mu\nu}$ which is Minkowski metric.

So the Minkowski metric transforms in the following way:

$$\eta_{\mu\nu} = \eta_{\gamma\delta} \Lambda_\mu^\gamma \Lambda_\nu^\delta, \quad (2.17)$$

or simplified not using the Einstein notation⁴:

$$\eta = \Lambda^T \eta \Lambda. \quad (2.18)$$

⁴Like Equation 2.7, Equation 2.16 can be written in the normal form $ds'^2 = (\Lambda \vec{x})^T \eta \Lambda \vec{x}$.

2.3.1 How the time change

Suppose a particle is moving only on x axis with a velocity $\frac{dx}{dt}$ respective to the laboratory reference frame K and totally rest in another reference frame K' followed the particle tightly.

From Interval invariant, we can have:

$$-c^2 dt^2 + dx^2 = -c^2 dt'^2 + dx'^2 \quad (2.19)$$

$$c^2 dt^2 \left(1 - \frac{dx^2}{c^2 dt^2}\right) = c^2 dt'^2 \left(1 - \frac{dx'^2}{c^2 dt'^2}\right) \quad (2.20)$$

Where $\frac{dx'^2}{c^2 dt'^2} = 0$, so that we have:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{dx'^2}{c^2 dt'^2}}} = \gamma \Delta t' > \Delta t. \quad (2.21)$$

which is Lorentz Transformation of time. $\Delta t'$ is the time interval of the particle measured by a clock set on the particle. t' is so-called **Proper time** and often be marked as $d\tau$. You can see time elapses slower on a train pass through before you if it's velocity is significant close to the speed of light. That's also why scientist always observe high energy particles from the cosmic rays lives longer than its average life.

Counter-intuitively, If you set K' to be at rest and K to move in opposite direction, you will find $\Delta t' < \Delta t$.

We will call the factor $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ as **Lorentz factor** γ later. γ increases to the infinity when velocity approaches to the light speed c . Here comes some interesting phenomenon like twins orthodox (You can search it in wikipedia). Generally, the square of infinitesimal proper time is

$$d\tau^2 = dt^2 - \frac{1}{c^2} dx^2 - \frac{1}{c^2} dy^2 - \frac{1}{c^2} dz^2 = -\frac{1}{c^2} ds^2. \quad (2.22)$$

When some books adopt opposite signature $(+, -, -, -)$ and $c = 1$, $d\tau^2 = ds^2$. They prefer to use proper time than the interval (e.g. Weinberg).

2.3.2 Space/Time-Like Interval and Light Cone

Definition 2.4. We call a Interval space-like if $I > 0$, and time-like if $I < 0$. $I = 0$ denotes the light propagation or things propagate at speed of c .

We draw a Light Cone whose boundary is delineated by light. The 4-D space in the up-part of the cone is absolutely future, the counterpart is absolutely past. Outside is absolutely separate. A particle, like you, can only travel along a curve upward inside the cone.

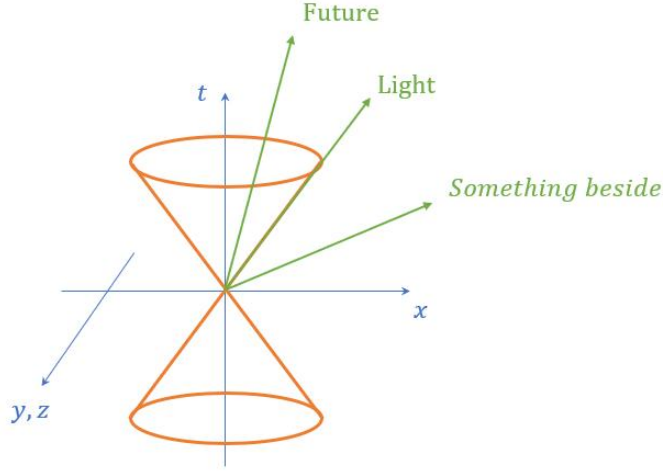


Figure 2.2: Light cone, actually is a 4-D super-cone

2.3.3 Lorentz Matrix and Hyperbolic Function

The invariance of the length $l^2 = x^2 + y^2 + z^2$ in Euclidean space provides the counter-clockwise rotation in xy plane:

$$\begin{cases} x' = x \cos \phi - y \sin \phi \\ y' = x \sin \phi + y \cos \phi \\ z' = z \end{cases} \quad (2.23)$$

It's because $\sin^2 \phi + \cos^2 \phi = 1$.

Now we know the invariance of the interval $s^2 = -c^2t^2 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 = s'^2$. Like the rotation in the Euclidean space, we can do the classical rotation between spatial coordinates, in other words, in xyz sub-space. The rotation is called **Lorentz Rotation**.

For example, The matrix form of the rotation in xy plane is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

The minus sign before t in the sum of interval allows us do a pseudo-rotation:

$$\begin{cases} x' = x \cosh \zeta + ct \sinh \zeta \\ ct' = x \sinh \zeta + ct \cosh \zeta \end{cases} \quad (2.24)$$

It's because $\cosh^2 \zeta - \sinh^2 \zeta = 1$. The initial configuration is $\tanh \zeta = \frac{v}{c}$, which is $\cosh^2 \zeta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\sinh \zeta = \frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$.

Here we can find out the matrix form of Λ_{β}^{α} is $\begin{pmatrix} \cosh \zeta & \sinh \zeta & 0 & 0 \\ \sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We call this

kind of Lorentz Transformation the **Lorentz Boost**, a translation with a constant velocity in x direction. If the direction of 'Boost' changes oppositely, the matrix would be

$$\begin{pmatrix} \cosh \zeta & -\sinh \zeta & 0 & 0 \\ \sinh \zeta & -\cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There are 3 operations (x, y, z) in Lorentz Boost and 3 operations (xy, yz, xz) in Lorentz Rotation.

2.3.4 How the length change

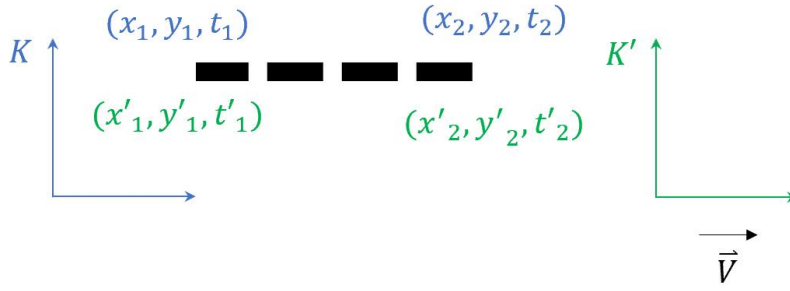


Figure 2.3:

From figure 2.3, the length of the ruler measured in K is $\Delta l = x_1 - x_2$, and :

$$\begin{cases} x_1 = \frac{x'_1 + Vt'_1}{\sqrt{1 - \frac{v^2}{c^2}}} \\ x_2 = \frac{x'_2 + Vt'_2}{\sqrt{1 - \frac{v^2}{c^2}}} \end{cases} \quad (2.25)$$

The measuring in K' is instant, so $t'_1 = t'_2 = t$, then:

$$x_1 - x_2 = \gamma(v)(x'_1 - x'_2), \quad (2.26)$$

which also means:

$$l' = \sqrt{1 - \frac{v^2}{c^2}} l = \frac{l}{\gamma} < l. \quad (2.27)$$

That is Lorentz Transformation of **length**. The length in moving direction decrease when you move. However, the height or width, perpendicular to the moving direction remains the same.

If you set K' rather than K to be at rest, you will find $l' > l$.

2.4 Relativistic Mechanics

2.4.1 4-Velocity

The Newtonian velocity $\vec{v} = \frac{d\vec{x}}{dt}$, here t is a scalar.

The Newtonian velocity can varies under SO(3) group (Rotation) $\vec{x}_i = O_{ij}\vec{x}_j$, which means $\vec{v}_i = O_{ij}\vec{v}_j$. This is simply because $\frac{dO_{ij}}{dt} = 0$ (Expand the formula before you will find why).

4-velocity require Lorentz covariant $u^\alpha = \Lambda_\beta^\alpha u^\beta$ under $x^\alpha = \Lambda_\beta^\alpha x^\beta$:

$$\begin{aligned} u^\alpha &= \frac{dx^\alpha}{dh} = \frac{d\Lambda_\beta^\alpha x^\beta}{dh} \\ &= \Lambda_\beta^\alpha \frac{dx^\beta}{dh} + x^\beta \frac{d\Lambda_\beta^\alpha}{dh} \\ &= \Lambda_\beta^\alpha u^\beta. \end{aligned} \quad (2.28)$$

So the second term $\frac{d\Lambda_\beta^\alpha}{dh}$ should be 0. The only reasonable quantity h in relativity to get 4-velocity is also a scalar — the Interval s , finally:

Definition 2.5. The 4-velocity

$$u^\alpha = \frac{dx^\alpha}{ds}. \quad (2.29)$$

And 4-acceleration

$$w^\alpha = \frac{d^2 x^\alpha}{ds^2}. \quad (2.30)$$

We can get

$$u^\alpha u_\alpha = \frac{dx^\alpha}{ds} \frac{dx_\alpha}{ds} = \frac{ds^2}{ds^2} = 1. \quad (2.31)$$

2.4.2 Action and Lagrangian for a Free Relativistic Particle

The Action should be a scalar, and a differential of the first order, the only reasonable quantity is the integral of the interval ds (the ds below is the infinitesimal interval rather than the infinitesimal action)

$$S = -\alpha \int_a^b ds = -\alpha \int \sqrt{\eta_{ij} dx^i dx^j}, \quad (2.32)$$

here α is a constant (why $-\alpha$? see Landau §8). Inspired by the Action of classical free particle $S = \int \frac{1}{2}mv^2 dt$, we subtract out a dt :

$$S = -\alpha \int_{t_1}^{t_2} \sqrt{\eta_{ij} dx^i dx^j} dt = -\alpha \int_{t_1}^{t_2} c \sqrt{1 - \frac{d\vec{x}}{c^2 dt}} dt = -\alpha \int_{t_1}^{t_2} c L dt. \quad (2.33)$$

And we know that the Lagrangian should become $\frac{1}{2}mv^2$ when v approaches to 0. We expand L respect to v , we can find $\alpha = mc$. So,

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}, \quad (2.34)$$

and

$$S = -mc \int ds \quad (2.35)$$

2.4.3 Momentum and Energy

By definition in Classical Mechanics, the Momentum can be directly derived from Lagrangian:

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = m\gamma\vec{v}. \quad (2.36)$$

And the energy

$$E = \vec{p}\vec{v} - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = m\gamma c^2 \approx mc^2 + 0 + \frac{1}{2}mv^2, \quad (2.37)$$

The first term is stationary energy which is well known by human. The second derivative is 0, the third term is kinetic energy.

2.4.4 Force and Acceleration

$$\begin{aligned} \vec{F} &= \frac{d\vec{p}}{dt} = m\gamma\dot{\vec{v}} + m\dot{\gamma}\vec{v} \\ &= m\gamma\vec{a} + m\gamma^3 \frac{(\vec{v} \cdot \vec{a})}{c^2} \vec{v} \\ &\neq \vec{a}, \end{aligned} \quad (2.38)$$

which is totally different with classical situation. (Not only different in modulus but also in direction.)

We can also define a 4-Force using 4-acceleration just like classical equation

$$F^\alpha = m \frac{d^2 x^\alpha}{d\tau^2} = m\gamma^2 \frac{d^2 x^\alpha}{dt^2}. \quad (2.39)$$

2.4.5 4-Momentum

Somebodies find a interesting quantity

$$\vec{p}^2 - \frac{E^2}{c^2} = m^2 \gamma^2 v^2 - \frac{m^2 \gamma^2 c^4}{c^2} = -m^2 c^2. \quad (2.40)$$

So we define a 4-momentum.

Definition 2.6. The 4-momentum is

$$p^\alpha = \left(\frac{E}{c}, \vec{p} \right). \quad (2.41)$$

Which have the quantity that $p^\alpha p_\alpha = -m^2 c^2$. So

$$p^\alpha = \Lambda^\alpha_\beta p^\beta. \quad (2.42)$$

Just like 4-velocity. The convenient transformation under the Lorentz's framework is the reason to pursue 4-quantity like this.

And if $m = 0$,

$$\vec{p}^2 - \frac{E^2}{c^2} = 0. \quad (2.43)$$

It is

$$E = c|p|, \quad (2.44)$$

which is reasonable for massless particle (like photon).

2.4.6 Least Action Principle

By $ds = \sqrt{dx_i dx^i}$,

$$\begin{aligned} \delta S &= -mc \delta \int_a^b ds \\ &= -mc \int_a^b \frac{dx_i \delta dx^i}{ds} \\ &= -mc \int_a^b u_i dx^i \\ &= -mc \left(u_i \delta x^i \Big|_a^b - \int_a^b \delta x^i \frac{du_i}{ds} ds \right) \\ &= 0 + mc \int_a^b \delta x^i \frac{du_i}{ds} ds \\ &= 0. \end{aligned} \quad (2.45)$$

Which derives the consistency of a free particle's 4-velocity:

$$\frac{du_i}{ds} = 0, \quad (2.46)$$

or just $du_i = 0$.

Chapter 3

Relativistic particle in Electromagnetic Field*

3.1 Action

The total action is

$$S = - \int_a^b \left(\underbrace{mc ds}_{\text{relativistic}} - \underbrace{\frac{e}{c} A_i dx^i}_{\text{particle in EM}} - \underbrace{\frac{1}{4} d^4 x F_{\mu\nu} F^{\mu\nu}}_{\text{Field}} \right). \quad (3.1)$$

Potential in 4 dimension is: $A^\alpha = (\phi, \vec{A})$.¹

The field part action is formed in such a way that only reasonable scalar which also possesses linear property of electromagnetic field (superposition of charge or field is allowed) is $F_{\mu\nu} F^{\mu\nu}$.²

3.2 Particle Hamiltonian

From what we have talked before, we can get

$$S = \int_{t^1}^{t^2} \left(-mc^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{e}{c} \phi + \frac{e}{c} \vec{A} \cdot \vec{v} \right) dt, \quad (3.2)$$

Then

$$L = \frac{dS}{dt} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{e}{c} \phi + \frac{e}{c} \vec{A} \cdot \vec{v}. \quad (3.3)$$

Because momentum $\vec{p} = \frac{\partial L}{\partial \vec{v}}$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c} \vec{A}, \quad (3.4)$$

¹You can find the potential in the second half of Griffith's *Introduction to Electrodynamics*.

²For detail see Landau §27.

and Hamiltonian is

$$H = \vec{v} \frac{\partial L}{\partial \vec{v}} - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\phi. \quad (3.5)$$

Then lets calculate. By using 2 equations above, we get

$$\begin{aligned} \frac{(H - e\phi)^2}{c^2} &= m^2 c^2 + \left(\vec{p} - \frac{e}{c} \vec{A}\right)^2 \\ &= m^2 c^2 + \frac{m^2 v^2}{1 - \frac{v^2}{c^2}} \\ &= \frac{m^2 c^2 - m^2 c^2 \frac{v^2}{c^2} + m^2 v^2}{1 - \frac{v^2}{c^2}} \\ &= \frac{m^2 c^2}{1 - \frac{v^2}{c^2}}. \end{aligned} \quad (3.6)$$

So that

$$H = \sqrt{m^2 c^4 + c^2 \left(\vec{p} - \frac{e}{c} \vec{A}\right)^2} + e\phi. \quad (3.7)$$

3.3 From Action to EOM

3.3.1 Particle EOM

Take the EOM of charged particle in the electromagnetic field as an example. We have got Equation 3.4. Remember 1.1 the Euler-Lagrangian Equation $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ we can get

$$\begin{aligned} \frac{\partial L}{\partial \dot{\vec{q}}} &= \frac{m\dot{\vec{q}}}{\sqrt{1 - \frac{\dot{\vec{q}}^2}{c^2}}} + \frac{e}{c} \vec{A} = \vec{p} + \frac{e}{c} \vec{A}, \\ \frac{\partial L}{\partial \vec{q}} &= \frac{e}{c} \nabla(\vec{A} \cdot \dot{\vec{q}}) - e\nabla\phi = \frac{e}{c} (\dot{\vec{q}} \cdot \nabla) \vec{A} + \frac{e}{c} \dot{\vec{q}} \times (\nabla \times \vec{A}) - e\nabla\phi. \end{aligned} \quad (3.8)$$

Then Euler-Lagrangian Equation can be written as

$$\frac{d}{dt} \left(\vec{p} + \frac{e}{c} \vec{A} \right) = \frac{e}{c} (\dot{\vec{q}} \cdot \nabla) \vec{A} + \frac{e}{c} \dot{\vec{q}} \times (\nabla \times \vec{A}) - e\nabla\phi. \quad (3.9)$$

Because $\frac{d}{dt} \vec{A} = \frac{\partial \vec{A}}{\partial t} + \nabla \vec{A} \cdot \dot{\vec{q}}$, we can finally get

$$\frac{d\vec{p}}{dt} = -\frac{e}{c} \frac{\partial \vec{A}}{\partial t} - e\nabla\phi + \frac{e}{c} \dot{\vec{q}} \times (\nabla \times \vec{A}), \quad (3.10)$$

which is the EOM. The left part $\frac{d\vec{p}}{dt}$ represents the force \vec{F} , and in the right part, $-\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \nabla\phi$ is the electric field strength \vec{E} and $\nabla \times \vec{A}$ is the magnetic field strength \vec{B} .

Finally we successfully obtained the EOM that we have learned in our high school, the expression of **Lorentz Force** :

$$\vec{F} = \vec{E}e + \vec{v} \times \vec{B}e. \quad (3.11)$$

3.3.2 Field EOM

Also you can calculate the EOM of the field from the field part of Action. This is one of your exercises. The result should be Maxwell equations(Both Homogeneous and non-homogeneous).

3.4 Electromagnetic Tensor

Definition 3.1. Electromagnetic Tensor is:

$$\begin{aligned} F^{\alpha\beta} &= \partial^\alpha A^\beta - \partial^\beta A^\alpha \\ &= -F^{\beta\alpha} \end{aligned} \quad (3.12)$$

We can get the matrix form of Electronic Tensor:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}. \quad (3.13)$$

We can see that it is **anti-symmetrical**, satisfying $F^{\alpha\beta} = -F^{\beta\alpha}$. If we do an inner product, we will get a result which is Lorentz invariant. That's also why we want the tensor.

$$F_{\alpha\beta}F^{\alpha\beta} = 2(B^2 - \frac{E^2}{c^2}) = \text{invariant}. \quad (3.14)$$

Relationship with Electromagnetism

From equation (3.1) we know the total action, in which $-\frac{1}{4} \int d^4x F_{\mu\nu}F^{\mu\nu}$ is electromagnetic field term. Then we can get Lagrangian:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{2\mu_0} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \end{aligned} \quad (3.15)$$

Take it into Lagrange Equation, we can get:

$$\partial_\nu F^{\mu\nu} = 0 \quad (3.16)$$

Maxwell Equation in Electromagnetic Tensor Form

In Maxwell Equation, $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and $\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$ can be written as:

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta \quad (3.17)$$

Where $J^\alpha = (c\rho, \vec{J})$ is four-current.

Also, $\nabla \cdot \vec{B} = 0$ and $\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0$ can be written as:

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \quad (3.18)$$

Where $F_{\alpha\beta,\gamma} = \frac{\partial F_{\alpha\beta}}{\partial x^\gamma}$, and this formula is called **Bianchi Identity**.
Try to prove it.

Part II

General Relativity

Chapter 4

Fundamental

The rough difference between 'general' and 'special' is the existence of gravitation.

4.1 2 Axioms of General Relativity

- The space-time is a 4-Dimensional **manifold**;
- Local Inertial Reference Frame(LIRF) can be always found in the neighborhood of any point in the space-time.

4.1.1 Manifold

Definition 4.1. A manifold is a topological space that **locally** resembles Euclidean space near each point.

For example, the surface of the Earth is a 2-D manifold, where locally is seemed flat but generally a sphere. Please refer to the Appendix to seek strict definition.

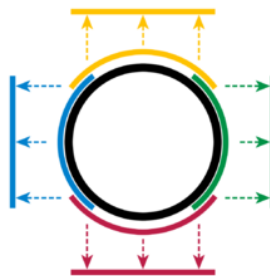


Figure 4.1: The four charts each map part of the circle to an open interval, and together cover the whole circle.

4.1.2 Equivalence Principle (EP)

The natural state is the Free-Fall. A stationary reference frame in a gravitational field is non-inertial. The EP claims that an inertial reference frame for Einstein Free-Fall Reference Frame is indistinguishable from a Newtonian inertial frame located far from any gravitational source. There are two levels to describe the principle:

- Weak EP: only influence the motion of particles due to gravity.
- Strong EP: the equivalence is extended to all physical phenomenon.

EP is only local because gravitational fields can be non-uniform. The LIRF given by the second Axiom has the following properties:

- In LIRF the physics is locally Lorentzian.
- The LIRF is identified by the absence of gravity.

A inertial reference frame is a frame that the body inside it isn't subject to the force. Now we needn't to remove that are universal (gravity).

Proof. Let's consider 2 situations.



Figure 4.2: A



Figure 4.3: B

For picture A:

$$m\ddot{\vec{y}} = R\vec{y} - mg\vec{y} \quad (4.1)$$

Where R is normal force. For $\ddot{y} = 0$ we can get:

$$R = mg \quad (4.2)$$

For picture B:

$$m\ddot{\vec{y}} = \vec{R} + \vec{F}_{eff} \quad (4.3)$$

Where $F_{eff}^{\vec{y}} = ma_{rel}^{\vec{y}}$, y' is the direction of the reference frame according to the rocket. So we can conclude that $\ddot{\vec{y}} = 0$. Also we have: $\vec{a} = \vec{a}' + a_{rel}^{\vec{y}}$, where $\vec{a} = g\vec{y}$, $\vec{a}' = \ddot{\vec{y}} = 0$. So that $a_{rel}^{\vec{y}} = \vec{a} = g\vec{y}$. Then we can conclude

$$R = mg, \quad (4.4)$$

which means $A = B$. □

EP tells us the gravitational force is equivalent to the acceleration of a reference frame that is not effected by the gravity.

EP also tells us the gravitational mass m_g , the one in the Newtonian gravitational formula $F = \frac{Gm_gM_g}{r^2}$ is the same as the one creating the inertia $F = m_t a$.

For every point in the spacetime, we can choose a free falling reference frame at every point whether there is gravitational field or not, Although the reference frame could be valid in very small region near the point due to strong homogeneous of the gravitational field. The first Axiom, which could be seen reasonable here: the spacetime is locally flat.

4.1.3 So what is gravity?

LIRF is free of the concept of the gravitation.

Since we can choose a LIRF at every point in a gravitational field according to EP, the gravitation at a point can be replaced by some property determined by the LIRF of the point.

The continuous changing of gravitational force in the space is replaced by the continuous changing of LIRFs in the space.

Therefore, we just need to study the property of the spacetime manifold, i.e. the difference of choosing coordinate systems at every point on the manifold, if we want to study the gravity. The details on a manifold can be described by sort of concepts like curvatures, covariant derivatives in Differential Geometry which will be introduced in Chapter 5.

Now we can answer the question at the very beginning of Chapter 1: General Relativity is a symmetry theory under general coordinate transformation and here the coordinates transformation is not we choosing other coordinate systems globally but the coordinate changing between different points on the manifold (locally).

But first let us just play a model by global coordinate transformation in Section 4.2.

4.2 See some definition in a nutshell from a new metric

We are just considering a new metric called **Rindler Metric**¹

$$ds^2 = \eta^2 d\xi^2 - d\eta^2 - dy^2 - dz^2. \quad (4.5)$$

Roughly the matrix form is $M_4 = M_2(\xi, \eta) \times E_2(y, z)$. E_2 means 2-D Euclidean.

The metric is based on the observer who is accelerating at a constant acceleration undergoing a hyperbolic motion due to special relativity.

The metric has some properties:

- Diagonal;
- $\xi = \text{const}$, $|\eta| = \left| \int_0^\eta d\eta' \right|$;
- $\eta = \text{const}$, $d\tau = |\eta| d\xi$.

Rindler spacetime is flat because it can fully transfer into Minkowski Metric using coordinate transformation. See Subsection 4.2.3.

4.2.1 Light Geodesic

Definition 4.2. **Geodesic** is the “shortest” path between two points on a manifold.

The special case is a straight line in flat Euclidean space, or a segment of a great circle on the surface of the Earth. You must have known the light geodesic in Minkowski Spacetime is the border of the Light cone.

We have proved that $ds^2 = 0$ is satisfied in any coordinates, so it is also a truth in (ξ, η, y, z) coordinate. Lets consider the light is traveling only in x-t plane ($dy = dz = 0$), then:

$$\eta d\xi = \pm d\eta. \quad (4.6)$$

After an integration, we can get:

$$\eta = \alpha e^{\pm \xi}, \quad (4.7)$$

it is same to the form:

$$\xi = \pm \ln \eta + \text{const}. \quad (4.8)$$

which is the trajectory(geodesic) of the light in this coordinates, as the picture shows. You can see you will get a **singularity** when $\eta = 0$.

4.2.2 Red Shift

You can easily find the relation $\Delta\xi_1 = \Delta\xi_2 = \Delta\xi$, suppose that:

$$\begin{cases} \Delta\tau_1 &= \eta_1 \Delta\xi \\ \Delta\tau_2 &= \eta_2 \Delta\xi \end{cases}. \quad (4.9)$$

¹https://en.wikipedia.org/wiki/Rindler_coordinates

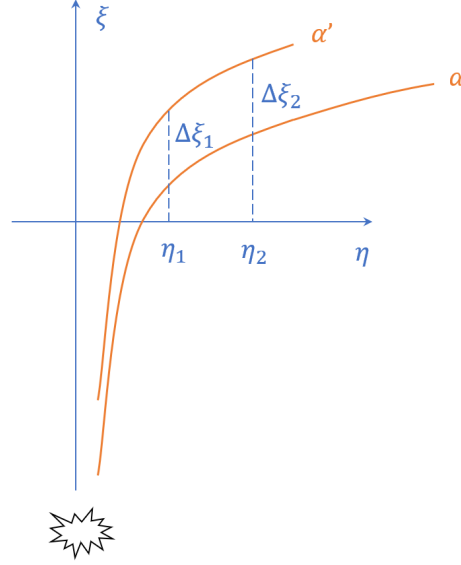


Figure 4.4: Geodesic of Light

If $\eta_1 \approx \eta_2$ and $\eta_2 = \eta_1 - \Delta\eta$. We can get:

$$\Delta\tau_2 = \left(1 + \frac{\Delta\eta}{\eta}\right)\Delta\tau_1, \quad (4.10)$$

let N be the number of wave peaks of one light beam,

$$\nu_1\Delta\tau_1 = \nu_2\Delta\tau_2 = N. \quad (4.11)$$

So

$$\nu_2 \simeq \left(1 - \frac{\Delta\eta}{\eta}\right)\nu_1 < \nu_1. \quad (4.12)$$

which represents **red shift**.

Einstein realized that $\Delta\eta \sim h$ which is the distance the light travel, and $\eta \sim \frac{1}{g}$.

$$\nu_2 \simeq \left(1 - \frac{gh}{c^2}\right)\nu_1 < \nu_1, \quad (4.13)$$

c^2 is for dimensional reason.

The redshift caused by accelerating is equivalence to the gravitational redshift, see the redshift caused by a stationary spherical blackhole at Subsection 6.3.4².

²See also https://en.wikipedia.org/wiki/Gravitational_redshift

4.2.3 Eliminating The Singularity

Consider a coordinates transformation $(\xi, \eta, y, z) \rightarrow (t, x, y, z)$: If $\eta > 0$,

$$\begin{cases} x = \eta \cosh \xi \\ t = \eta \sinh \xi, \end{cases} \quad (4.14)$$

if $\eta < 0$,

$$\begin{cases} x = \eta \sinh \xi \\ t = \eta \cosh \xi, \end{cases} \quad (4.15)$$

we get Minkowski spacetime

$$ds^2 = \eta^2 d\xi^2 - d\eta^2 - dy^2 - dz^2 \rightarrow \underbrace{dt^2 - dx^2 - dy^2 - dz^2}_{\text{Minkowski}}. \quad (4.16)$$

The coordinate transformation do not depends on the position, so the system is globally Minkowski.

What's the physical meaning of the result? Only by coordinates transformation, time and space are detached, and there's **no singularity!**

4.2.4 Horizon

$$x^2 - t^2 = \eta^2 \quad (4.17)$$

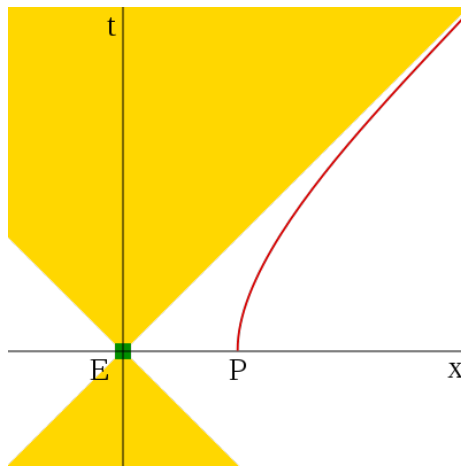


Figure 4.5: Horizon of a accelerating particle

4.3 Rotating Coordinates

A rotating frame along z axis at a angular velocity Ω can be described by coordinates transformation:

$$\begin{cases} x &= x' \cos \Omega t - y' \sin \Omega t \\ y &= x' \sin \Omega t + y' \cos \Omega t . \\ z &= z' \end{cases} \quad (4.18)$$

Then the interval can be described as

$$ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt'^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt' - 2\Omega x' dy' dt'. \quad (4.19)$$

Now the metric is not diagonal.

However, you should now that both Rindler and Rotating coordinates don't change the flatness of the Minkowski Spacetime.

Remark 4.3. Superluminal?

Try to compute the ds^2 of the star 1 light-year away under the reference frame stationary with a 1 r/s-revolving observer. Some may opine that the star moves at the speed of 2π ly/s. Explain if the star is real superluminal.

Certainly, the answer is no.

Why we do so many coordinate transformations? Because our nature properties do not depend on the man-made coordinates. We would like to seek the different properties through out different coordinate systems and find the true one.

4.4 Distance Measuring under General Relativity

How to measure the length of AB generally? We use photon as our messenger, travel toward the target and then back immediately. The total time it required is

$$d\tau = \frac{\sqrt{g_{00}}}{c} dt = \frac{\sqrt{g_{00}}}{c} (dx^{\alpha(2)} - dx^{\alpha(1)}), \quad (4.20)$$

then the distance will be

$$dl^2 = \frac{c^2}{4} d\tau^2. \quad (4.21)$$

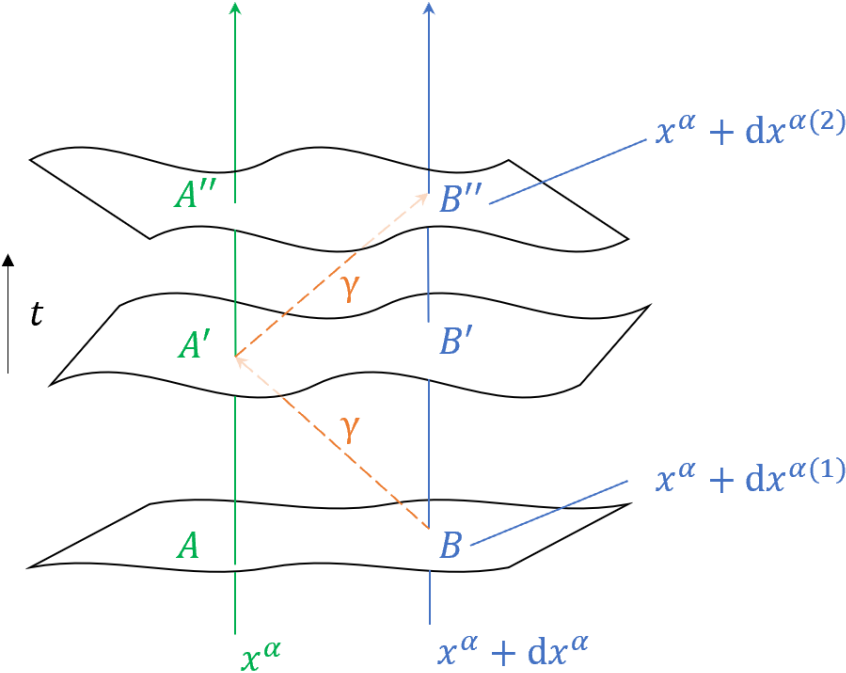


Figure 4.6: Measuring the distance

Chapter 5

Mathematical Formulation

Differential Geometry would not be requisite but it would be better if you have handled it well. Concepts here are mostly counter-intuitive so be careful.

5.1 Curvature demystified

The property that is meant by the "curvature" of space is not curvature in the usual sense, or even in the sense of simple mathematics (classical differential geometry). Actually it is an internal property of a manifold. Blame C. F. Gauss for this, as we owe the terminology to him. Just to make this point explicitly, "curve" originally meant a curved line, but in Gauss' terms a 1-D space (line) cannot be curved! Besides, the surface of a cylinder, which is thought to be curved in our daily lives, is actually flat. Here are some examples of curvature:

- Spacetime Curvature caused by Tidal Force:

Tidal Force is a phenomenon caused by gradient in gravitational field. The reason you see the sea surges up and down is the tidal forces generated by the Sun and the Moon.

We begin with a tiny free-falling elevator located near a point-like gravitational source at the beginning.

$$\begin{aligned} m(R+z)\ddot{z} &= -\frac{GMm}{(R+z)^2}\hat{r} \\ &= -\frac{GMm}{R^2\left(1+\frac{z}{R}\right)^2}\hat{r} \\ &\approx \underbrace{-\frac{GMm}{R^2}\hat{r}}_{m\ddot{r}} + \underbrace{\frac{2GMmz}{R^3}\hat{r}}_{m\ddot{z}}. \end{aligned} \tag{5.1}$$

Then you can find the acceleration is depending on the vertical distance from the center of the elevator, smaller at the top and bigger at the bottom; and dependent

on the absolute value of the horizontal distance from the center:

$$\begin{aligned}\ddot{z} &= \frac{2GM}{R^3} \hat{z} \\ \ddot{y} &= -\frac{GM}{R^3} \hat{y} \\ \ddot{x} &= -\frac{GM}{R^3} \hat{x},\end{aligned}\tag{5.2}$$

you can find there's a locally flat spacetime (no g-force) near the center, and the elevator is going to be longer and thinner if the elevator is soft enough.

The equation can be rewrite as

$$\frac{d^2 \xi^i}{dt^2} = -R^i_{0k0} \xi^k,\tag{5.3}$$

where $\xi^i = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and the components of the curvature are

$$\begin{aligned}R^1_{010} &= R^2_{020} = -\frac{GM}{R^3}; \\ R^3_{030} &= \frac{2GM}{R^3}.\end{aligned}\tag{5.4}$$

- Curvature on the surface of a cylinder with radius R :

Theorem 5.1. Gauss Curvature is the product of two principal curvatures $K = \kappa_1 \kappa_2$.¹

Therefore, cylinder surface is flat:

$$K = \kappa_1 \kappa_2 = \frac{1}{R} \frac{1}{\infty} = 0.\tag{5.5}$$

In fact, we can roll a flat paper into a cylinder, the process is actually a coordinate transformation: change x to θ and keep y .

- Curvature on a sphere with radius a :

On a sphere,

$$\sin \frac{\xi}{2a} = \sin \frac{\xi_0}{2a} \cos \phi,\tag{5.6}$$

when ξ is small,

$$\xi = \xi_0 \cos \phi = \xi_0 \cos \frac{s}{a}.\tag{5.7}$$

Now there's a non-zero Gaussian Curvature

$$K = \frac{1}{a^2} = -\frac{1}{\xi} \frac{d^2 \xi}{ds^2}.\tag{5.8}$$

Obviously you can not cut, stretch the paper shell of a globe to turn it into a flat paper. However, if we cut a very small piece of paper at any point on the shell, we get a approximately flat paper. Remember the concept of 'manifold'.

¹Please refer to https://en.wikipedia.org/wiki/Gaussian_curvature

5.2 Covariant Differentiation

First Let's see how the vector changes on a manifold, then we can precisely define the curvature at any point on an arbitrary manifold.

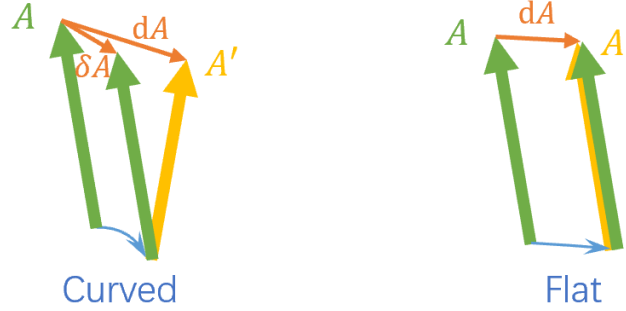


Figure 5.1: How vector changes in space

A covariant vector is transformed according to

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k. \quad (5.9)$$

Take the derivation we naturally get

$$\begin{aligned} dA_i &= \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k d\frac{\partial x'^k}{\partial x^i} \\ &= \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l. \end{aligned} \quad (5.10)$$

The quantity that caused by the translation on a curved manifold is written in

$$\delta A_i = A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l = \Gamma_{il}^k A_k dx^l. \quad (5.11)$$

Here Γ_{il}^k is called **Connection Coefficient** or **Christoffel Symbol**, which lets tensor analysis becoming concise. We will go into detail about it in Section 5.3.

We now have the quantity

$$\begin{aligned} DA_i &= dA_i - \delta A_i \\ &= \left(\partial_l A_i - \Gamma_{il}^k A_k \right) dx^l \\ &= A_{i;l} dx^l, \end{aligned} \quad (5.12)$$

then we define **Covariant Derivative** of a vector A_i in parenthesis in the equation above

$$\nabla_l A_i = A_{i;l} = \partial_l A_i - \Gamma_{il}^k A_k. \quad (5.13)$$

Just like normal derivatives, you can also prove the correctness of Leibniz rule of covariant derivative $(A_i B_k)_{;l} = A_{i;l} B_k + A_i B_{k;l}$.

Besides, the DA_i is a vector which is easy to prove, then

$$g_{ik} DA^k = DA_i = D(g_{ik} A^k) = g_{ik} DA^k + (Dg_{ik}) A^k, \quad (5.14)$$

this reads

$$Dg_{ik} = 0. \quad (5.15)$$

The covariant derivative of a tensor A_{ik} is²

$$\nabla_l A_{ik} = A_{ik;l} = \partial_l A_{ik} - \Gamma_{il}^m A_{mk} - \Gamma_{kl}^m A_{im}. \quad (5.16)$$

5.3 Christoffel Symbol

We also call Christoffel Symbol ‘‘Affine Connection’’ or ‘‘Connection Coefficient’’.

5.3.1 Properties of Christoffel Symbol

- In Flat spacetime

Obviously, if the spacetime is Galilean or Minkowski then $\Gamma_{il}^k = 0$, and the covariant derivative is degenerated into normal derivatives, which looks reasonably.

- In LIRF

According to the Equivalence Principle, at any point on the manifold we can choose a LIRF where $\Gamma_{il}^k = 0$.

- **Not** a tensor

If a tensor is zero then any coordinate transformation could not change its value to be non-zero. As we discussed above, when we shift from LIRF to other reference frame, this law for tensor transformation is no longer valid.

- Symmetry

$$\Gamma_{il}^k = \Gamma_{li}^k. \quad (5.17)$$

The relationship between metric tensor $g_{\mu\nu}$ and Christoffel Symbol can be simply obtained from the fact that the metric is invariant under the covariant derivative in Equation 5.15: $Dg_{ik} = 0$. We can list three equations:

$$\begin{cases} g_{ik;l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk} \Gamma_{il}^m - g_{im} \Gamma_{kl}^m = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,il} - \Gamma_{i,kl} = 0 \\ g_{li;k} = \frac{\partial g_{li}}{\partial x^k} - g_{mi} \Gamma_{lk}^m - g_{lm} \Gamma_{ik}^m = \frac{\partial g_{li}}{\partial x^k} - \Gamma_{i,lk} - \Gamma_{l,ik} = 0 \\ g_{kl;i} = \frac{\partial g_{kl}}{\partial x^i} - g_{ml} \Gamma_{ki}^m - g_{km} \Gamma_{li}^m = \frac{\partial g_{kl}}{\partial x^i} - \Gamma_{l,ki} - \Gamma_{k,li} = 0. \end{cases} \quad (5.18)$$

²Find Proof in Landau §85.

Sum the first and the second equations and minus the third equation, then use the symmetry of the Christoffel Symbol we get the expression of Christoffel Symbol in terms of metric tensor:

$$\Gamma_{kl}^i = \frac{1}{2}g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \quad (5.19)$$

For simplicity, we often write $\Gamma = g^{-1}\partial g$, which can notice you that each Christoffel Symbol carries one metric derivative.

Christoffel Symbol somehow represents the gravitational field strength, you will find this in derivation of the geodesics in Equation 5.21.

5.4 Geodesics in General Relativity

For massive particle under curvilinear coordinates, by same principle (the least action principle), we generalize the result in subsection 2.4.6 where the coordinates is not curved, $du^i = 0$, into

$$Du^i = 0. \quad (5.20)$$

This equation is “good” under general coordinates transformation (general covariant), in contrast, $du^i = 0$ is “good” only under Lorentzian Transformation. Divided by ds ,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma_{\sigma\nu}^\alpha \frac{dx^\sigma}{ds} \frac{dx^\nu}{ds} = 0. \quad (5.21)$$

This is called **Geodesic Equation** for massive particle. Making an analogy between this and $a - \frac{F}{m} = 0$, we can say $-m\Gamma_{\sigma\nu}^\alpha \frac{dx^\sigma}{ds} \frac{dx^\nu}{ds}$ is the 4-Force. That’s why we consider the Christoffel Symbol as the field strength.

You can interpret that if the spacetime is flat, which means $\Gamma = 0$, then the geodesic equation degenerates to the law that 4-velocity is consistent in special relativity.

For massless particle like photon, $ds = 0$ so the Equation 5.21 is not valid. We just need to change ds into another affine parameter $d\lambda$.

5.4.1 Killing Vectors

Here ‘Killing’ is a name. The **Killing vector** is the vector that can preserve the metric on a manifold at some points, or in other words, the zero Lie derivative of the metric by a vector field η : $\mathcal{L}_\eta g_{\mu\nu} = 0$. Killing vectors describe infinitesimal symmetries of the metric.

For example, in the world described by following metric $ds^2 = -cdt^2 + a_1(t)dx_1^2 + a_2(t)dx_2^2 + a_3(t)dx_3^2$, there are three global Killing vectors because the metric’s matrix elements are not influenced by the value of x_1, x_2, x_3 :

$$\begin{aligned} \eta^1 &= (0 \ 1 \ 0 \ 0) \\ \eta^2 &= (0 \ 0 \ 1 \ 0) \\ \eta^3 &= (0 \ 0 \ 0 \ 1). \end{aligned} \quad (5.22)$$

5.5 Curvature Tensors³

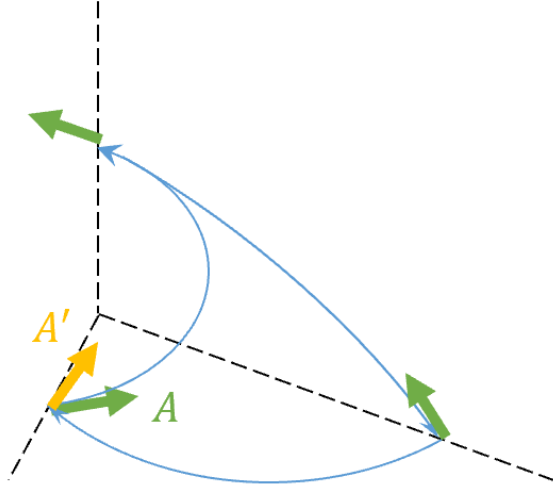


Figure 5.2: Vector moves in a loop on a curved manifold

Looking at the figure 5.2, the green arrow is moving along the triangular edge. After the arrow completes its traveling (the yellow arrow), it is different from the original arrow. The difference between beginning vector A and ending vector A' is $\Delta A = A - A'$, and the area that the contour loop encloses is Δf^{lm} . The components of ΔA (here we choosing a covariant vector) are

$$\Delta A_k = \oint \delta A_k = \oint \underbrace{\Gamma_{kl}^i}_{V_{kl}} A_i dx^l. \tag{5.23}$$

By Stokes' Theorem, saying that the integral of a differential form ω over the boundary of some orientable manifold is equal to the integral of its exterior derivative $d\omega$ over the whole of Ω , we get

$$\begin{aligned} \Delta A_k &= \oint V_{kl} dx^l \\ &= \frac{1}{2} \int df^{lm} \left[\frac{\partial(\Gamma_{km}^i A_i)}{\partial x^l} - \frac{\partial(\Gamma_{kl}^i A_i)}{\partial x^m} \right] \\ &\cong \Delta f^{lm} \left[\frac{\partial \Gamma_{km}^i}{\partial x^l} A_i - \frac{\partial \Gamma_{kl}^i}{\partial x^m} A_i + \Gamma_{km}^i \frac{\partial A_i}{\partial x^l} - \Gamma_{kl}^i \frac{\partial A_i}{\partial x^m} \right] \\ &= \frac{1}{2} R_{klm}^i A_i \Delta f^{lm}. \end{aligned} \tag{5.24}$$

³Most of formulas about Riemannian geometry could be found in https://en.wikipedia.org/wiki/List_of_formulas_in_Riemannian_geometry

Finally we using the symbol R^i_{klm} to substitute the complex expression in the middle of equation 5.24 ⁴, called Riemann Tensor:

Definition 5.2. The **Riemann Curvature Tensor** is

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{kn} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}, \quad (5.27)$$

which has 256 components describing the most detailed information about the manifold's curvature at every point.

A Covariant Riemann Curvature tensor is simply

$$R_{iklm} = g_{in} R^n_{klm}. \quad (5.28)$$

Another way to express Riemann Tensor is

$$A_{k;l;m} - A_{k;m;l} = [\nabla_l - \nabla_m] A_k = A_i R^i_{klm}. \quad (5.29)$$

5.5.1 Sufficient and necessary condition of Flat Spacetime

Theorem 5.3. $R^i_{klm} = 0$ is sufficient and necessary condition of the flatness of the spacetime.

Proof. Necessity:

If the spacetime is flat, we can find coordinates such that globally

$$g_{\mu\nu} = \eta_{\mu\nu},$$

so $\Gamma = g^{-1} \partial g = 0$, then $\partial \Gamma = 0$.

So $R^i_{klm} = 0$, and Riemann Tensor obey the Lorentzian transformation law,

$$R'^a_{bcd} = \Lambda^a_i \Lambda^k_b \Lambda^l_c \Lambda^m_d R^i_{klm} = 0,$$

so it is always 0 no matter how we choose coordinate systems.

⁴Equation 5.24 is based on covariant vectors. Here you can see contro-variant vectors can do the same job:

We know $A^k B_k$ is a scalar that do not change along any contour on any manifold. so

$$\begin{aligned} \Delta A^k B_k &= B_k \Delta A^k + A^k \Delta B_k \\ &= B_k \Delta A^k + A^k \left(\frac{1}{2} R^i_{klm} B_i \Delta f^{lm} \right) \\ &= B_k \left(\Delta A^k + \frac{1}{2} A^i R^k_{ilm} \Delta f^{lm} \right) \\ &= 0. \end{aligned} \quad (5.25)$$

So if we use contro-variant vector to determine the curvature, formula 5.24 would slightly changes into

$$\Delta A^k = -\frac{1}{2} R^k_{ilm} A^i \Delta f^{lm}. \quad (5.26)$$

Sufficiency:

At one point on the manifold you can choose an appropriate coordinate system to have a Minkowski spacetime.

If $R_{klm}^i = 0$, then you can parallel transport the coordinate system to any other point and it also shows Minkowski spacetime. So the manifold is flat everywhere. \square

5.5.2 Properties of Riemann Curvature Tensor

- Symmetry and anti-symmetry:

$$R_{iklm} = R_{lmik}. \quad (5.30)$$

$$R_{iklm} = -R_{kilm} = -R_{ikml}. \quad (5.31)$$

Write them down and expand them to prove this!

- Cyclic property:

$$R_{iklm} + R_{imkl} + R_{ilmk} = R_{i[klm]} = 0. \quad (5.32)$$

Easy to prove by using symmetry and anti-symmetry property.

Here the new notation $[klm]$ is an abbreviated form of $\begin{vmatrix} k & l & m \\ k & l & m \\ k & l & m \end{vmatrix} = +klm - kml - lkm + lmk + mkl - mlk$. (The signs should stand before the tensor not the indices)⁵.

- Bianchi identity, you have must seen it in Equation 3.18:

$$R_{klm;n}^i + R_{knl;m}^i + R_{kmn;l}^i = R_{k[lm;n]}^i = 0. \quad (5.33)$$

5.5.3 Contraction to Ricci Tensor and Scalar

Definition 5.4. The **Ricci Tensor** is the contraction of Riemann Tensor in terms of the second and the fourth indices:

$$\begin{aligned} R_{\mu\nu} &= g^{\rho\sigma} R_{\mu\rho\nu\sigma} \\ &= \frac{\partial \Gamma_{km}^i}{\partial x^l} - \frac{\partial \Gamma_{kl}^i}{\partial x^m} + \Gamma_{kn}^i \Gamma_{km}^n - \Gamma_{nm}^i \Gamma_{kl}^n, \end{aligned} \quad (5.34)$$

or

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = -g^{\rho\sigma} R_{\rho\nu\sigma\mu} = -g^{\rho\sigma} R_{\mu\rho\sigma\nu}. \quad (5.35)$$

The **Field Scalar** or **Ricci Scalar** is the contraction of Ricci Tensor:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\rho\sigma} R_{\mu\rho\nu\sigma}. \quad (5.36)$$

⁵Also see Appendix 10.2.

5.5.4 Another form of Bianchi Identity and Einstein Tensor

Contracting two indices in Bianchi Identity, and using Equation 5.35 to figure out each term, we obtain

$$g_i^l \left(R_{klm;n}^i + R_{knl;m}^i + R_{kmn;l}^i \right) = R_{km;n} - R_{kn;m} + R_{kmn;i}^i = 0, \quad (5.37)$$

then we can do another contraction on the equation,

$$g^{km} \left(R_{km;n} - R_{kn;m} + R_{kmn;i}^i \right) = R_{;n} - R_{n;m}^m - R_{n;i}^i = 0. \quad (5.38)$$

Actually this is the same form of

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)_{;\mu} = G_{\mu\nu;\mu} = 0, \quad (5.39)$$

which indicates an invariant quantity during covariant derivation. So we hereby define this invariant as the Einstein Tensor:

Definition 5.5. Einstein Tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad (5.40)$$

and the trace of Einstein Tensor under 4-D spacetime is the same as a minus Ricci Scalar:

$$G = g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = R - 2R = -R. \quad (5.41)$$

5.6 Derivation of Gravitational Field Equation⁶ (without a cosmological constant Λ)

Again, like the derivation of Euler-Lagrangian Equation, the EOM in classical mechanics, the variational principle is used here to derive the EOM of Gravitational field.

The total action is:

$$S = S_g + S_m, \quad (5.42)$$

where S_m is the action of matter and S_g is **Einstein-Hilbert Action** for the gravity, it was first proposed by Hilbert in 1915:

$$S_g = \frac{2}{k^2} \int R d^4 x \sqrt{-g}. \quad (5.43)$$

k is a constant we will substitute later.

The reason for this form is the same as the electromagnetic field action's and a free relativistic particle's in subsection 2.4.2 and section 3.1: requiring a reasonable scalar to form the Action⁷.

⁶See Weinberg *Gravitation and Cosmology* section 7.1 for another derivation, a conjecture directly from the Equivalence Principle (EP).

⁷For detail reason see Landau §93.

Remark 5.6. Here we are going to explore the variation of a metric.

We make a infinitesimal coordinate transformation: $x^i \rightarrow x'^i = x'^i(x^j), x'^i = x^i + \xi^i$. The metric changes in following way:

$$\begin{aligned} g'^{ik}(x') &= g'^{lm} \frac{\partial x'^i}{\partial x^l} \\ &= g'^{lm}(x) \left(\delta^l_x + \frac{\partial \xi^i}{\partial x^l} \right) \left(\delta^k_m + \frac{\partial \xi^k}{\partial x^m} \right) \\ &= g'^{ik}(x') + \xi^k \frac{\partial g'^{ik}}{\partial x^\alpha} + o(\xi^{(2)}). \end{aligned} \quad (5.44)$$

...

$$\delta g_{ik} = -\nabla^k \xi^i - \nabla^i \xi^k. \quad (5.45)$$

Remember the Killing vector mentioned in subsection 5.4.1, here we introduce the **Killing's Equation**:

$$\nabla^k \xi^i + \nabla^i \xi^k = 0, \quad (5.46)$$

which has the same meaning of $\mathcal{L}_\xi g_{ik} = 0$. The Killing vectors are the solutions to the equation.

First, we take the variation of S_g with respect to the metric. If there is no matter (vacuum), then $\delta S_g = 0$, from which we can get the vacuum EOM.

We know that $\delta g = -g g_{ik} \delta g^{ik}$, so

$$\begin{aligned} \delta \sqrt{-g} &= \frac{1}{2\sqrt{-g}} (-) \delta g \\ &= -\frac{1}{2\sqrt{-g}} (-g) g_{ik} \delta g^{ik} \\ &= -\frac{1}{2} \sqrt{-g} g_{ik} \delta g^{ik}, \end{aligned} \quad (5.47)$$

then the variation of S_g is

$$\begin{aligned} \delta S_g &= \frac{2}{k^2} \int \left(g^{ik} R_{ik} \delta \sqrt{-g} + \sqrt{-g} R_{ik} \delta g^{ik} + \sqrt{-g} g^{ik} \delta R_{ik} \right) d^4 x \\ &= \frac{2}{k^2} \int \left[g^{ik} R_{ik} \left(-\frac{1}{2\sqrt{-g}} \delta g \right) + \sqrt{-g} R_{ik} \delta g^{ik} + \sqrt{-g} g^{ik} \delta R_{ik} \right] d^4 x \\ &= \frac{2}{k^2} \int \left[R \left(-\frac{\sqrt{-g}}{2} g_{ik} \delta g^{ik} \right) + \sqrt{-g} R_{ik} \delta g^{ik} + \sqrt{-g} g^{ik} \delta R_{ik} \right] d^4 x \\ &= \frac{2}{k^2} \int \left[\left(-\frac{1}{2} R g_{ik} + R_{ik} \right) \delta g^{ik} + g^{ik} \delta R_{ik} \right] d^4 x \sqrt{-g} \\ &= \frac{2}{k^2} \int \left(G_{ik} \delta g^{ik} + g^{ik} \delta R_{ik} \right) d^4 x \sqrt{-g} \\ &= 0. \end{aligned} \quad (5.48)$$

The second term in the result $g^{ik}\delta R_{ik} = 0$ because

...

The first term must be 0, then

$$G_{\mu\nu} = 0. \quad (5.49)$$

This is the **Einstein Equation in vacuum**.

Then we add external matter and energy, the matter part of action debuts:

$$S_m = \frac{1}{c} \int d^4x \sqrt{-g} \mathcal{L}(\phi, g_{ik}, \partial g_{ik}). \quad (5.50)$$

The Lagrangian of matter is under the influence of the curvature of the manifold, i.e. the gravitation potential and its derivative.

Take the variation of the matter action with respect to the metric, by integral by part we get

$$\begin{aligned} \delta S_m &= \frac{1}{c} \int \delta(\mathcal{L} \sqrt{-g}) d^4x \\ &= \frac{1}{c} \int \left[\frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial g^{ik}} \delta g^{ik} + \frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial(\frac{\partial g^{ik}}{\partial x^l})} \delta\left(\frac{\partial g^{ik}}{\partial x^l}\right) \right] d^4x \\ &= \frac{1}{c} \int \left\{ \frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial g^{ik}} \delta g^{ik} + \frac{\partial}{\partial x^l} \left[\frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial(\frac{\partial g^{ik}}{\partial x^l})} \delta g^{ik} \right] - \frac{\partial}{\partial x^l} \left[\frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial(\frac{\partial g^{ik}}{\partial x^l})} \right] \right\} d^4x \\ &= \frac{1}{c} \int \left\{ \frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial g^{ik}} + 0 - \frac{\partial}{\partial x^l} \left[\frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial(\frac{\partial g^{ik}}{\partial x^l})} \right] \right\} \delta g^{ik} d^4x \\ &= \frac{1}{2c} \int \sqrt{-g} T_{ik} \delta g^{ik} d^4x. \end{aligned} \quad (5.51)$$

Here we define the Energy-Momentum Tensor:

Definition 5.7. Energy-Momentum Tensor or Stress-Energy Tensor, sometimes Stress-Energy-Momentum tensor is

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \left[\frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^l} \frac{\partial(\sqrt{-g} \mathcal{L}_m)}{\partial(\frac{\partial g^{\mu\nu}}{\partial x^l})} \right]. \quad (5.52)$$

It is easy to see that $T_{\mu\nu}$ a symmetric tensor since $g_{\mu\nu}$ is symmetric.

Combining with the previous results, $\delta S = \delta S_g + \delta S_m = 0$ gives us the general **Einstein Equation**:

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.} \quad (5.53)$$

The coefficient is set to be $k^2 = \frac{32G\pi}{c^3}$.

Qualitatively $G_{\mu\nu} \propto T_{\mu\nu}$. The equation tells us that how much has the spacetime been curved is only depending on the energy-momentum tensor.

It is the EOM of the gravitational field. It is **non-linear** because intuitively the right part $T_{\mu\nu}$ contains gravitation which will affect the left part $G_{\mu\nu}$, the gravitation field itself and then the gravitation affects $T_{\mu\nu}$ again. In contrast, the EOM in Newtonian physics: $F = ma$ which is linear.

5.6.1 Set the coefficient $k^2 = \frac{32G\pi}{c^3}$

5.6.2 Sufficient and necessary condition of Vacuum Spacetime

Theorem 5.8. $R = 0$ and $R_{\mu\nu} = 0$ is sufficient and necessary condition of $G_{\mu\nu} = 0$ or $T_{\mu\nu} = 0$ which means the spacetime is vacuum.

Proof. Sufficiency is easy to prove.

Necessity:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0,$$

so

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R,$$

applying a metric to both sides, we get

$$g^{\mu\nu}R_{\mu\nu} = \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R.$$

The contraction produces 4, $R = 2R$, so $R = 0$, then $R_{\mu\nu} = 0$. □

5.7 Energy-Momentum Tensor*

We have known Energy-Momentum Tensor from Definition 5.7. For generality it is just

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}. \quad (5.54)$$

5.7.1 Conservation Law of Energy-Momentum Tensor

If we only consider $\delta S_m = 0$, from Killing's Equation and the symmetry of T_{ij} and integral by part, we can get ⁸

$$\begin{aligned}
 \delta S_m &= \frac{1}{2c} \int \sqrt{-g} T_{ik} (\xi^{i;k} + \xi^{k;i}) d^4x \\
 &= \frac{1}{c} \int \sqrt{-g} T_{ik} \xi^{i;k} d^4x \\
 &= \frac{1}{c} \int \sqrt{-g} \left[(T_i^k \xi^i)_{;k} - T_{i;k}^k \xi^i \right] d^4x \\
 &= \frac{1}{c} \int \sqrt{-g} (0 - T_{i;k}^k \xi^i) \sqrt{-g} d^4x \\
 &= 0.
 \end{aligned} \tag{5.55}$$

For arbitrary ξ^i the equation above should be satisfied so

$$T_{i;k}^k = 0, \tag{5.56}$$

this is the general covariant version for energy-momentum conservation. Variation respect to the metric is called diffeomorphism transformation.

We can also have Weyl Transformation $g_{\mu\nu} \rightarrow g_{\mu\nu} e^{2a}$ and take the variation respect of a . We will get

$$T_{\mu}^{\mu} = 0. \tag{5.57}$$

5.7.2 Examples of Energy-Momentum Tensor

- Massive particle
- Perfect fluid

$$T_k^i = (p + \epsilon) u^i u_k - p \delta_k^i. \tag{5.58}$$

- Dust

If we consider dust, which has no pressure between each of them. This means $p = 0$, then we have

$$T_k^i = \epsilon u^i u_k. \tag{5.59}$$

- Continuous Equation and Euler's Equation of perfect fluid From energy-momentum conservation

⁸In Equation 5.55 the reason for $(T_i^k \xi^i)_{;k} = 0$ is shown at Landau's §94, Equation (94.6).

- Scalar field ϕ

$$S(\phi, g_{\mu\nu}) = \int d^4x \sqrt{-g} \left[\frac{1}{2} g_{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right]. \quad (5.60)$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} (\partial\phi)^2 + V \right]. \quad (5.61)$$

- Electromagnetic field

$$p = \frac{1}{3} \phi \quad (5.62)$$

Chapter 6

Solutions to Einstein Equation

6.1 Vacuum Solutions

If the situation is **vacuum**(gravitational source is out of the region you selected). Then $T_{\mu\nu} = 0$ in the region.

$$G_{\mu\nu} = 8\pi GT_{\mu\nu} = 0. \quad (6.1)$$

By Theorem 5.2,

$$R_{\mu\nu} = 0. \quad (6.2)$$

By the way, the spacetime in the vacuum region with this property is called Ricci-flat Spacetime (Not flat space! Remember: Only $R_{\mu\nu\sigma\rho} = 0$ can equal to flat space).

- One trivial solution is our old friend, Minkowski Spacetime, which is empty and flat every where and is a special case of Ricci-flat Spacetime.
- There are two spacetime named after de Sitter and Anti-de Sitter spacetime. They are solutions of Einstein Equation with a positive or negative cosmological constant Λ respectively.
- If you study vacuum regions outside the “source”, you will also get vacuum solutions. Different solutions depend on different properties of the “source”. In following sections we will study several solutions of this kind.

6.2 Blackholes

It contains somewhere with some infinity physical properties. It tears the spacetime dramatically. Never be seen by human-beings, black hole is fascinating. Natural Black holes have mass M , spin J and charge Q . By 'No-hair Theorem' (Wiki it!), only fixing these 3 properties can totally determine how a black hole looks like.

6.3. SWARZCHILD METRIC (SPHERICAL SYMMETRY SO(3) AND TIME-INVARIANT)51

Types of Black holes		
	Without J	With J
Without Q	Swarzchild	Kerr
With Q	Reissner-Nordstrom	Kerr-Newmann

First lets begin with the simplest one, a perfectly spherical massive stationary object. See how this body influences the spacetime nearby and which situation would the ball turn into a black hole.

6.3 Swarzchild Metric (Spherical symmetry SO(3) and time-invariant)

This metric we are going to know differs from that we solved the Minkowski Metric under a rotating coordinates, which doesn't have mass at anywhere. The latter has matter with mass M at the system center steadily or stationarily, and gravity is exist with spherical symmetry, like the gravity generated by the sun, spherical enough, or those generated by any gigantic celestial bodies with nearly empty surrounding. The metric describes the spacetime outside the center object.

It had never been worked out, even Einstein didn't either, until Swarzchild's solving 50 years after general relativity's advent.

If you just want to know the application of Swarzchild Metric, you can jump the subsection 6.3.1.

6.3.1 Solve the Metric out¹

Totally there are 10 degrees of freedom in $g_{\mu\nu}$ (Symmetrical matrix with n dimensions has $\frac{(n-1)^2}{2} + n$ degrees of freedom), but we can apply other symmetries to decrease the number of free components from 10 to only 4.

Because the metric has a mass center, We can use (r, θ, ϕ, t) to replace the original coordinates (t, x, y, z) . To symplify the notation, let me just use $(1, 2, 3, 0)$ to represent (r, θ, ϕ, t) in this chapter.

First, the time reversal symmetry is preserved:

$$ds^2(t) = ds^2(-t). \tag{6.3}$$

So if we do the coordinate transformation $(r, \theta, \phi, t) \rightarrow (r, \theta, \phi, -t)$, then $g'_{\mu 0} = g_{\mu 0}$.

Recall the coordinate transformation formula of metric tensor:

$$g'_{\mu 0} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^0} g_{\alpha\beta} = \delta_\mu^\alpha \cdot -\delta_0^\beta g_{\alpha\beta} = -g_{\mu 0} = g_{\mu 0}, \quad \mu \neq 0, \tag{6.4}$$

we immediately get

$$g_{\mu 0} = 0, \quad \mu \neq 0. \tag{6.5}$$

¹For more details please refer to https://en.wikipedia.org/wiki/Deriving_the_Schwarzschild_solution.

Besides, the SO(3) symmetry allows the metric invariant under the other two reversal transformations: $(r, \theta, \phi, t) \rightarrow (r, \theta, -\phi, t)$ and $(r, \theta, \phi, t) \rightarrow (r, -\theta, \phi, t)$. In the same way, we get

$$\begin{aligned} g_{\mu 3} &= 0, & \mu &\neq 3, \\ g_{\mu 2} &= 0, & \mu &\neq 2. \end{aligned} \quad (6.6)$$

More strongly, the spacetime is static, in other words, all the metric components are independent on time t :

$$\frac{\partial g_{\mu\nu}}{\partial t} = 0. \quad (6.7)$$

Now, congratulations! Only 4 non-zero components $g_{11}(r, \theta, \phi)$, $g_{22}(r, \theta, \phi)$, $g_{33}(r, \theta, \phi)$, $g_{00}(r, \theta, \phi)$ are left, and they are functions of the radius r , the pitch angle θ and the azimuth angle ϕ . The infinitesimal interval is

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2. \quad (6.8)$$

Besides, if we fixed ϕ and θ , $d\theta^2 = d\phi^2 = 0$. Then by spherical symmetry,

$$\frac{\partial g_{00}}{\partial \theta, \phi} = \frac{\partial g_{11}}{\partial \theta, \phi} = 0, \quad (6.9)$$

so g_{00} and g_{11} are only functions of r . Let us just call g_{00} as $B(r)$ and g_{11} as $A(r)$ for simplicity, A' and B' are their derivatives respect to r .

Also, if we fixed t and r , by spherical symmetry it is required that the metric be a 2-D spherical shell. Using knowledge in elementary geometry, the infinitesimal length on a 2-D spherical shell is $dl^2 = D \cdot r^2(d\theta^2 + \sin^2 \theta d\phi^2)$. D is a constant factor due to isotropic gravitation, without loss of generality the factor can be set to 1. Thus,

$$g_{22} = r^2 \quad g_{33} = r^2 \sin^2 \theta. \quad (6.10)$$

Finally we get

$$\begin{aligned} ds^2 &= Bdt^2 + Adr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \\ &= Bdt^2 + Adr^2 + r^2 d\Omega^{(2)}. \end{aligned} \quad (6.11)$$

We use $\Omega^{(2)}$ to refer to the solid angle.

Next, we can utilize the vacuum condition $R_{\mu\nu} = R = 0$. First we need to find out the Christoffel Symbol. Remember the equation 5.19

$$\Gamma_{kl}^m = \frac{1}{2} g^{mk} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right), \quad (6.12)$$

6.3. SWARZCHILD METRIC (SPHERICAL SYMMETRY SO(3) AND TIME-INVARIANT)⁵³

we can obtain 4 components of Christoffel Symbol²:

$$\begin{aligned} \Gamma_{kl}^1 &= \begin{pmatrix} \frac{A'}{2A} & 0 & 0 & 0 \\ 0 & -\frac{r}{A} & 0 & 0 \\ 0 & 0 & -\frac{r \sin^2 \theta}{A} & 0 \\ 0 & 0 & 0 & -\frac{B'}{2A} \end{pmatrix} & \Gamma_{kl}^3 &= \begin{pmatrix} 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & \cot \theta & 0 \\ \frac{1}{r} & \cot \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \Gamma_{kl}^2 &= \begin{pmatrix} 0 & \frac{1}{r} & 0 & 0 \\ \frac{1}{r} & 0 & 0 & 0 \\ 0 & 0 & -\sin \theta \cos \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \Gamma_{kl}^0 &= \begin{pmatrix} 0 & 0 & 0 & \frac{B'}{2B} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{B'}{2B} & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (6.13)$$

Using the vacuum condition

$$R_{\mu\nu} = \frac{\partial \Gamma_{\mu\nu}^l}{\partial x^l} - \frac{\partial \Gamma_{\mu l}^{\nu}}{\partial x^\nu} + \Gamma_{\mu\nu}^m \Gamma_{lm}^l - \Gamma_{\mu l}^m \Gamma_{\nu m}^l = 0, \quad (6.14)$$

we get a set of equations:

$$0 = \begin{cases} 4A'B^2 - 2rB''AB + rA'B'B + rB'^2A \\ rA'B + 2A^2B - 2AB - rB'A \\ -2rB''AB + rA'B'B + rB'^2A - 4B'AB, \end{cases} \quad (6.15)$$

then we can start to solve A and B out.

Subtract the third equation from the first equation we get

$$A'B + B'A = 0 = (AB)' \implies AB = \text{const.} \equiv k, \quad B = \frac{k}{A}. \quad (6.16)$$

Then substitute it into the second equation we get

$$krA' + kA(A - 1) = 0. \quad (6.17)$$

So by solving the simple differential equation,

$$\begin{cases} A = \frac{1}{1 + \frac{cc}{r}} \\ B = k \left(1 + \frac{cc}{r}\right), \quad cc = \text{const.} \end{cases} \quad (6.18)$$

At last, we determine the constants by **weak-field approximation**³:

$$\begin{cases} k = -1 \\ cc = -\frac{2MG}{c^2} = -2MG. \end{cases} \quad (6.19)$$

here c is the light speed.

²Every element of Christoffel Symbol and Ricci Tensor is showed in <https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf>

³Please check Landau's book Equation (87.10)-(87.12)

Finally, these are the Schwarzschild Metric and its infinitesimal interval:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2MG}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{2MG}{r}} & 0 & 0 \\ 0 & 0 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}, \quad (6.20)$$

$$ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \frac{1}{1 - \frac{2MG}{r}} dr^2 + r^2 d\Omega^{(2)}. \quad (6.21)$$

6.3.2 Singularity

If we take $r = r_g = 2MG$ or $r = 0$, the metric tensor or ds^2 would not be well defined.

Definition 6.1. Schwarzschild Radius $r_g = 2MG$.

These two positions are singularities, one is the 2-D sphere with radius r_g and another is the center point. But are they real singularities or just the result from bad-choosing of coordinates? For example, when you measure the slope of a hillside, you use $\tan \theta$ to substitute θ , the pseudo-singularity appears when there is a cliff, $\theta = \frac{\pi}{2}$.

The Schwarzschild situation is shown in figure 6.1.

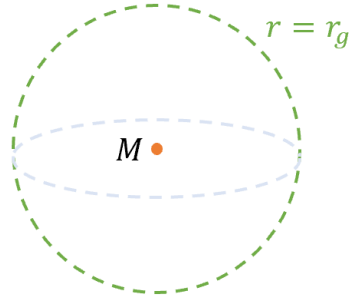


Figure 6.1: The situation described by Schwarzschild Metric

- The first singularity is called the **Event Singularity**, the sphere with radius r_g is called the **Event Horizon**, which we will expand in subsection 6.3.3. Here $g_{00} = 0$. Event Singularity can be eliminated by lots of coordinates transformation.

e.g. Kruskal–Szekeres Coordinates Transformation

The Kruskal–Szekeres coordinates transformation is given below, which is seemed sophisticated.

$$\begin{cases} T &= (\pm \frac{r}{2GM} \mp 1)^{\frac{1}{2}} e^{\frac{r}{4MG}} \sinh \frac{t}{4GM} \\ X &= (\pm \frac{r}{2GM} \mp 1)^{\frac{1}{2}} e^{\frac{r}{4MG}} \cosh \frac{t}{4GM} \end{cases}. \quad (6.22)$$

6.3. SWARZCHILD METRIC (SPHERICAL SYMMETRY SO(3) AND TIME-INVARIANT) 55

Then the interval becomes

$$ds^2 = \frac{32G^3 M^3}{r} e^{-\frac{r}{2GM}} (-dT^2 + dX^2) + r^2 d\Omega^{(2)}. \quad (6.23)$$

There is no singularity at $r = 2MG$ comparing to Swarzschild's. Actually it is an analytic continuation of Swarzschild Metric. $r = 0$ singularity still remains.

You can see finding such a subtle transformation is a tough job. But the curvatures provides us with a method to measure if the gravity is strong enough at some point to produce a singularity. Remember $R = R_{\mu\nu} = 0$. The only non-zero(probably) quantity is $R_{\mu\nu\sigma\rho}$.

So we construct a quadratic scalar invariant called **Kretschmann Scalar**

$$K = R^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho}. \quad (6.24)$$

This quantity at Event Singularity is $K|_{r=r_g} = \frac{48G^2 M^2}{r^6}|_{r=r_g} = \frac{48G^2 M^2}{64G^6 M^6} < \infty$, so it is not a real singularity.

- Another singularity is a point at the center of the system, which is a real physical singularity. No coordinate transformation could eliminate the center singularity. You can check that $K|_{r=0} = \infty$. How to interpret a physical singularity has puzzled physicists for decades.

6.3.3 Collapse and Event Horizon

If there is another small testing object that is originally static is being attracted by the center object, let us calculate the equation of motion of the testing object.

Lets first see the action of a massive particle

$$S = -m \int ds = -m \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau, \quad (6.25)$$

then

$$\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{mg_{\mu\nu} \dot{x}^\nu}{\sqrt{\dot{x}^2}}. \quad (6.26)$$

Particularly, following quantity is conserved, because the Lagrangian isn't a function of t and according to the Euler-Lagrangian Equation,

$$\frac{\partial L}{\partial t} = -\frac{mg_{00} \dot{t}}{\sqrt{\dot{x}^2}} = \text{const.} = E. \quad (6.27)$$

If we set $\sqrt{\dot{x}^2} = 1$, then

$$\dot{t} = -\frac{E}{m} \frac{1}{1 - \frac{r_g}{r}} = \frac{e}{1 - \frac{r_g}{r}}. \quad (6.28)$$

Also, from $ds^2/d\tau^2 = 1$, we get

$$g_{00}\dot{t}^2 + g_{rr}\dot{r}^2 = 1, \quad (6.29)$$

then

$$e^2 = \dot{r}^2 + 1 - \frac{r_g}{r}. \quad (6.30)$$

For $r \rightarrow \infty$ and $\dot{r} = 0$, the constant $e = 1$, so

$$\frac{dr}{d\tau} = \pm \sqrt{\frac{r_g}{r}}. \quad (6.31)$$

If it requires infinite proper time duration for the testing object to reach the center singularity, then we do not need to care about the singularity (In other words, it is geodesic complete). However, the fact is frustrating. Lets calculating the proper time duration:

$$\int d\tau = - \int \frac{dr}{\sqrt{\frac{r_g}{r}}}, \quad (6.32)$$

if we set $r(\tau = 0) = r_0$, after the integration,

$$\tau(r) = \frac{2}{3} \left(\frac{r_0^{\frac{3}{2}}}{r_g^{\frac{1}{2}}} - \frac{r^{\frac{3}{2}}}{r_g^{\frac{1}{2}}} \right). \quad (6.33)$$

For a object started at event horizon, after $\tau = \frac{2}{3} \frac{r_g}{c} = \frac{4}{3} \frac{MG}{C^3}$ of time it reached the center. For a blackhole with solar mass M_\odot , $\tau \sim 10^{-6}$ s and for a typical galactic center black-hole $M = 10^{10}M_\odot$, $\tau \sim 10^4$ s. (Actually not only the particle, but the whole reference frame is collapsing. There is no way to establish a rigid reference frame inside the horizon.)

Lets then draw the lightcones in the **Lemaitre $\tau - R$ diagram** in Figure 6.2, R is defined by **Lemaitre Transformation**

$$\begin{cases} dR = dt + \sqrt{\frac{r}{r_g}} \frac{dr}{1 - \frac{r_g}{r}} \\ d\tau = dt + \sqrt{\frac{r_g}{r}} \frac{dr}{1 - \frac{r_g}{r}}. \end{cases} \quad (6.34)$$

$$ds^2 = -d\tau^2 + \frac{r_g}{r} dR^2 + r^2 d\Omega^{(2)}. \quad (6.35)$$

For $r = \text{const.}$, they are lines with 45 degrees slope in the diagram. Especially the line represents $r = 0$, it crosses the origin.

The slope of those lightcones' slants can be solved from $ds^2 = 0$:

$$\frac{d\tau}{dR} = \pm \sqrt{\frac{r_g}{r}}. \quad (6.36)$$

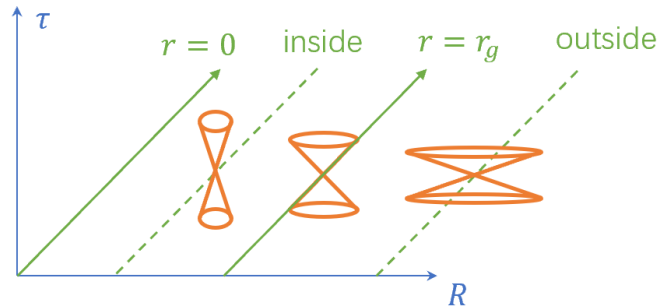


Figure 6.2: How things move in Schwarzschild spacetime

From Figure 6.2 you can see the object could only escape (increase r) outside the Schwarzschild radius. Inside, the object falls into the center singularity inevitably, even light can not avoid being attracted into the center. Observers far away can only see event happened outside the Schwarzschild radius. This is the reason that people call the spherical shell with r_g radius ‘**Event Horizon**’, which was elaborated in subsection 6.3.2.

Remark 6.2. There exists some ways to derive Schwarzschild Radius by Newtonian Method. The subject is mainly about using light speed to substitute the classical escape velocity $v = \sqrt{\frac{2MG}{r}}$ to see the minimum escaping radius. But the problem is if there is an object with velocity of light speed, then Newtonian framework will not valid from the very beginning. Also, the escaping trajectory is a parabola but not a orbit limited in the horizon spherical shell.

Here is an typical wrong example: <https://arxiv.org/ftp/gr-qc/papers/0611/0611104.pdf>

Anyway, you can use this method as a tip to remember Schwarzschild Radius since there is indeed no light can escapes from inside of Schwarzschild Radius.

6.3.4 Time and Gravitational RedShift

Assume there is an object traveling towards the mass center outside of Schwarzschild radius $r > r_g$. From Schwarzschild metric, we now know the object’s proper time is

$$d\tau = \sqrt{-g_{00}}dt = \sqrt{1 - \frac{2MG}{r}}dt < dt. \quad (6.37)$$

Although the proper time of the object remains its original elapsing rate in its view, the time elapses more slowly in the views of observers who is far away from the ‘Event Horizon’. More approaching to ‘Event Horizon’ the object is, slower the movement is.

Remark 6.3. Remember that in special relativity, the time dilation is relative. Two observers will both have the version that the other one's time slows down (Each frame is inertial, flat respect to the other). However, in general relativity, the time-slowed guy will see the other guy with normal time pace (Two frames are not equivalent, because they suffer different gravity. Meanwhile, you should recognize that this is similar to the situation in the twin paradox.)

Naturally, a beam of light with frequency ω_0 (measured in light source's proper time) will decrease its frequency in the views of observers far away from the 'Event Horizon'. Closer to the 'Event Horizon', lower the frequency is.

$$\omega = \frac{\omega_0}{\sqrt{-g_{00}}} > \omega_0. \quad (6.38)$$

Lower frequency means **Redshift** happened, and this is only attributed to gravitational effect rather than Doppler Effect. We call it Gravitational Redshift. So we will not only observe the object moving slower but also redder, finally be frozed and disappear in a dark shadow (redshift sequence: γ -ray, X-ray, ultraviolet, blue, red, then infrared, eventually radio wave...) on the 'Event Horizon'.

6.3.5 Penrose Diagram

To show the Schwarzschild spacetime of the whole universe seems impossible because we don't have an infinitely large paper. However, we can pull back the "infinities" to the finite scale we can manage by coordinates transformation.

In $t - r$ coordinate, the diagram can be divided in following region:

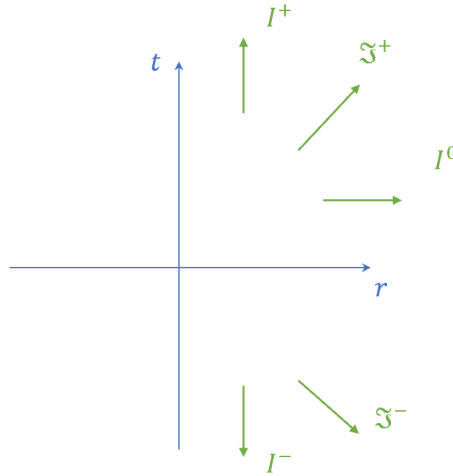


Figure 6.3: Infinities

6.3. SWARZCHILD METRIC (SPHERICAL SYMMETRY $SO(3)$ AND TIME-INVARIANT) 59

- I^+ \equiv "future time-like infinity": $t \rightarrow \infty$ and $r \rightarrow \text{const.}$.
- I^- \equiv "past time-like infinity": $t \rightarrow -\infty$ and $r \rightarrow \text{const.}$.
- I^0 \equiv "space-like infinity": $t \rightarrow \text{const.}$ and $r \rightarrow \infty$.
- \mathcal{J}^+ \equiv "future null infinity": $t + r \rightarrow +\infty$ and $t - r \rightarrow \text{finite.}$
- \mathcal{J}^- \equiv "past null infinity": $t + r \rightarrow \text{const.}$ and $t - r \rightarrow -\infty$.

If we choose the singature $(- + ++)$, for spherical coordinate we have

$$ds^2 = -dt^2 + dr^2 + r^2(\sin^2 \theta d\phi^2 + d\theta^2). \quad (6.39)$$

If we coordinate transform the diagram from (t, r, θ, ϕ) into $(\phi, \xi, \theta, \phi)$, where

$$\begin{aligned} t + r &= \tan\left[\frac{1}{2}(\phi + \xi)\right] \\ t - r &= \tan\left[\frac{1}{2}(\phi - \xi)\right], \end{aligned} \quad (6.40)$$

we can get the metric

$$\begin{aligned} ds^2 &= \frac{-d\phi^2 + d\xi^2}{4 \cos^2\left[\frac{1}{2}(\phi + \xi)\right] \cos^2\left[\frac{1}{2}(\phi - \xi)\right]} \\ &= \Omega^2(-d\phi^2 + d\xi^2) + r^2 d\Omega^{(2)}. \end{aligned} \quad (6.41)$$

Here we have

- I^+ : $\xi = 0, \phi = \pi$.
- I^- : $\xi = 0, \phi = -\pi$.
- I^0 : $\xi = \pi, \phi = 0$.
- \mathcal{J}^+ : $\phi = \pi - \xi$.

Then the diagram becomes **Penrose Diagram**

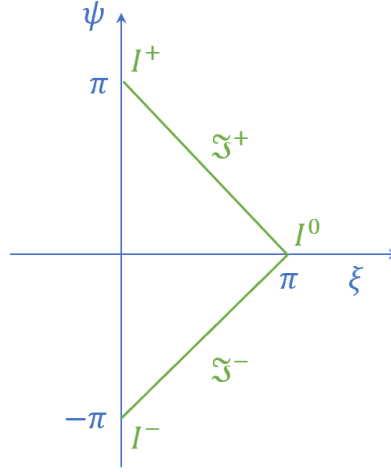


Figure 6.4: Penrose diagram for Minkowski spacetime

Here we can see we avoid the infinity ∞ appearing in the diagram using the coordinate transformation.

Penrose Diagram for Schwarzschild Spacetime

Schwarzschild Spacetime in Kruskal-Szekeres coordinate can be written as

$$ds^2 = \frac{32\pi M^2}{r} e^{\frac{\bar{r}}{2M}} (-dv^2 + du^2) + r^2 d\Omega^{(2)}. \quad (6.42)$$

In $u - v$ diagram, the spacetime looks like Here we use coordinate transformation to get Penrose Diagram for Schwarzschild case:

$$\begin{aligned} u + v &= \tan \frac{1}{2}(\phi + \xi) \\ u - v &= \tan \frac{1}{2}(\phi - \xi). \end{aligned} \quad (6.43)$$

The metric becomes

$$ds^2 = \frac{32\pi^2 e^{\frac{\bar{r}}{2M}}}{r} \frac{(-d\phi^2 + d\xi^2)}{4 \cos^2[\frac{1}{2}(\phi + \xi)] \cos^2[\frac{1}{2}(\phi - \xi)]}. \quad (6.44)$$

Here we need to find the position of horizon and singularity of Schwarzschild spacetime in Penrose diagram.

- $\boxed{r = 2\pi} \rightarrow \phi = -\xi$ or $\phi = \xi$
- $\boxed{r = 0} \rightarrow \phi = \frac{\pi}{2}$ or $\phi = \frac{-\pi}{2}$.

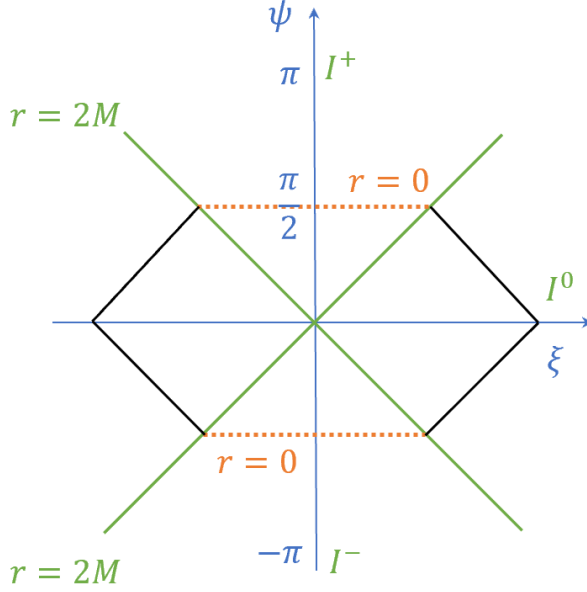


Figure 6.5: The Penrose diagram for Schwarzschild

The Penrose diagram for Schwarzschild spacetime is shown in Figure 6.5.

For collapsing Schwarzschild black hole, we can draw the Penrose diagram as Figure 6.6 The figure is composed of two Penrose diagrams. Before the matter collapsed into a black hole, the spacetime is Minkowski. After the collapsing, spacetime would be Schwarzschild. In figure 6.6 we can see below the orange line, which is light shell, the spacetime is Minkowski. Above the orange line, the spacetime is Schwarzschild.

6.3.6 How Massive Particle Moves

6.3.7 How Light is Bend

Assume the light action is

$$S = - \int \left(\frac{1}{2} \frac{\dot{x}^2}{e} + \frac{1}{2} m^2 e \right) d\tau. \quad (6.45)$$

e is a Lagrangian multiplier.

Taking the variation in terms of e ,

$$\delta_e S = - \int \delta e \left(-\frac{1}{2} \frac{\dot{x}^2}{e^2} + \frac{1}{2} m^2 \right) d\tau = 0. \quad (6.46)$$

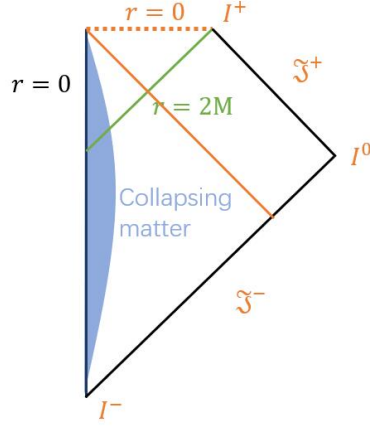


Figure 6.6: The Penrose diagram for Collapsing Schwarzschild Black Hole

Then

$$e^2 = \frac{\dot{x}^2}{m^2}. \quad (6.47)$$

After taking $m = 0$, The light's action from Equation 6.3.7 becomes

$$S = \int \mathcal{L} d\lambda = \int -\frac{1}{2} e^{-1} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda, \quad (6.48)$$

the Lagrangian in explicit form is

$$\mathcal{L} = -\frac{1}{2e} \left[\left(1 - \frac{2M}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2M}{r}} - r^2 \dot{\phi}^2 \right]. \quad (6.49)$$

e transforms in sort of way:

$$e'^{-1}(\lambda') = \Lambda e^{-1}(\lambda) \quad (6.50)$$

Previously, $\delta_e S = 0$ leads to $-\frac{\dot{x}^2}{2e^2} = 0$, which is equivalent to $ds^2 = 0$ for light.

Using the conservation law, $\frac{\partial \mathcal{L}}{\partial t} = \text{const.}$ and $\frac{\partial \mathcal{L}}{\partial \phi} = \text{const.}$ we get

$$\begin{cases} \left(1 - \frac{2M}{r}\right) \dot{t} = \frac{E}{M_p} = \epsilon \\ r^2 \dot{\phi} = \frac{J}{M_p} = J, \end{cases} \quad (6.51)$$

after we setting the gauge

$$e = \frac{1}{M_p}. \quad (6.52)$$

6.3. SWARZCHILD METRIC (SPHERICAL SYMMETRY SO(3) AND TIME-INVARIANT)63

We replace \dot{t} and $\dot{\phi}$ in the Lagrangian by ϵ and J .

$$\mathcal{L} = \frac{1}{2} \frac{\epsilon^2}{(1 - 2M/r)} - \frac{1}{2} \frac{r^2 \dot{\phi}^2}{(1 - 2M/r)} - \frac{1}{2} \frac{J^2}{r^2}. \quad (6.53)$$

Because we have $ds = 0$ for light, we have

$$-\dot{t} + \epsilon^2 - (1 - \frac{2M}{r}) \frac{J^2}{r^2} = 0. \quad (6.54)$$

If we do the coordinate transformation $\lambda \rightarrow \lambda'$, we have

$$(1 - \frac{2M}{r}) \frac{dt}{d\lambda'} = (1 - \frac{2M}{r}) \frac{dt}{d\lambda} \frac{d\lambda}{d\lambda'} = (1 - \frac{2M}{r}) \frac{dt}{d\lambda} \frac{1}{\alpha} = E. \quad (6.55)$$

Here $1/\alpha = \epsilon$, we can set $\epsilon = 1$. We can also define $J = b$, then the equation above can be derived as

$$\dot{t} + (1 - \frac{2M}{r}) \frac{b^2}{r^2} = 1. \quad (6.56)$$

with

$$E = 1 \quad (6.57)$$

and

$$\frac{r^2 d\phi}{dr} = -b. \quad (6.58)$$

These three equation respectively indicate:

- When $r \rightarrow \infty$, $\dot{t} = 1$.
- $\frac{dt}{d\lambda} = 1$.
- $d\phi = \frac{b}{r^2} dr$.

By integrating the final equation we get

$$\phi = \frac{b}{r}. \quad (6.59)$$

While b is a small number, we can approximately consider $\phi \approx \sin \phi$. Then we have

$$b = r \sin \phi. \quad (6.60)$$

As what is shown in the picture, b is the **Impact Factor** in scattering theory.

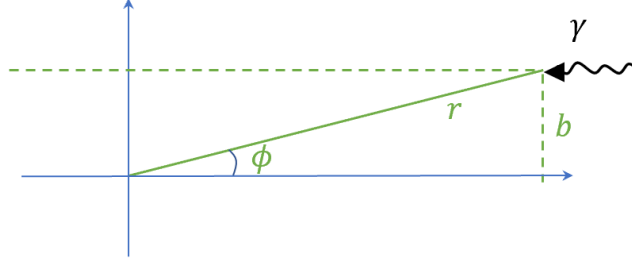


Figure 6.7: Impact factor

6.4 Kerr Metric⁴

The Swartzchild Metric is designed for a spherical stationary object. However, in nature, there is few things locomote without angular momentum. How to deal a object with angular momentum? Then here comes the Kerr Metric. We often call the black holes' angular momentum 'spin'.

(Notice: Whilst the mass has a standard Newtonian interpretation, for black holes it is nonsense to talk about the revolve or rotation. Because scientists do not and hitherto can not care the inter structure of black holes. The spin that black holes possess is simply considered as an intrinsic property succeed from the angular momentum of the original star.)

The system has an angular momentum J , and an angular momentum density $a = \frac{J}{M}$. a has the same unit as length because $c = 1$ (Remember angular momentum has a unit of $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$ under natural units).

The metric is too complicated to solve and even very difficult to check the correctness in Einstein Equation, we can start directly:

$$ds^2 = -\left(1 - \frac{r_g r}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2 \theta\right) \sin^2 \theta d\phi^2 - \frac{r_g r a \sin^2 \theta}{\rho^2} d\phi dt. \quad (6.61)$$

Here $\Delta = r^2 - r r_g + a^2$ and $\rho^2 = r^2 + a^2 \cos^2 \theta$.

You can find from the last item of the interval that $g_{\phi t} \neq 0$ which means $g_{\mu\nu}$ is not diagonal.

If $a = 0$, then Kerr Metric degenerate to Swarchild's, which is reasonable because a represents the spin.

⁴For more detail about Kerr Metric you can refer to <https://arxiv.org/pdf/0706.0622v3.pdf> page=35

If $M \rightarrow 0$, $r_g \rightarrow 0$. the metric reasonably approaches to Minkowski Metric.

$$\begin{aligned} ds^2 &\rightarrow -dt^2 + \frac{\rho^2}{r^2 + a^2} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \\ &= -dt^2 + dx^2 + dy^2 + dz^2. \\ &= \eta_{\mu\nu} dx^\mu dx^\nu \end{aligned} \quad (6.62)$$

Transformation to Cartesian coordinates are represented below

$$\begin{cases} t &= t \\ x &= \sqrt{r^2 + a^2} \sin \theta \cos \phi \\ y &= \sqrt{r^2 + a^2} \sin \theta \sin \phi \\ z &= r \cos \theta \end{cases} \quad (6.63)$$

which is the proper coordinates transformation that satisfies equation (6.62). You can find that the radius r is stretched when the spin exists.

6.4.1 Singularity and horizons

If we take $\rho = 0$ or $\Delta = 0$, interval ds^2 would not be well defined.

- If

$$\rho^2 = r^2 + a^2 \cos^2 \theta = 0. \quad (6.64)$$

It is true when $r = 0$ and $\theta = \frac{\pi}{2}$. Then equation (6.63) turns into

$$\begin{cases} x &= a \cos \phi \\ y &= a \sin \phi \\ z &= 0 \end{cases}, \quad (6.65)$$

which means there is a ring shaped singularity:

$$x^2 + y^2 = a^2. \quad (6.66)$$

The scalar $R^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} \Big|_{\rho=0} \simeq \frac{1}{\rho^6} \Big|_{\rho=0} = \infty$. So the ring is a real singularity.

- If

$$\Delta = r^2 - rr_g + a^2 = 0, \quad (6.67)$$

then

$$r_{\pm} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}. \quad (6.68)$$

- If $g_{00} = 0$,

$$r^2 - rr_g + a^2 \cos^2 \theta = 0, \quad (6.69)$$

then

$$r_{0\pm} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2 \cos^2 \theta}. \quad (6.70)$$

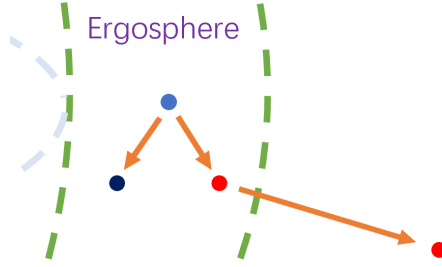


Figure 6.9: Energy extraction

In the picture, we have a pair of particles with total energy $E = E_1 + E_2 = -|E_1| + E_2$ in the ergosphere and particle 1 has a negative energy $E_1 < 0$. The negative-energized particle cannot live outside the ergosphere, so the only possibility is you get particle 2 with energy $E_2 = E + |E_1|$ and meanwhile particle 1 falls into the blackhole. You extract $|E_1|$ from the rotation energy from the Kerr blackhole.

6.4.4 Closed Timelike Curve (CTC)

$$\left\{ \begin{array}{l} r = \text{const.} < 0 \\ \theta = \frac{\pi}{2} \\ t = \text{const.} \\ ds^2 = -\left(r^2 + a^2 + \frac{2ma^2}{r}\right)d\phi^2. \end{array} \right. \quad (6.74)$$

If r is small and negative, then

$$-\frac{2ma^2}{r}d\phi^2 > 0 \quad (6.75)$$

is possible. ϕ is time-like now. So when $\phi \in [0, 2\pi]$, it is a Closed Timelike Curve.

6.4.5 Penrose Diagram

The Penrose Diagram for Kerr Metric is as following

6.5 Reissner-Nordstrom Metric

Let's study the situation with electrical charge but without spin. Unlike Swarchild's and Kerr's, the charge drive the right side of Einstein equation non-zero.

$$\left\{ \begin{array}{l} G_{\mu\nu} = 8\pi GT_{\mu\nu} = 2(F_{ul}F_{\nu}^l - \frac{1}{4}g_{\mu\nu}F_{\rho\nu}F^{\rho\nu}). \\ \nabla_{\mu}F^{\mu\nu} = 0 \end{array} \right. \quad (6.76)$$

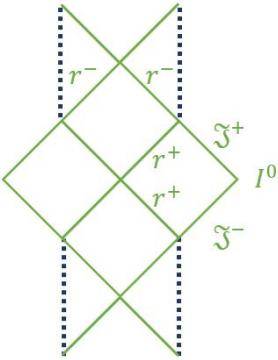


Figure 6.11: Penrose Diagram for Reissner-Nordstrom Metric

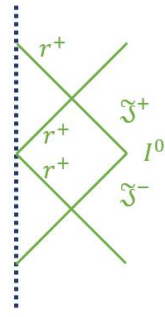


Figure 6.12: Penrose Diagram for External Reissner-Nordstrom Black Hole

Chapter 7

The Spacetime's Perturbation —— the Gravitational Waves

Since GW150914 (means Gravitational Wave Event on 14 Sep. 2015, the picture shown on cover) first discovered the Gravitational Waves, this topic was going to be under the spotlight.

7.1 Perturbation on Vacuum

In vacuum, $R_{\mu\nu} = 0$. Assume a **Linear Perturbation** $|h_{\mu\nu}| \ll 1$ on Minkowski Metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (7.1)$$

For simplify the calculation, let

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}, \quad (7.2)$$

here h is the contraction of $h_{\mu\nu}$, and the \bar{h} is the contraction of $\bar{h}_{\mu\nu}$,

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h\eta^{\mu\nu}\eta_{\mu\nu} = h - 2h = -h. \quad (7.3)$$

Now by using this result,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \frac{1}{2}\bar{h}\eta_{\mu\nu}, \quad (7.4)$$

we get

$$\boxed{h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}.} \quad (7.5)$$

In the following part, we will find out the symmetry of $h_{\mu\nu}$ by calculating the Einstein tensor of $g_{\mu\nu}$. Please note that because the space-time is flat, we can be sure that

$G_{\mu\nu} = 0$.

To get Einstein tensor, we should firstly find out Ricci tensor and Ricci scalar. For $g_{\mu\nu}$, the Ricci tensor is

$$R_{km} = g^{il} \frac{1}{2} \left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l} \right) + g^{il} g_{mp} (\Gamma_{kl}^m \Gamma_{im}^p - \Gamma_{km}^m \Gamma_{il}^p). \quad (7.6)$$

Take in $g_{ik} = \eta_{ik} + h_{ik}$, notice that h_{ik} is small amount and for η_{ik} we have $R_{ik} = 0$. Then we get

$$\begin{aligned} R_{km} &= \frac{1}{2} \eta^{il} \left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l} \right) + O(h^2) \\ &= \frac{1}{2} \left(\frac{\partial^2 h_m^l}{\partial x^k \partial x_l} + \frac{\partial^2 h_k^l}{\partial x^i \partial x_m} - \frac{\partial^2 h}{\partial x^k \partial x_m} - \square h_{km} \right) + O(h^2) \\ &= \frac{1}{2} (\bar{h}_{m,lk}^l + \bar{h}_{k,lm}^l - \square \bar{h}_{km} + \frac{1}{2} \square \bar{h} \eta_{km} + O(h^2)). \end{aligned} \quad (7.7)$$

Easily we can get Ricci scalar

$$\begin{aligned} R &= \eta^{km} R_{km} \\ &= \frac{1}{2} (2\bar{h}_{lm}^{lm} - \square \bar{h} - 2\square \bar{h}) \\ &= \bar{h}_{lm}^{lm} + \frac{1}{2} \square \bar{h}. \end{aligned} \quad (7.8)$$

Therefore, Einstein tensor is

$$G_{km} = R_{km} - \frac{1}{2} g_{km} R = -\frac{1}{2} \square \bar{h}_{km} = 0. \quad (7.9)$$

This equation impose a gauge $\partial_k \bar{h}^{ki} = 0$, or it can be also written as $\partial_k h^{ki} - \frac{1}{2} \partial^i h = 0$. Because h_{ik} is small amount, it will be small amount in any reference frame. Now we try to do a reference transformation. If we change x^i into $x'^i = x^i + \xi_i$, where ξ_i is also small amount, then h_{ik} change into:

$$h'_{ik} = h_{ik} + \frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i}, \quad (7.10)$$

$$h' = \eta^{ik} h'_{ik} = h + 2 \frac{\partial \xi^i}{\partial x_k}. \quad (7.11)$$

As a result,

$$\partial_k h'^{ki} - \frac{1}{2} \partial^i h' = \partial_k h^{ki} - \frac{1}{2} \partial^i h + \square \xi^i = 0, \quad (7.12)$$

$$\square h'_{ik} = \square h_{ik} + \frac{\partial \square \xi_\nu}{\partial x_\mu} + \frac{\partial \square \xi_\nu}{\partial x_\mu}. \quad (7.13)$$

If we choose

$$\square \xi^i = 0. \quad (7.14)$$

Take it into the equation above, we get

$$\partial_i h'^{ik} - \frac{1}{2} \partial^i h' = 0, \quad (7.15)$$

$$\boxed{\square' h_{ik} = 0}, \quad (7.16)$$

which is a general wave equation, representing that magnetic field travel at the speed of light in vacuum.

7.1.1 TT Gauge

7.2 GW with Sources

Now $T_{\mu\nu} \neq 0$.

$$G_{\mu\nu} = -\frac{1}{2} \square \bar{h}_{\mu\nu} = 8\pi G_N T^{\mu\nu}. \quad (7.17)$$

7.2.1 GW from 2 Black holes

Chapter 8

Cosmology

Assume the metric is depending on time,

$$ds^2 = dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^{(2)} \right]. \quad (8.1)$$

At any time, the spacetime is a sphere.

Homogeneous and Isotropic:

8.1 Perfect Fluid

$$\dot{\rho} + \frac{3\dot{a}}{a} (P + \rho) = 0. \quad (8.2)$$

- **Dust**
- **Radiation**

8.1.1 Friedman Equations

Definition 8.1. Hubble Constant

$$H = \frac{\dot{a}}{a}. \quad (8.3)$$

Set

$$\Omega = \frac{8\pi G\rho}{3H^2} = \frac{\rho}{\rho_{critical}}, \quad (8.4)$$

the Friedman Equation can written into

$$\Omega - 1 = \frac{\kappa^2}{H^2 a^3}. \quad (8.5)$$

- If $\rho < \rho_{critical}$, $\kappa < 0$;

- If $\rho = \rho_{critical}$, $\kappa = 0$;
- If $\rho > \rho_{critical}$, $\kappa > 0$;

Now $H \approx 70\text{km/s/Mpc}$.

8.2 Cosmological constant Λ

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.} \quad (8.6)$$

Chapter 9

Introduction to Nonlocal Gravity*

Part III
Appendix

Chapter 10

Notations*

10.1 Einstein Sum Convention

Thanks to Einstein's indolence, we have Einstein notation

$$a^i b_i = \sum_{i=1}^n a^i b_i = a^1 b_1 + a^2 b_2 + \dots + a^n b_n. \quad (10.1)$$

Upper indices represent components of contravariant vectors (vectors), lower indices represent components of covariant vectors (covectors).

In GR cases, $i = 1, 2, 3, 4$.

$$a^i b_i = a^i \eta_{ij} b^j = a^1 b^1 - a^2 b^2 - a^3 b^3 - a^4 b^4, \quad (10.2)$$

when we are in Minkovsky Spacetime, for example.

10.2 Ricci calculus

Parentheses () denote the symmetry of a tensor with respect to those indices, for example

$$T_{(\alpha\beta\gamma\dots)} = \frac{1}{n!} \sum_{p \in \text{permutations}} T_{p(\alpha\beta\gamma\dots)}. \quad (10.3)$$

In contrast, square brackets [] denote the anti-symmetry,

$$T_{[\alpha\beta\gamma\dots]} = \frac{1}{n!} \sum_{p \in \text{permutations}} (-1)^{n_p} T_{p(\alpha\beta\gamma\dots)}. \quad (10.4)$$

10.3 Scalar, Vector and Tensor

Scalar:

$$\underbrace{\phi'(x')}_{\text{in } K'} = \underbrace{\phi(x(x'))}_{\text{in } K}$$

Vector:

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \text{ (CONTROVARIANT).}$$

$$B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta} \text{ (COVARIANT).}$$

Tensors of rank 2

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \text{ (CONTROVARIANT)}$$

$$G'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} G_{\gamma\delta} \text{ (COVARIANT)}$$

Suppose: $x^{\mu} = f^{\mu}(x'^{\mu'})$, $x'^{\mu'} = g^{\mu'}(x^{\mu})$, $x^{\mu} = x^{\mu}(x'^{\mu'})$, $x'^{\mu'} = x'^{\mu'}(x^{\mu})$.

We define vector transform in the form as:

$$V^{\mu} = \Lambda^{\mu}_{\mu'} V'^{\mu'}, \quad (10.5)$$

where $\Lambda^{\mu}_{\mu'} = \frac{\partial x^{\mu}}{\partial x'^{\mu'}}$ (Contravariant).

The definition can be found in Landau and Wiki¹.

10.4 Dual Space

The space $(E, g_{\mu\nu}, \vec{e}_{\mu})$ has a dual space $(E^*, g^{\mu\nu}, \vec{e}^{\mu})$.

Definition 10.1. $g_{\mu\lambda} g^{\lambda\nu} = \delta_{\mu}^{\nu}$

Then we have $g^{\mu\nu} \vec{e}_{\nu} = \vec{e}^{\mu}$. From the definition we can get:

$$\vec{e}^{\mu} \cdot \vec{e}_{\nu} = \delta_{\nu}^{\mu} \quad (10.6)$$

So that we can calculate that:

$$\begin{aligned} \vec{e}^{\mu} \cdot \vec{e}^{\nu} &= g^{\mu\alpha} \vec{e}_{\alpha} \cdot g^{\nu\beta} \vec{e}_{\beta} \\ &= g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta} \\ &= g^{\mu\alpha} \cdot \delta_{\alpha}^{\nu} \\ &= g^{\mu\nu} \end{aligned} \quad (10.7)$$

10.5 Covariant and Contravariant

Assume there is a function ϕ from manifold M to manifold N .

¹https://en.wikipedia.org/wiki/Covariance_and_contravariance_of_vectors

Chapter 11

Differential Geometry*

This chapter is written with the help of the slides of Differential Geometry class held in SUSTech by Professor Yidun Wan. It can help you to understand more about the basis of General Relativity and help you to be familiar with those notation which you have to handle within the calculation. Some results may overlap with those in previous text, then you can regard them as a review.

11.1 Topological Space

Topology is Preliminaries of Differential Geometry. Topological space is something having a 'shape' of the same kind.

Definition 11.1. The Topological Space is a pair (X, \mathcal{T}) , where X is a set and $\mathcal{T} = \{U_i | i \in I\}$ denotes a collection of subsets of X (I is a index set), with the following properties:

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
- If J is a subcollection of I , $\bigcup_{j \in J} U_j \in \mathcal{T}$;
- If K is a **finite** subcollection of I , $\bigcap_{k \in K} U_k \in \mathcal{T}$.

X alone is often called topological space, \mathcal{T} is said to be the topology on X .

Definition 11.2. Any set $U_i \in \mathcal{T}$ is open.

11.1.1 Some Basic Definition

Before we introduce the concept 'connectness' and 'continuity', we have to introduce some basic definition:

Definition 11.3. Interior: The interior A^0 of a set A is the largest open set in A . If $A^0 = A$, A is open.

Definition 11.4. Closed set: A set $A \subset X$ is closed if $A^c = X \setminus A$ is open, where $A^c = \{x \in X \mid x \notin A\}$ is the complement of A .
This depends on the topology of X .

Definition 11.5. Closure: The closure \bar{A} of A in X is the smallest closed set containing A .

Definition 11.6. Boundary: The boundary ∂A of A in X is $\partial A = \bar{A} \setminus A^0$.

Definition 11.7. Dense: $A \subset X$ is dense if $\bar{A} = X$.

Definition 11.8. Neighbourhood: Let T be a topological space, $N(x)$ is a neighbourhood of $x \in X$ if $N(x)$ is a subset of T and contains at least one open subset $U(x) \ni x$ of T .

For example, in \mathbb{R} , $[a, b]$ is a neighbourhood of any $x \in (a, b)$.

Theorem: A subset $A \subset X$ is open if and only if A is neighbourhood of every points in it.

11.1.2 Continuity

Definition 11.9. A map $f : X \rightarrow Y$ between two topological spaces X and Y is continuous if
 \forall open set V in Y ,
 $f^{-1}(V)$ is an open set in X .

For example,

$X = Y = \mathbb{R}$, $f(x) = x^2$ is **continuous**.

$X = \mathbb{R}$, $Y = \{0, 1\}$, $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is **discontinuous**.

11.1.3 Connectness

Definition 11.10. A topological space X is **connected** if it cannot be written as $X = X_1 \cup X_2$ with X_1, X_2 open and $X_1 \cap X_2 = \emptyset$.

Definition 11.11. A topological space is **pathwise connected** if any $x, y \in X$, \exists a continuous map $f : [0, 1] \rightarrow X$.

For example, $f(0) = x$, $f(1) = y$.

Pathway connected means connected, but connected does not mean pathway connected because of the condition topologist's sine curve as figure shows.

Definition 11.12. If $f(0) = f(1)$, f is called a **loop**. Then, X is **simply connected** if any loop in X can be continuously shrunk to a point.

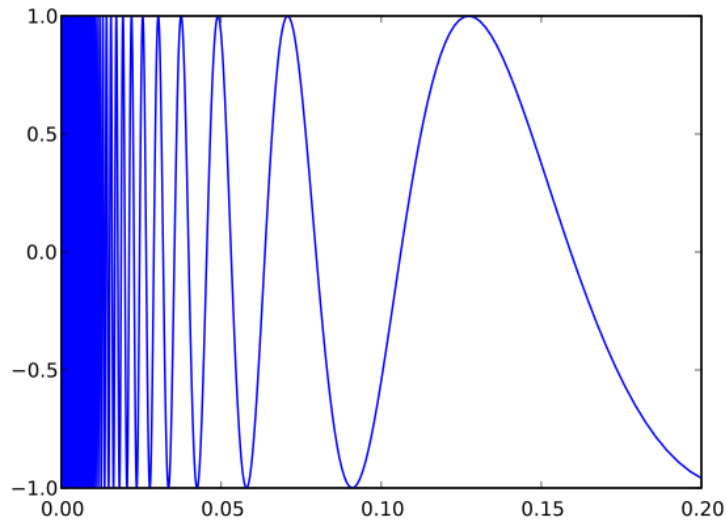


Figure 11.1: Topologist's Sine Curve

11.1.4 Homeomorphic

In a words, Homeomorphic is a way to define an **equivalence** between topological spaces.

Definition 11.13. A **homeomorphism** between two topological spaces X and Y is a **continuous** map $f : X \rightarrow Y$ that has a **continuous** inverse $f^{-1} : Y \rightarrow X$.

We say X is homeomorphic to Y , and vice versa.

A homeomorphism is indeed an equivalence relation, because

- (1) $X \rightarrow id_X \rightarrow X$. (reflectivity)
- (2) $X \rightarrow f \rightarrow Y \Rightarrow Y \rightarrow f^{-1} \rightarrow X$. (symmetry)
- (3) $X \rightarrow f \rightarrow Y, Y \rightarrow g \rightarrow Z \Rightarrow X \rightarrow g \circ f \rightarrow Z$. (transitivity)

11.1.5 Homotopy

Definition 11.14. $X \rightarrow f \rightarrow Y, Y \rightarrow g \rightarrow X, f, g$ both continuous. If

$$\begin{aligned} f \circ g &\sim id_Y \\ g \circ f &\sim id_X, \end{aligned} \tag{11.1}$$

it means X is homotopy to Y , where \sim is the **homotopy of maps**. But what is homotopy of maps?

Definition 11.15. Let $X \rightarrow f_0 \rightarrow Y, Y \rightarrow f_1 \rightarrow X$ be continuous maps. $f_0 \sim f_1$ if \exists a continuous family of functions

$X \times I \rightarrow F_t \rightarrow Y, I = [0, 1]$. That is $F_0(x) = f_0(x), F_1(x) = f_1(x)$.
The family F_t is called a **homotopy**.

For example, $X = (0, 1)$ is a line, $Y = \{0\}$ is a point, show that $X \sim Y$.

Proof. We can define

$$\begin{aligned} f : X \rightarrow Y : f(x) &= 0 \\ g : Y \rightarrow X : g(0) &= \frac{1}{2}. \end{aligned} \quad (11.2)$$

It means

$$\begin{aligned} f \circ g &= \text{id}_Y, \\ g \circ f &= \frac{1}{2} \neq \text{id}_X. \end{aligned} \quad (11.3)$$

From $f \circ g = \text{id}_Y$ we can directly get $f \circ g \sim \text{id}_Y$. As a result, we only need to show $g \circ f \sim \text{id}_X$.

Construct F_t as

$$F_t = tx + \frac{1-t}{2}. \quad (11.4)$$

Clearly

$$\begin{aligned} F_0 &= \frac{1}{2} = g \circ f \\ F_1 &= x = \text{id}_X, \end{aligned} \quad (11.5)$$

which means $g \circ f \sim \text{id}_X$.

That is $(0, 1) \sim \{0\}$.

□

11.2 Differential Manifold

We have learnt homeomorphism between two topological spaces, a equivalence relation. In this section we will focus on an equivalence relation between topological spaces that even finer than homeomorphism.

Remember that a homeomorphism is an invertible continuous map between two topological spaces, but what if we demand the map f be not only continuous but also **smooth (infinitely differentiable)**?

However, because topological space is abstract, we can not directly define smoothness. As a result, we need to endow a topological space with **coordinate**.

11.2.1 Differentiable Manifold and Differentiable Map

Definition 11.16. An m -dimensional topological space M is a **differentiable manifold** if M is endowed in a family of pairs $\{(U_i, \phi_i)\}$, that is
(1) $\{U_i\}$ is a family of open set of M and $\bigcup_i U_i = M$;

- (2) $\phi_i : U_i \rightarrow U'_i \subset \mathbb{R}^m$ is a **homeomorphism**;
- (3) For any U_i and U'_j , if $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$ is **smooth**, that is infinitely differentiable.

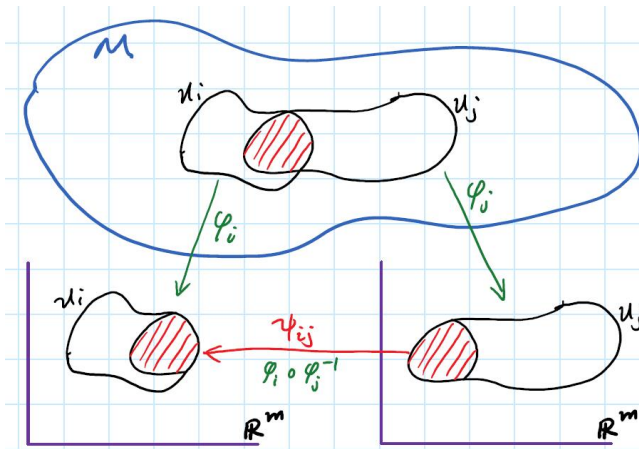


Figure 11.2: Differentiable Manifold

Note that each ϕ_i is represented by a set $\{x^1, x^2, \dots, x^m\}$ of m functions: **the coordinate!**

- (1) A U_i is called a **coordinate neighbourhood**.
- (2) A pair (U_i, ϕ_i) is a **chart**.
- (3) $\{(U_i, \phi_i)\}$ is called an **atlas**.

Definition 11.17. Here are $y(x) = \psi \circ f \circ \phi^{-1}(x)$ or $y^\nu = f^\nu(x^\mu)$. If $y(x)$ is infinitely differentiable, f is said to be **differentiable** or **smooth** at P . The set of all smooth

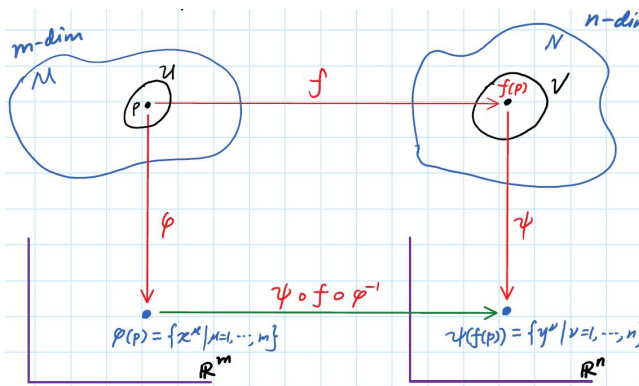


Figure 11.3: Differentiable Map

functions over M is denoted by $F(M)$.

11.2.2 Tangent Space

A generic manifold has no origin, and a straight arrow from origin to a point, as a vector.

In a manifold, one moves along curves. A **curve** is just a map.

$$\begin{aligned} C &: (a, b) \rightarrow M \\ &: t \in (a, b) \rightarrow c(t) \in M. \end{aligned} \quad (11.6)$$

We consider **Jordan curves**, which do not self intersect.

Then, we study the change rate of change of function $M \rightarrow \mathbb{R}$ along a curve $c(t)$ in M .

$$\begin{aligned} \frac{df(c(t))}{dt} \Big|_{t=0} &= \frac{d}{dt} [f \circ \phi^{-1} \circ \phi \circ c(t)] \Big|_{t=0} \\ &= \frac{d}{dt} (f \circ \phi^{-1})(\phi \circ c(t)) \Big|_{t=0}. \end{aligned} \quad (11.7)$$

Remember the \circ does not mean \times , and $f \circ \phi^{-1} \circ \phi \circ c(t)$ actually means $f \circ \phi^{-1}(\phi \circ c(t))$, which is composite function, so that

$$\begin{aligned} \frac{df(c(t))}{dt} \Big|_{t=0} &= \frac{\partial(f \circ \phi^{-1})}{\partial x^\mu} \frac{dx^\mu(c(t))}{dt} \Big|_{t=0} \\ &= \frac{\partial f}{\partial x^\mu} \frac{dx^\mu(c(t))}{dt} \Big|_{t=0} \\ &= (X^\mu \frac{\partial}{\partial x^\mu}) f \\ &= X[f], \end{aligned} \quad (11.8)$$

where we firstly change $f \circ \phi^{-1}$ to f in convenience, and define $X^\mu = \frac{dx^\mu(c(t))}{dt} \Big|_{t=0}$. Then we define that $X = X^\mu \frac{\partial}{\partial x^\mu}$ is the **tangent vector** of M at $P = c(0)$.

Definition 11.18. All tangent vectors at $p \in M$ from a vector space $T_p M$ is the **tangent space** at P .

- (1) The set $\{e_\mu = \frac{\partial}{\partial x^\mu}\}_P$ is a coordinate basis of $T_p M$.
- (2) Any vector V in $T_p M$ can be expanded as

$$V = V^\mu e_\mu = V^\mu \frac{\partial}{\partial x^\mu}. \quad (11.9)$$

- (3) V is independent of the coordinate chosen, because $\phi \circ \phi^{-1} = \phi' \circ \phi'^{-1}$.

Because of law (3), let $p \in U_i \cap U_j$, $\phi_i = \{x^\mu\}$, $\phi_j = \{y^\nu\}$. Then $V = V^\mu \frac{\partial}{\partial x^\mu} = \tilde{V}^\nu \frac{\partial}{\partial y^\nu}$. Then we can get

$$\tilde{V}^\nu = V^\mu \frac{\partial y^\nu}{\partial x^\mu}. \quad (11.10)$$

- (4) $V[f] = V^\mu \frac{\partial f}{\partial x^\mu}$ is the change of f along V .

11.2.3 Cotangent Space

In quantum mechanics we have learnt that $\langle \psi | \psi \rangle$ is a number. $\langle \psi |$ is the dual vector of $|\psi\rangle$, so that dual vector can change vector into a number.

We have calculated the derivative of smooth function

$$\begin{aligned} df(p) &= d[f \circ \phi^{-1} \circ \phi(p)] \\ &= \frac{\partial f}{\partial x^\mu} dx^\mu. \end{aligned} \quad (11.11)$$

From that we can see that, df is not a number. Actually, df is something can be expanded in the basis $\{dx^\mu\}$. But what is dx^μ ?

Definition 11.19.

$$\langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \frac{\partial x^\mu}{\partial x^\nu} = \delta_\nu^\mu. \quad (11.12)$$

It means that

$$\langle \frac{\partial f}{\partial x^\mu} dx^\mu, V^\nu \frac{\partial}{\partial x^\nu} \rangle = \frac{\partial f}{\partial x^\mu} V^\nu \delta_\nu^\mu = \frac{\partial f}{\partial x^\mu} V^\mu, \quad (11.13)$$

which means that $\langle df, V \rangle = V[f] \in \mathbb{R}$ is a number.

Definition 11.20. Cotangent Space:

- (1) df is called 1-form.
- (2) All 1-form at $p \in M$ form the **cotangent space** T_p^*M , with a coordinate basis $\{dx^\mu\}$.
- (3) Any 1-form $\omega \in T_p^*M$ is written as $\omega = \omega_\mu dx^\mu$.
- (4) Similarly ω is **coordinate independent** by the construction.

$$\begin{aligned} \omega &= \omega_\mu dx^\mu = \tilde{\omega}_\nu dy^\nu \\ \Rightarrow \tilde{\omega}_\nu &= \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}, \end{aligned} \quad (11.14)$$

where ω is called covariant.

(5) $\langle \cdot, \cdot \rangle : T_p^*M \times T_pM \rightarrow \mathbb{R}$ is called inner product

$$\langle \omega, v \rangle = \langle \omega_\mu dx^\mu, v^\nu \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu v^\nu \delta_\nu^\mu = \omega_\mu v^\mu. \quad (11.15)$$

11.3 Riemann Geometry

We know that the manifold is equipped with certain geometry defined by metric.

Among all the metrics that can be defined on a manifold, there is a special type called (pseudo) Riemannian metrics, which of particular interest because they describe the spacetime governed by Einstein gravity.

11.3.1 (Pseudo) Riemannian Manifold

Definition 11.21. A metric on a manifold is generalization of such inner products and is clearly a $(0, 2)$ -tensor.

$$\begin{aligned} g_P : T_P^*M \otimes T_P M &\rightarrow \mathbb{R} \\ U \otimes V &\rightarrow g(U, V). \end{aligned} \quad (11.16)$$

For example, recall that in \mathbb{R}^3 , $\vec{u} \cdot \vec{v} = u^1 v^1 + u^2 v^2 + u^3 v^3$, it can be represented by

$$\begin{pmatrix} u^1 & u^2 & u^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}, \quad (11.17)$$

where $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is metric g ,

$$g(\vec{U}, \vec{V}) = g_{ij} U^i V^j \in \mathbb{R}. \quad (11.18)$$

Definition 11.22. A **Riemann metric** on M is a $(0, 2)$ -tensor $g \in \mathcal{J}_2^0(M)$ that satisfies the axioms:

- (1) $g_P(U, V) = g_P(V, U)$ (**symmetric**)
 - (2) $g_P(U, V) \geq 0$ and $= 0$ only if $u = 0$, for all $p \in M$. (**positive definite**).
- The pair (M, g) is called a **Riemannian manifold**.

Definition 11.23. A **pseudo Riemannian metric** on M is a $(0, 2)$ -tensor $g \in \mathcal{J}_2^0(M)$ that satisfies the axioms:

- (1) symmetric
- (2) If $g_P(U, V) = 0 \forall U \in T_P M$, then $V = 0$.

The pair (M, g) is called **pseudo-Riemannian manifold**.

We know that $g_P(U, \cdot) : T_P M \rightarrow \mathbb{R}$, which means $g_P(U, \cdot) = \omega_\mu \in T_P^* M$.

In a coordinate chart (U, ϕ) with coordinate $\{x^\mu\}$

$$\begin{aligned} g_P &= g_{\mu\nu}(p) dx^\mu \otimes dx^\nu \\ \Rightarrow g_P\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) &= g_{\mu\nu} = g_{\nu\mu}. \end{aligned} \quad (11.19)$$

where $g_{\mu\nu}$ can be seen as amplitude and $dx^\mu \otimes dx^\nu$ can be seen as direction in cotangent space.

$g_{\mu\nu}$ can be taken as inevitable matrix with inverse matrix $g^{\alpha\beta}$. That is

$$\begin{aligned} g_{\mu\nu} g^{\nu\rho} &= \delta_\mu^\rho \\ g^{\alpha\beta} &= g_{\mu\nu} g^{\mu\alpha} g^{\nu\beta}. \end{aligned} \quad (11.20)$$

That is, $g_{\mu\nu}$ and $g^{\mu\nu}$ can be used to **raise** and **lower** the tensor indices!
An infinitesimal displacement vector is $dx^\mu \frac{\partial}{\partial x^\mu}$.

$$\begin{aligned} ds^2 &= g(dx^\mu \frac{\partial}{\partial x^\mu}, dx^\nu \frac{\partial}{\partial x^\nu}) \\ &= dx^\mu dx^\nu g(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}) \\ &= g_{\mu\nu} dx^\mu dx^\nu. \end{aligned} \quad (11.21)$$

Often, ds^2 is a metric! The infinitesimal length is $\sqrt{ds^2} = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ from Riemann metric.

11.3.2 Covariant Derivative, Parallel Transport and Geodesic

In Euclidean space, consider a vector field $V(x)$ at two nearby points x and $x' = x + \delta x$.

$$V^y(x') = V^y(x) + \partial_\mu V^y(x) \delta x^\mu + \dots, \quad (11.22)$$

where $\partial_\mu V^y(x)$ is a tensor and δx^μ is a vector.

However, in curve space, a vector field $W(x)$ in the same condition

$$W^y(x') \neq W^y(x) + \partial_\mu W^y(x) \delta x^\mu + \dots, \quad (11.23)$$

where $\partial_\mu W^y(x)$ is not a tensor. As a result, we should build a nontrivial way to "parallelly" transport $W(x)$ to $\tilde{W}(x')$, that is

$$\tilde{W}^\mu(x') - W^\mu(x) \propto \delta x^\mu. \quad (11.24)$$

Also

$$\widetilde{W_1 + W_2}(x') = \tilde{W}_1(x') + \tilde{W}_2(x'). \quad (11.25)$$

To satisfy these two relationship, we can set

$$\tilde{W}^\mu(x') = W^\mu(x) - W^\nu \Gamma_{\nu\rho}^\mu(x) \delta x^\rho. \quad (11.26)$$

Then our target is finding what $\Gamma_{\nu\rho}^\mu$ is. Now because $\tilde{W}(x')$ and $\tilde{W}(x')$ are at x' , we can compare them. Suppose that $W(x) = W^\mu e_\mu$,

$$\begin{aligned} &\lim_{\delta x \rightarrow 0} \frac{W(x') - \tilde{W}(x')}{\delta x^\sigma} \\ &\lim_{\delta x \rightarrow 0} \frac{W^\mu(x) + \partial_\nu W^\nu(x) \delta x^\nu - W^\mu(x) + W^\rho \Gamma_{\rho\nu}^\mu(x) \delta x^\nu}{\delta x^\sigma} e_\mu|_x \\ &= (\partial_\sigma W^\mu + \Gamma_{\rho\sigma}^\mu W^\rho) e_\mu|_x \\ &= \nabla_\sigma W^\mu e_\mu \\ &= \nabla_\sigma W \\ &= W^\mu_{;\sigma} e_\mu, \end{aligned} \quad (11.27)$$

where $\partial_\sigma W^\mu + \Gamma_{\rho\sigma}^\mu W^\rho = \nabla_\sigma W^\mu$ is definition, which is called covariant derivative. Here we have covariant derivative of W with respect to any vector field V .

$$\nabla_\nu W = V^\sigma \nabla_\sigma W = v^\sigma \nabla_\sigma W^\mu e_\mu, \quad (11.28)$$

which means

$$\nabla_\nu W^\mu = V^\sigma \nabla_\sigma W^\mu = V^\sigma W_{;\sigma}^\mu. \quad (11.29)$$

Suppose V is the tangent vector along a curve $x(t)$, that is

$$V^\mu = \frac{dx^\mu(t)}{dt}. \quad (11.30)$$

We have the covariant derivative of W along the curve

$$\begin{aligned} \nabla_\mu W^\mu &= \frac{dx^\sigma(t)}{dt} \nabla_\sigma W^\mu \\ &= \frac{dx^\sigma}{dt} (\partial_\sigma W^\mu + \Gamma_{\rho\sigma}^\mu W^\rho) \\ &= \frac{dW^\mu}{dt} + \Gamma_{\rho\sigma}^\mu W^\rho \frac{dx^\sigma}{dt}. \end{aligned} \quad (11.31)$$

If it = 0, we can say W^μ is **parallely transported** along the curve. Moreover, if W happens to be the tangent vector along $x(\lambda)$, that is $W^\mu = \frac{dx^\mu}{d\lambda} = V^\mu$. Then

$$\begin{aligned} \nabla_\nu V &= \frac{dV^\mu}{d\lambda} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} \\ &= \frac{d^2 X^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}. \end{aligned} \quad (11.32)$$

If it is = 0, it is **geodesic equation**. That is, a geodesic is a curve on which its own tangent vector is parallely transported.

11.3.3 Affine Connection

Relook the covariant derivative

$$\begin{aligned} \nabla_\mu e_\nu &= \nabla_\mu (\delta_\nu^\rho e_\rho) \\ &= (\partial_\mu \delta_\nu^\rho) e_\rho + \Gamma_{\mu\rho}^\sigma \delta_\nu^\rho e_\sigma \\ &= \Gamma_{\mu\nu}^\sigma e_\sigma. \end{aligned} \quad (11.33)$$

As a result, the covariant derivative $\nabla_X Y$ is more formally understood as a map $\nabla : \Xi(M) \times \Xi(M) \rightarrow \Xi(M)$, that is change $(X, Y) \rightarrow \nabla_X Y$ called the affine connection, satisfying

- (1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
- (2) $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$
- (3) $\nabla_{(fX)} Y = f \nabla_X Y$
- (4) $\nabla_X(fY) = X[f]Y + f \nabla_X Y$.

11.3.4 Metric Connection

Definition 11.24. An affine connection ∇ is a **metric connection**, or metric connection if

$$\nabla_{\lambda} g_{\mu\nu} = 0, \quad (11.34)$$

that is

$$\nabla_{\nu}(g(X, Y)) = 0 \quad (11.35)$$

11.4 From Equivalence Principle Directly to General Relativity

As we all know, equivalence principle is the most important basis of General Relativity. Actually we can directly get many items of GR using equivalence principle. In this section, we will introduce details of this process. The definition of equivalence principle has been shown in 4.1.2.

To sum up, equivalence principle has two implication:

- (1) Gravity acts on everything uniformly.
- (2) Inertial mass = gravitational mass.

11.4.1 Geodesic Equation From the Equivalence Principle

Consider a free particle in the Minkovsky space. We can get

$$\frac{d^2 y}{dt^2} = 0. \quad (11.36)$$

Actually, in the condition that there is no outside force acting on the object, the object will move along a straight line. As a result, it describe a straight line in the Minkovsky space (Remember this conclusion!). We describe the coordinate system of this Minkovsky space as X .

After that, we introduce a changing of coordinate from X to Y : $y = x - \frac{1}{2}at^2$, with which we can get

$$\frac{d^2 x}{dt^2} = a \quad (11.37)$$

Now we consider image in Y , it can be locally consider as Minkovsky space, which means that

$$d\tau^2 = \eta_{\mu\nu} dy^{\mu} dy^{\nu}. \quad (11.38)$$

Then we consider a locally free particle, from which we can get

$$\frac{d^2 y}{d\tau^2} = 0 \quad (11.39)$$

Suppose y^{μ} is related to general x^{μ} (by coordinate transformation). That is $y^{\mu} = y^{\mu}(x)$. Then

$$\frac{dy^{\nu}}{d\tau} = \frac{dy^{\nu}}{dx^{\rho}} \frac{dx^{\rho}}{d\tau}, \quad (11.40)$$

which means

$$\begin{aligned} \frac{d^2 y^\nu}{d\tau^2} &= \frac{dy^\nu}{dx^\rho} \frac{dx^\rho}{d\tau^2} + \frac{d^2 y^\nu}{dx^\rho dx^\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} \\ &= 0. \end{aligned} \quad (11.41)$$

Derive the equation, we can get

$$\frac{d^2 x^\nu}{d\tau^2} + \left(\frac{\partial x^\nu}{\partial y^\mu} \frac{\partial^2 y^\mu}{\partial x^\rho \partial x^\sigma} \right) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0. \quad (11.42)$$

Now, our target is to remove y in the equation, which only appear in the term $\left(\frac{\partial x^\nu}{\partial y^\mu} \frac{\partial^2 y^\mu}{\partial x^\rho \partial x^\sigma} \right)$. Then we should introduce Equivalence Principle. Because $y^\nu = y^\nu(x)$, we have

$$\eta_{\mu\nu} dy^\mu dy^\nu = g_{\rho\sigma} dx^\rho dx^\sigma. \quad (11.43)$$

That is

$$g_{\rho\sigma} = \eta_{\mu\nu} \frac{dy^\mu}{dx^\rho} \frac{dy^\nu}{dx^\sigma}, \quad (11.44)$$

which means

$$\partial_\alpha g_{\rho\sigma} = \eta_{\mu\nu} \left(\frac{d^2 y^\mu}{dx^\rho dx^\alpha} \frac{dy^\nu}{dx^\sigma} + \frac{dy^\mu}{dx^\rho} \frac{d^2 y^\nu}{dx^\sigma dx^\alpha} \right). \quad (11.45)$$

From that we have

$$-\partial_\alpha g_{\rho\sigma} + \partial_\rho g_{\sigma\alpha} + \partial_\sigma g_{\alpha\rho} = 2\eta_{\mu\nu} \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial^2 y^\nu}{\partial x^\rho \partial x^\sigma}. \quad (11.46)$$

Then

$$\begin{aligned} \eta^{\beta\lambda} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial x^\gamma}{\partial y^\lambda} (-\partial_\alpha g_{\rho\sigma} + \partial_\rho g_{\sigma\alpha} + \partial_\sigma g_{\alpha\rho}) &= 2\delta_\nu^\lambda \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial^2 y^\nu}{\partial x^\rho \partial x^\sigma} \\ &= 2 \frac{\partial^2 y^\lambda}{\partial x^\rho \partial x^\sigma}. \end{aligned} \quad (11.47)$$

That is

$$\begin{aligned} \frac{1}{2} \eta^{\beta\lambda} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial x^\gamma}{\partial y^\lambda} (-\partial_\alpha g_{\rho\sigma} + \partial_\rho g_{\sigma\alpha} + \partial_\sigma g_{\alpha\rho}) &= \frac{1}{2} g^{\alpha\gamma} (-\partial_\alpha g_{\rho\sigma} + \partial_\rho g_{\sigma\alpha} + \partial_\sigma g_{\alpha\rho}) \\ &= \Gamma_{\rho\sigma}^\lambda \\ &= \frac{\partial x^\gamma}{\partial y^\lambda} \frac{\partial^2 y^\lambda}{\partial x^\rho \partial x^\sigma}, \end{aligned} \quad (11.48)$$

which is **Christoff symbols!**

Take it into the original equation, we can get

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma_{\rho\sigma}^\nu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0, \quad (11.49)$$

which is **geodesic equation**. Remember it is locally a straight line in the Y coordinate.

11.4.2 Gravitational Action From Equivalence Principle

Using Equivalence Principle and Least Action Principle, we can also get geodesic equation.

We know

$$S_{SR} = -m \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} \rightarrow S_{GR} = -m \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}. \quad (11.50)$$

The left part means locally flat and the right part is generic metric. Then we deal with action S_{GR} ,

$$\begin{aligned} S_{GR} &= -m \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \\ &= -m \int d\lambda \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \\ &= \int d\lambda L. \end{aligned} \quad (11.51)$$

Then from $\delta S_{GR} = 0$, we can get Euler Lagrange equation

$$\frac{d}{d\lambda} \left(\frac{2}{L} g_{\mu\rho} \frac{dx^\mu}{d\lambda} \right) - \frac{1}{L} \partial_\rho g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \quad (11.52)$$

By letting $\lambda = \tau$, where τ is proper time.

$$\frac{d}{d\lambda} \left(2g_{\mu\nu} \frac{dx^\mu}{d\tau} - \partial_\rho g_{\mu\nu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \quad (11.53)$$

From that equation we can also get

$$\frac{d^2 x^\nu}{d\tau^2} + \frac{1}{2} g^{\rho\gamma} (\partial_\mu g_{\sigma\rho} + \partial_\sigma g_{\mu\rho} - \partial_\rho g_{\mu\sigma}) \frac{dx^\mu}{d\tau} \frac{dx^\sigma}{d\tau}. \quad (11.54)$$

It means

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma_{\rho\sigma}^\nu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0, \quad (11.55)$$

which is **geodesic equation**. It gives you a simpler way to calculate Christoffel symbols.

11.4.3 Newtonian Limit

In this section, we will recover the Newtonian physics from Einstein's physics by introducing Newtonian Limit.

First of all, Newtonian Limit is

(1) The particle moves very slowly, as compared with the speed of light.

$$\frac{dx^0}{d\tau} \gg \frac{dx^i}{d\tau} \Rightarrow \frac{dx^i}{dx^0} = v^i \ll 1 = c. \quad (11.56)$$

(2) The gravitational field is weak. That is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h), \quad (11.57)$$

where $h_{\mu\nu}$ is very small.

(3) The gravitational field is static. (Because Newton don't know $g_{\mu\nu}$ could be vary in time.)

$$\partial_0 h_{\mu\nu} = 0. \quad (11.58)$$

Now we impose these conditions to the geodesic equation:

$$\frac{d^2 X^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dX^\rho}{d\tau} \frac{dX^\sigma}{d\tau} = 0. \quad (11.59)$$

Inducing condition (1), we can get

$$\frac{d^2 X^\mu}{d\tau^2} + \Gamma_{00}^\mu \left(\frac{dX^0}{d\tau}\right)^2 \approx 0 \quad (11.60)$$

because other terms is negligible compared to $\frac{dX^0}{d\tau}$.

By inducing condition (2), we can get

$$\begin{aligned} \Gamma_{\rho\sigma}^\mu &= \frac{1}{2} g^{\mu\nu} (-g_{\rho\sigma,\nu} + g_{\sigma\nu,\rho} + g_{\nu\rho,\sigma}) \\ &\approx \frac{1}{2} (\eta^{\mu\nu} - h^{\mu\nu}) (-h_{\rho\sigma,\nu} + h_{\sigma\nu,\rho} + h_{\nu\rho,\sigma}) \\ &\approx \frac{1}{2} \eta^{\mu\nu} (-h_{\rho\sigma,\nu} + h_{\sigma\nu,\rho} + h_{\nu\rho,\sigma}). \end{aligned} \quad (11.61)$$

That is

$$\Gamma_{00}^\mu = \frac{1}{2} \eta^{\mu\nu} (-h_{00,\nu} + h_{0\nu,0} + h_{\nu 0,0}), \quad (11.62)$$

and from which we can finally get

$$\Gamma_{00}^\mu = -\frac{1}{2} \eta^{\mu\nu} h_{00,\nu}. \quad (11.63)$$

Also, from condition (3), we can get that in particular

$$\begin{aligned} \Gamma_{00}^0 &= -\frac{1}{2} h_{00,0} = 0 \\ \Gamma_{00}^i &= -\frac{1}{2} h_{00,i}. \end{aligned} \quad (11.64)$$

Take it into what we get from condition one

$$\frac{d^2 X^i}{d\tau^2} = \frac{1}{2} h_{00,i}, \quad (11.65)$$

where $g_{00} = \eta_{00} + h_{00} = -1 + h_{00}$.

Suppose $h_{00} = -2\Phi = \frac{2GM}{r}$, there is

$$\frac{d^2 X^i}{d\tau^2} = -\partial_i \Phi = -\vec{\nabla} \Phi. \quad (11.66)$$

It is **Newtonian!**

It implies that far from a spherical symmetric mass distribution, the gravitational field must be weak. That is

$$g_{00} \rightarrow -1 + \frac{2GM}{r}. \quad (11.67)$$