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PIGNA
1839

Lagrangian Formulation of classical Mechanics

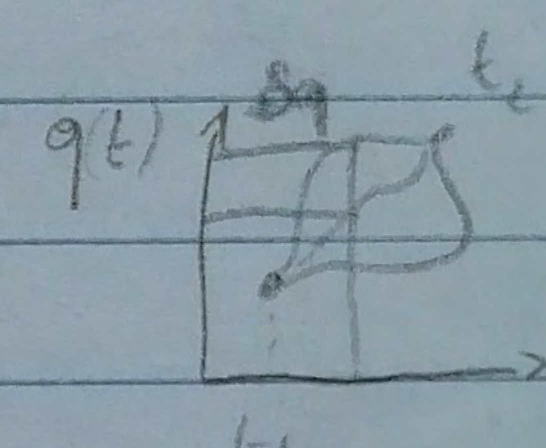
Point particle Lagrangian:

$\mathcal{L}(q, \dot{q}, t)$

Let the system occupy the positions $q^{(1)}$ at t_1, $q^{(2)}$ at t_2, the the system moves between these positions in such a way that the integral:

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt$$

takes the minimal possible value.



Let's consider a variation of $q(t)$, namely

$$q(t) + \delta q(t).$$

At t_1 and t_2 $\quad \delta q(t_1) = \delta q(t_2) = 0$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \mathcal{L}(q + \delta q, \dot{q} + \delta \dot{q}, t)\, dt - \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt .$$

We Now expand (Taylor expand) to the First order, in δq:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \right) dt = 0$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \cdot \delta q \right) dt =$$

$$\frac{\delta \mathcal{L}}{\delta \dot q}\delta q\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q}\right)\delta q\, dt$$

$\|$

0 because
$\delta q(t_1) = \delta q(t_2) = 0$

$$\delta S = 0 \ \forall \delta q \Rightarrow \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} = 0$$

For more d.o.f:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_i} = 0, \quad i = 1, \dots, s$$

$\uparrow$ for example # of particles.

If $\frac{\partial \mathcal{L}}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_i} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial v_i}$ const. (For $\mathcal{L} = \frac{1}{2}v^2$: Law of inertia)
EXERCISE

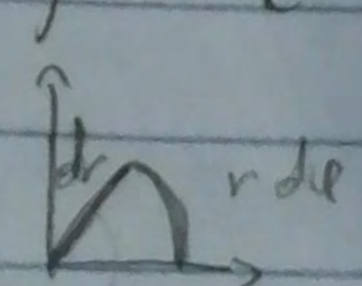
$\mathcal{L}(v^2) = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2)$ (*) in Cartesian coordinates.

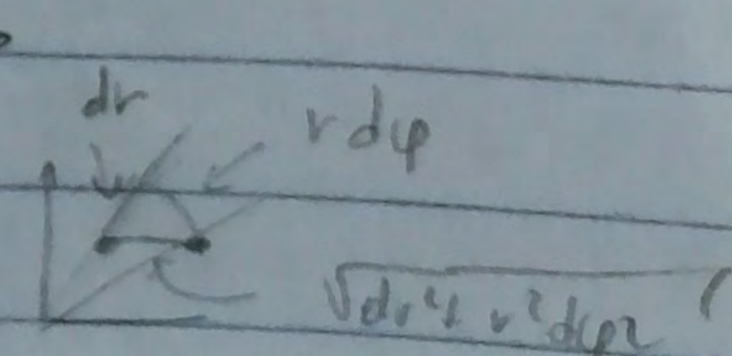
In cilindrical coordinates $\mathcal{L} = ?$

$v^2 = \left(\frac{dl}{dt}\right)^2$, but $dl^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\varphi^2 + dz^2$

$\Rightarrow \mathcal{L} = \frac{1}{2}m\left(\dot r^2 + r^2\dot\varphi^2 + \dot z^2\right)$, which in form is different from (*).
Indeed, suppose I define $r \equiv x, \varphi \equiv y, z \equiv z \Rightarrow$

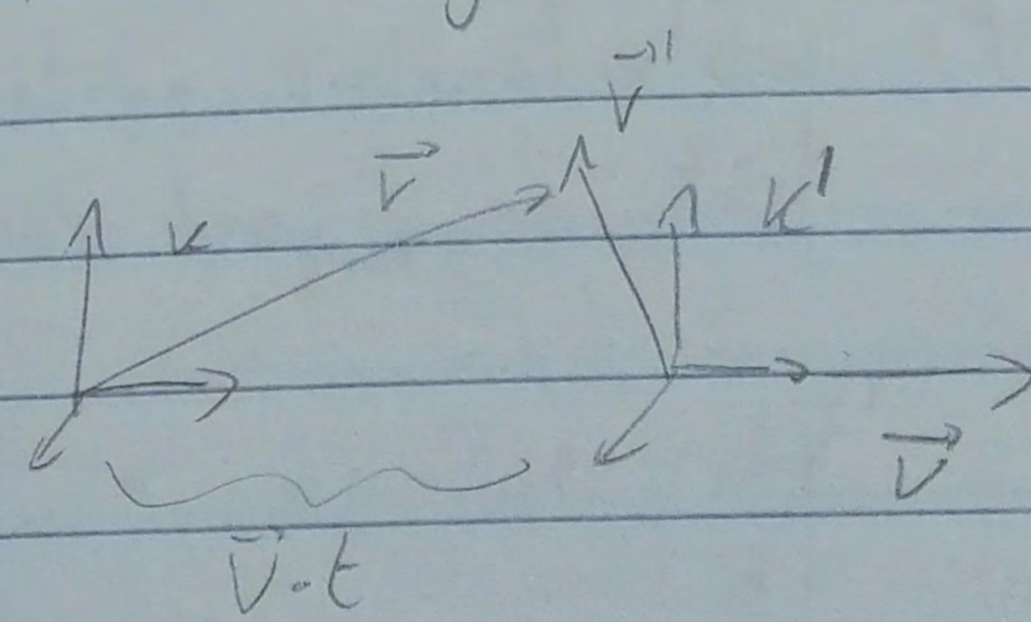
$$\mathcal{L} = \frac{1}{2}m\left(\dot x^2 + x^2\dot y^2 + \dot z^2\right) \neq (*).$$





GALILEO'S RELATIVITY PRINCIPLE

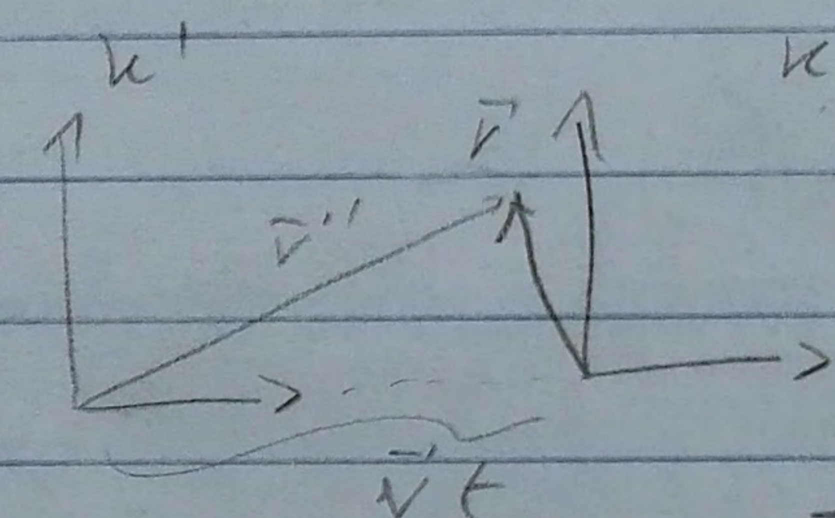
Given 2 reference systems K, K',



The laws of Mechanics are invariant under the transformation:

$$\vec{r} = \vec{r}' + \vec{V} t \qquad \text{or} \qquad \vec{r}' = \vec{r} - \vec{V} t \qquad (*)$$

$$t = t'$$



$\vec{r} + \vec{V} t = \vec{r}'$

$\vec{v} + \vec{V} = \vec{v}'$

Consider a free particle.

$\vec{v}' \quad \vec{v}$

$$\mathcal{L}(v^2) = \frac{1}{2} m v^2 \qquad \dot{\vec{r}}' = \dot{\vec{r}} + \vec{V}_{rel}$$

$$\vec{v} = \vec{v}' + \vec{V}_{rel}$$

$$\mathcal{L}(v'^2) = \frac{1}{2} m \left(v^2 + \vec{V}_{rel}^{\,2} + 2 \vec{v} \cdot \vec{V}_{rel} \right)$$

$$= \frac{1}{2} m v^2 + \underbrace{\frac{d}{dt} \left(m \vec{r} \cdot \vec{V}_{rel} + \frac{1}{2} m \vec{V}_{rel}^{\,2}\, t \right)}_{\text{total derivative.}}$$

If I use $\vec{V}_{rel} \to -\vec{\ }$

$$\Rightarrow \boxed{\mathcal{L}(v'^2) = \mathcal{L}(v^2)}$$

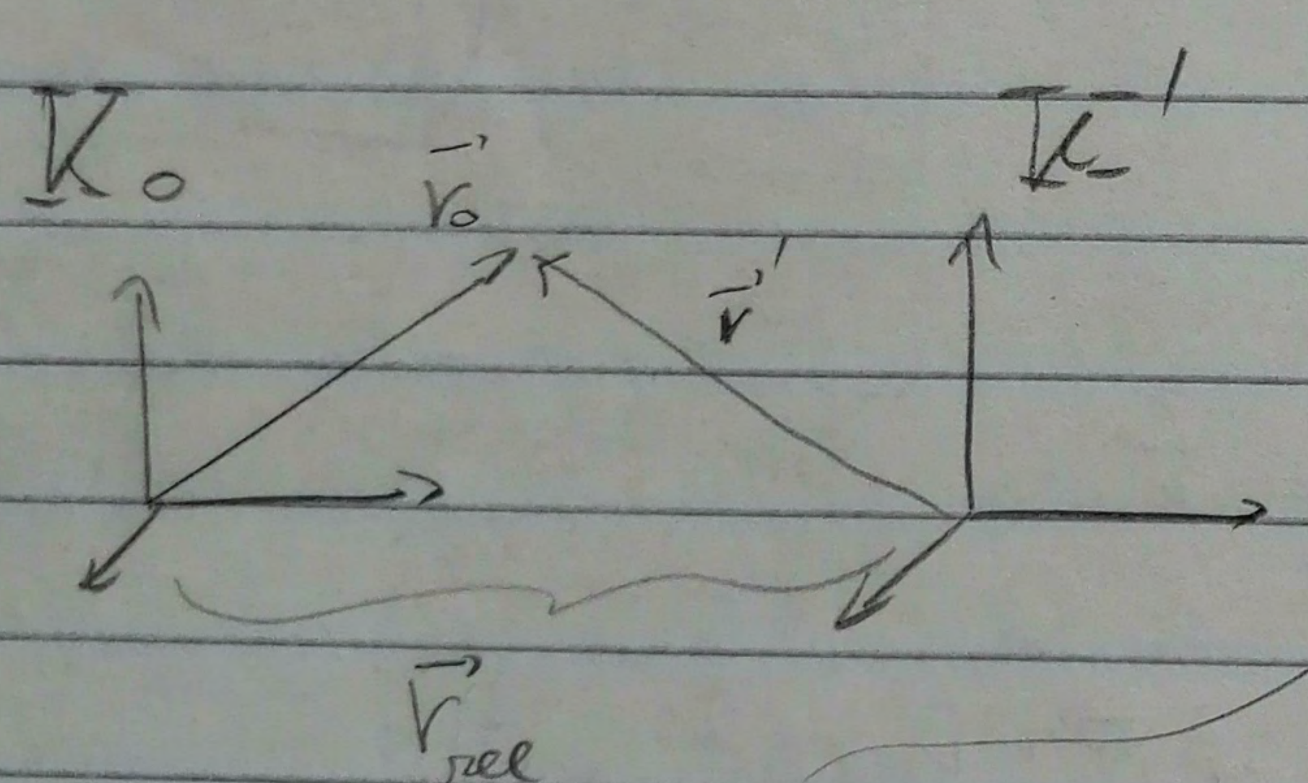
Lagrangian for a system of particles

$$\mathcal{L} = \sum_a \frac{1}{2} m_a v_a^2 - U(r_1, r_2 \dots r)$$

instantaneous interaction between particles.

EOM: $$m_a \frac{d\vec{v}_a}{dt} = \underbrace{-\frac{\partial U}{\partial \vec{r}_a}}_{\vec{F}_a}$$

Motion in a non-inertial frame



$\vec{r}_0 = \vec{r}' + \vec{V}_{rel}$

derive:

$$\frac{d\vec{r}_0}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{V}_{rel}}{dt}$$

$$\vec{v}_0 = \vec{v}' + \vec{V}_{rel}$$

$$\vec{v}_0 = \vec{v}' + \vec{V}_{rel}(t)$$

v_0: velocity of a particle respect to K_0,

v': " " the same particle " " K'.

$$\mathcal{L}_0 = \frac{1}{2} m v_0^2 - U$$

We replace $\vec{v}_0 = \vec{v}' + \vec{V}_{rel}$ in $\mathcal{L}_0$ to obtain $\mathcal{L}'$, namely
means that the functional dependence is not the same.

$$\mathcal{L}' = \frac{1}{2} m \left(\vec{v}' + \vec{V}_{rel}\right)^2 - U$$

$$= \frac{1}{2} m \vec{v}'^2 + \underbrace{m \vec{v}' \cdot \vec{V}_{rel}} + \frac{1}{2} m \vec{V}_{rel}^2 - U$$

$$m \vec{V}_{rel} \cdot \vec{v}' = m \vec{V}_{rel} \cdot \frac{d\vec{r}'}{dt} = \underbrace{\frac{d}{dt}\left(m \vec{V}_{rel} \cdot \vec{r}'\right)}_{\text{total derivative}} - m \frac{d\vec{V}_{rel}}{dt} \cdot \vec{r}'$$

$$\vec{V}_{rel}^2 = \vec{V}_{rel}^2(t) \equiv \frac{d}{dt}\left(\vec{f}(t)\right) \Rightarrow \text{total derivative.}$$

$$\Rightarrow \mathcal{L}' = \frac{1}{2} m \vec{v}'^2 - m \vec{r}' \cdot \underbrace{\frac{d\vec{V}_{rel}}{dt}}_{\equiv \vec{W}} - U$$

$\boxed{\text{different in form from } \mathcal{L}.}$

$$= \frac{1}{2} m \vec{v}'^2 - m \vec{v}' \cdot \vec{W} - U$$

EOM in R':

$$m \frac{d\vec{v}'}{dt} = \underbrace{-\frac{\partial U}{\partial \vec{r}'}}_{\vec{F}} \underbrace{- m \overbrace{\vec{W}(t)}^{+\vec{\Omega}_{rel}}}_{\vec{F}_{apparent}}$$

$- m \vec{\Omega}_{rel}$

$$\vec{W}(t) = \vec{\Omega}_{rel}$$

Analogy:

$$\vec{v}_0 = \vec{v}' + \vec{V}_{rel}$$

$$\frac{d\vec{v}_0}{dt} = \frac{d\vec{v}'}{dt} + \frac{d\vec{V}_{rel}}{dt} \Rightarrow$$

$$\vec{F} = m\frac{d\vec{v}'}{dt} + m\vec{\Omega}_{rel} \Rightarrow m\frac{d\vec{v}'}{dt} = \vec{F} \overbrace{- m \vec{\Omega}_{rel}}^{\vec{F}_{app}}$$

Lagrangian in general coordinates : $q_a \to x_a$

$a = 1, \dots, s$

$$x_a = f_a(q_1, \dots, q_s)$$

$$\dot{x}_a = \sum_k \frac{\partial f_a}{\partial q_k} \dot{q}_k$$

$$\mathcal{L}(x_a) = \frac{1}{2} \sum_i \dot{x}_i^2 - U(x)$$

$$= \frac{1}{2} \sum_i \dot{x}_i \dot{x}_i - U(x)$$

$$= \frac{1}{2} \sum_i \sum_j \sum_k \frac{\partial f_i}{\partial q_j} \frac{\partial f_i}{\partial q_k} \dot{q}_j \dot{q}_k - U(x(q))$$

$$= \frac{1}{2} \sum_{j,k} \underbrace{\left(\sum_i \frac{\partial f_i}{\partial q_j} \frac{\partial f_i}{\partial q_k} \right)} \dot{q}_j \dot{q}_k - U(q)$$

$$= \underbrace{\frac{1}{2} \sum_{j,k} \alpha_{jk}(q) \dot{q}_j \dot{q}_k} - U(q) .$$

still quadratic in velocities in general coordinates

<u>σ – Model</u>

EXT. 0

$$t = t'$$

$$x^i = x^{i'} + v^i t$$

$$x^0 = t \qquad x_0 = \eta_{00} t = \eta_{00} x^0$$

$$\partial'_\mu \partial'^\mu \psi =$$

$$= \frac{\partial}{\partial x'^\mu} \frac{\partial}{\partial x'_\mu} \psi$$

$$\partial_\mu \partial^\mu = \eta_{\mu\nu} \partial^\mu \partial^\nu$$

$$= \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \left[\frac{\partial x^e}{\partial x'_\mu} \frac{\partial}{\partial x^e} \psi \right]$$

$$= \frac{\partial x^0}{\partial x'^\mu} \frac{\partial}{\partial x^0} \left[\frac{\partial x^e}{\partial x'_\mu} \frac{\partial}{\partial x^e} \psi \right] +$$

$$+ \frac{\partial x^i}{\partial x'^\mu} \frac{\partial}{\partial x^i} \left[\frac{\partial x^e}{\partial x'_\mu} \frac{\partial}{\partial x^e} \psi \right] =$$

$$= \underbrace{\frac{\partial t}{\partial t'}}_{=1,\ t'=t} \frac{\partial}{\partial t} \left[\frac{\partial x^e}{\partial x'_0} \frac{\partial}{\partial x^e} \psi \right] + \cancel{\underbrace{\frac{\partial t}{\partial x'^i}}_{=0} \frac{\partial}{\partial t} \left[\frac{\partial x^e}{\partial x'_i} \frac{\partial}{\partial x^e} \psi \right]}$$

$$+ \frac{\partial x^i}{\partial t'} \frac{\partial}{\partial x^i} \left[\frac{\partial x^e}{\underbrace{\partial x'_0}_{= t'}} \frac{\partial}{\partial x^e} \psi \right] + \underbrace{\frac{\partial x^i}{\partial x'^j}}_{\delta^i_j} \frac{\partial}{\partial x^i} \left[\frac{\partial x^e}{\partial x'_j} \frac{\partial}{\partial x^e} \psi \right]$$

$$= \frac{\partial}{\partial t} \left[\frac{\partial x^e}{\partial t'} \eta_{00} \frac{\partial}{\partial x^e} \psi \right] + v^i \frac{\partial}{\partial x^i} \left[\frac{\partial x^0}{\underbrace{\partial t'}_{t}} \eta_{00} \frac{\partial}{\partial x^0} \psi \right]$$

$$+ v^i \frac{\partial}{\partial x^i} \left[\underbrace{\frac{\partial x^j}{\partial t}}_{v^j} \eta_{00} \frac{\partial}{\partial x^j} \psi \right] +$$

$$+ \delta^i_j \frac{\partial}{\partial x^i} \left[\delta^k_j \frac{\partial}{\partial x^k} \psi \right] =$$

$$= \frac{\partial}{\partial t} \left[- \frac{\partial}{\partial t} \psi - v^k \frac{\partial}{\partial x^k} \psi \right] + v^i \left[\frac{\partial}{\partial x^i} (-) \frac{\partial}{\partial t} \psi \right]$$

$$- v^i \frac{\partial}{\partial x^i} \left[v^j \frac{\partial}{\partial x^j} \psi \right] + \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^j} \psi =$$

$$\left[-\frac{\partial^2}{\partial t^2}\psi - 2v^i \frac{\partial}{\partial x^i}\frac{\partial}{\partial t}\psi - v^i \frac{\partial}{\partial x^i}\left[v^j \frac{\partial}{\partial x^j}\psi \right] + \right.$$

$$\left. + \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} \right]\psi = 0$$

Relativistic particle in Electro-magnetic Field.

$$S = \int_a^b \left(-mc\, ds - \frac{e}{c} A_i\, dx^i \right)$$

$$\begin{pmatrix} + & & \\ & - & \\ & & - \end{pmatrix}$$

$$A_i \equiv (\varphi, \vec{A})$$

$$\Rightarrow S = \int_a^b \left(-mc\, ds - \frac{e}{c} \varphi\, dt + \frac{e}{c} \vec{A} \cdot d\vec{x} \right) \quad \leftarrow \vec{v} = \frac{d\vec{x}}{dt}$$

$$= \int_{t_1}^{t_2} \left(-mc^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{e}{c} \varphi + \frac{e}{c} \vec{A} \cdot \vec{v} \right) dt$$

$$\underline{\vec{P}} = \frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c} \vec{A}$$

$$\vec{P} - \frac{e}{c}\vec{A} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\mathcal{H} = \vec{v} \cdot \frac{\partial \mathcal{L}}{\partial \vec{v}} - \mathcal{L} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\varphi$$

Let's calculate:

$$\frac{(\mathcal{H} - e\varphi)^2}{c^2} = m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A} \right)^2 = m^2 \gamma^2 \frac{c^4}{c^2}$$

$$m^2c^2 + \frac{m^2 v^2}{1 - \frac{v^2}{c^2}} = \frac{m^2c^2 - \cancel{m^2c^2 \frac{v^2}{c^2}} + \cancel{m^2 v^2}}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow \mathcal{H} - e\varphi = \sqrt{m^2c^4 + c^2\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}$$

$$\Rightarrow \mathcal{H} = \sqrt{m^2c^4 + c^2\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\varphi \, .$$

The laws of Physics must be the same in any Inertial reference frame. Different reference frames are related by Lorentz transformations. Therefore, the mathematical equations that describe the same Physics in different reference frames must be Lorentz invariant or covariant.

$$\int A_\mu(x(\lambda))\,\frac{dx^\mu}{d\lambda}$$

LORENTZ INVARIANT THEORY

$$\boxed{S = \underbrace{-mc^2\int ds}_{S_p} \underbrace{-\frac{e}{c}\int A_\mu dx^\mu}_{S_I} - \frac{1}{4\pi}\frac{1}{c}\int F_{\mu\nu}F^{\mu\nu}\,d^4x}$$

$F_{\mu\nu}(x)$ vs $F_{\mu\nu}(x(\lambda))$ no

Full fundamental theory ~ 100 years ago.

$$\delta_x(S_p + S_I) = 0 \Rightarrow mc\int \eta_{\mu\nu}\frac{du^\mu}{ds}\delta x^\nu ds - \frac{e}{c}\int A_\mu \delta dx^\mu - \frac{e}{c}\int \delta A_\mu dx^\mu$$

$$mc\int \eta_{\mu\nu}\frac{du^\mu}{ds}\delta x^\nu ds - \frac{e}{c}\int A_\mu d\delta x^\mu - \frac{e}{c}\int \partial_\nu A_\mu \delta x^\nu dx^\mu$$

$$+\frac{e}{c}\int dA_\mu \delta x^\mu - \frac{e}{c}\int \partial_\nu A_\mu \frac{dx^\mu}{ds}\delta x^\nu ds$$

$$+\frac{e}{c}\int \partial_\nu A_\mu dx^\nu \delta x^\mu - \frac{e}{c}\int \partial_\nu A_\mu u^\mu \delta x^\nu ds$$

$$+\frac{e}{c}\int \partial_\mu A_\nu u^\mu \delta x^\nu ds - \frac{e}{c}\int \partial_\nu A_\mu u^\mu \delta x^\nu ds$$

$$+\frac{e}{c}\int F_{\mu\nu} u^\mu \delta x^\nu ds$$

$$m\eta_{\mu\nu}\frac{du^\mu}{ds} = -\frac{e}{c}F_{\mu\nu}u^\mu \Rightarrow m\frac{du_\nu}{ds} = -\frac{e}{c}F_{\mu\nu}u^\mu$$

$$m\frac{du_\alpha}{ds} = \frac{e}{c}F_{\alpha\beta}u^\beta$$

$$\delta S_p = \delta\left(-mc\int ds\right) = \delta\left(-mc\int\sqrt{+\dot{x}^\mu \dot{x}_\mu}\,d\lambda\right) = \delta\left(-mc\int \frac{1}{2\left(\frac{ds}{d\lambda}\right)}(+2\dot{x}^\mu)\,\delta\dot{x}^\nu \eta_{\mu\nu}\,d\lambda\right)$$

$$= -mc\int \frac{d}{dt}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}^\mu \dot{x}_\mu}}\,\delta x^\nu \eta_{\mu\nu}\right) + mc\int \frac{d}{d\lambda}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}^\mu \dot{x}_\mu}}\right)\delta x_\mu$$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}}\right) \underset{s=\lambda}{\uparrow}= \frac{du^\mu}{ds}$$

$$\left[\ \frac{ds}{d\lambda} = \sqrt{\dot{x}^\mu \dot{x}_\mu} \qquad \delta\frac{dx}{d\lambda} = \frac{d}{d\lambda}\delta x\ \right] \ (*)$$

$$-\frac{1}{4}\delta\int F_{\mu\nu}F^{\mu\nu}\,d^4x\ \frac{1}{4\pi}$$

$$= -\frac{1}{4}\int\left(\delta F_{\mu\nu}\,F^{\mu\nu} + F_{\mu\nu}(\delta F^{\mu\nu})\right)d^4x\ \frac{1}{4\pi}$$

$$= -\frac{1}{2}\int F_{\mu\nu}(\delta F^{\mu\nu})\,d^4x\ \frac{1}{4\pi}$$

$$= -\frac{1}{2}\int F^{\mu\nu}\,\delta(\partial_\mu A_\nu - \partial_\nu A_\mu)\,d^4x\ \frac{1}{4\pi}$$

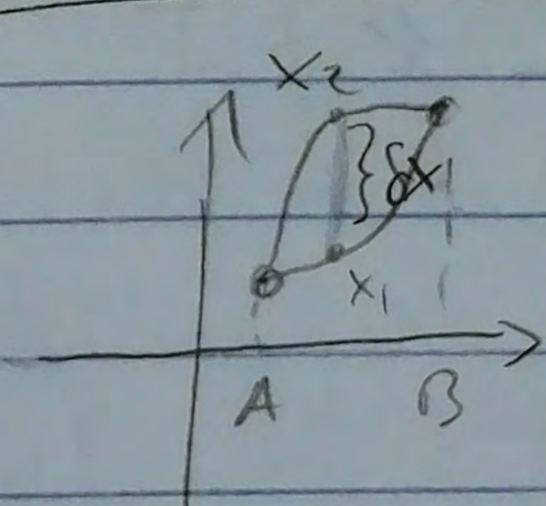
$$= -\frac{1}{2}\int\left[F^{\mu\nu}(\partial_\mu \delta A_\nu) - F^{\mu\nu}(\partial_\nu \delta A_\mu)\right]d^4x\ \frac{1}{4\pi}$$

$$= -\frac{1}{2}\int\left[(-\partial_\mu F^{\mu\nu})\,\delta A_\nu + (\partial_\nu F^{\mu\nu})\,\delta A_\mu\right]d^4x\ \frac{1}{4\pi}$$

$$= +\frac{1}{2}\int\left[(\partial_\mu F^{\mu\nu})\,\delta A_\nu - (\partial_\nu F^{\mu\nu})\,\delta A_\mu\right]d^4x\ \frac{1}{4\pi}$$

$$+\,\partial_{\overset{\mu}{\nu}} F^{\overset{\nu}{\mu}\overset{\mu}{\nu}}\,\delta A_{\overset{\nu}{\mu}}$$

$$= \int(\partial_\mu F^{\mu\nu})\,\delta A_\nu\,d^4x\ \frac{1}{4\pi}$$



(*) Notice: $\delta x = x_2(t) - x_1(t)$

$$\delta\left(\frac{dx}{dt}\right) = \dot{x}_2(t) - \dot{x}_1(t) = \frac{d}{dt}x_2(t) - \frac{d}{dt}x_1(t) =$$

$$= \frac{d}{dt}\left(x_2(t) - x_1(t)\right) = \frac{d}{dt}\delta x \Rightarrow \frac{d}{dt}\delta = \delta\frac{d}{dt}$$

$$de = \rho \, dV \quad \text{3-volume}$$

charge $\qquad$ 0,1,2,3 $\qquad$ scalar

$$de \, dx^\mu = \rho \, dV \, dx^i = \rho \, dV \frac{dx^\mu}{dt} dt$$

$\underbrace{dt \cdot dV}_{dV_{(4)}}$: scalar $\Rightarrow \rho \frac{dx^\mu}{dt}$ 4-vector.

invariant Lorentz

$$\Rightarrow \text{Def}: \quad J^\mu = \rho \frac{dx^\mu}{dt} = \left(\rho \frac{dx^0}{dt}, \rho \frac{d\vec{x}}{dt} \right) = (c\rho, \vec{J}) \qquad x^0 = ct$$

$$\delta_A \left(-\frac{e}{c} \int A_\mu \frac{dx^\mu}{dt} \cdot dt \right) = -\frac{e}{c} \int \frac{dx^\mu}{dt} (\delta A_\mu) \, dt$$

$$e = \int \rho \, dV \Rightarrow S_{int} = -\frac{1}{c} \int \int \rho \, dV A_\mu \frac{dx^\mu}{dt} dt$$

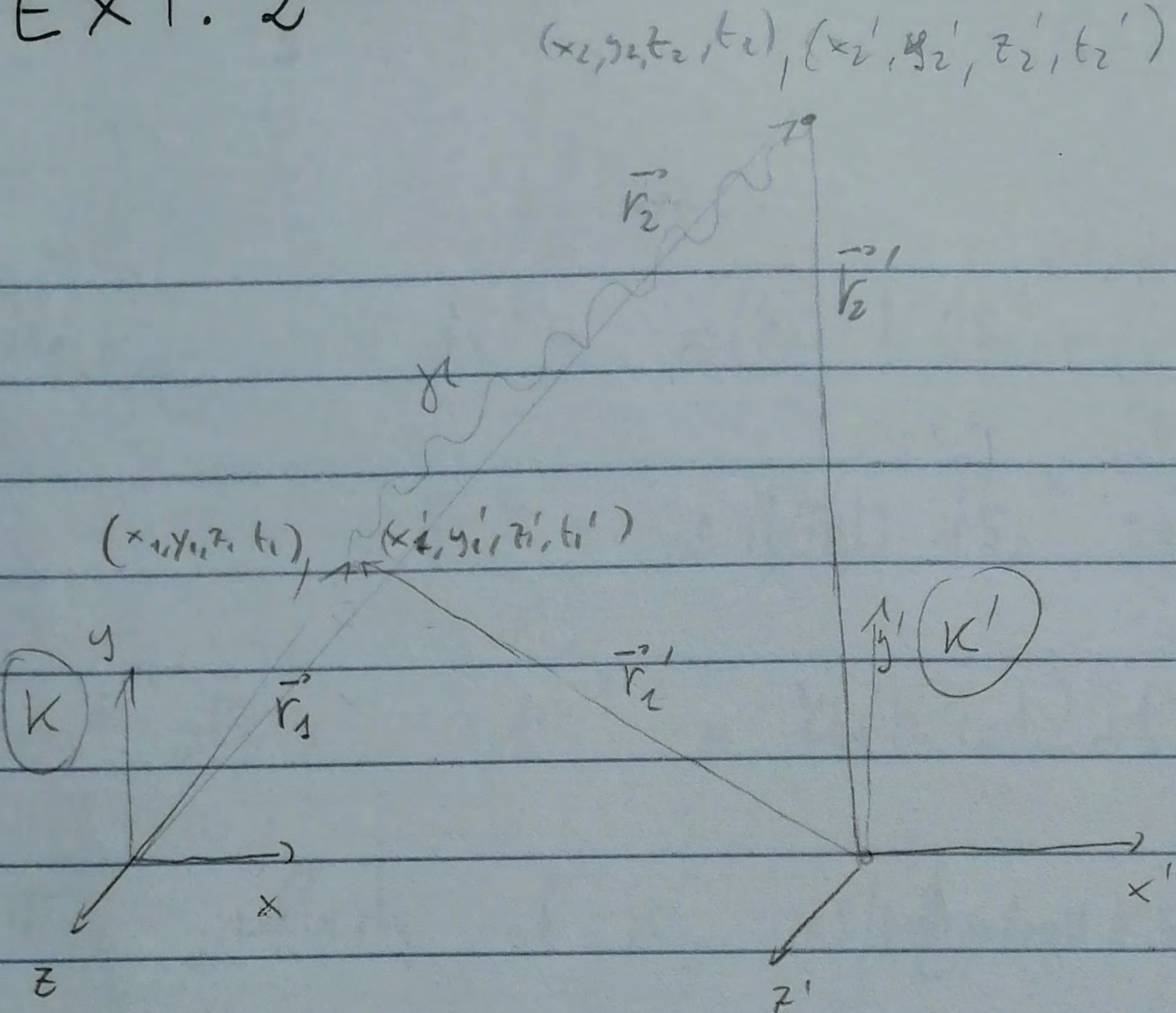
$$= -\frac{1}{c} \int \rho A_\mu \frac{dx^\mu}{dt} d^4x$$

$$= -\frac{1}{c^2} \int A_\mu J^\mu \, d^4x$$

$$= -\frac{1}{c^2} \int J^\mu \delta A_\mu \, d^4x = -\frac{1}{c^2} \int J^\mu \delta A_\mu \, d^4x$$

Total variation:

$$\frac{1}{4\pi} \partial_\mu F^{\mu\nu} - \frac{1}{c^2} \rho J^\nu = 0 \Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c^2} J^\nu}$$



FROM "c" Invariance follows:

event (1)

$E_1 = (x_1, y_1, z_1, t_1)$

$E_2 = (x_2, y_2, z_2, t_2)$

$\vec{r}_1 = (x_1, y_1, z_1)$ $\quad$ $\vec{r}_2 = (x_2, y_2, z_2)$

$\vec{r}_1\,' = (x_1', y_1', z_1')$ $\quad$ $\vec{r}_2\,' = (x_2', y_2', z_2')$

$$(\vec{r}_2 - \vec{r}_1)\cdot(\vec{r}_2 - \vec{r}_1) = c^2 (t_2 - t_1)^2$$

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} = c^2 (t_1 - t_2)^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = c^2 (t_2 - t_1)^2 \quad \text{in } K$$

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 = c^2 (t_2' - t_1')^2 \quad \text{in } K'$$

In general we define the distance of two EVENTS:

$$s_{12}^2 = \Delta\vec{r}\cdot\Delta\vec{r} - c^2 \Delta t^2 = \Delta r^2 - c^2 \Delta t^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2 t^2$$

From the invariance of c follows that if $s_{12}^2 = 0$ in K
$\Rightarrow s_{12}'^2 = 0$ in K' for any K'.

Infinitesimal distance:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

If $ds^2 = 0$ in K $\Rightarrow$ $ds'^2 = 0$ in K'

because of invariance of c. Namely:

$$\frac{d\vec{x}^2}{dt^2} = \underbrace{c^2 \quad , \quad \frac{d\vec{x}'^2}{dt'^2} = c^2}_{\text{same}}$$

but of course $\vec{x}' \neq \vec{x}$ and in particular $t' \neq t$.
← contrary to galileo.

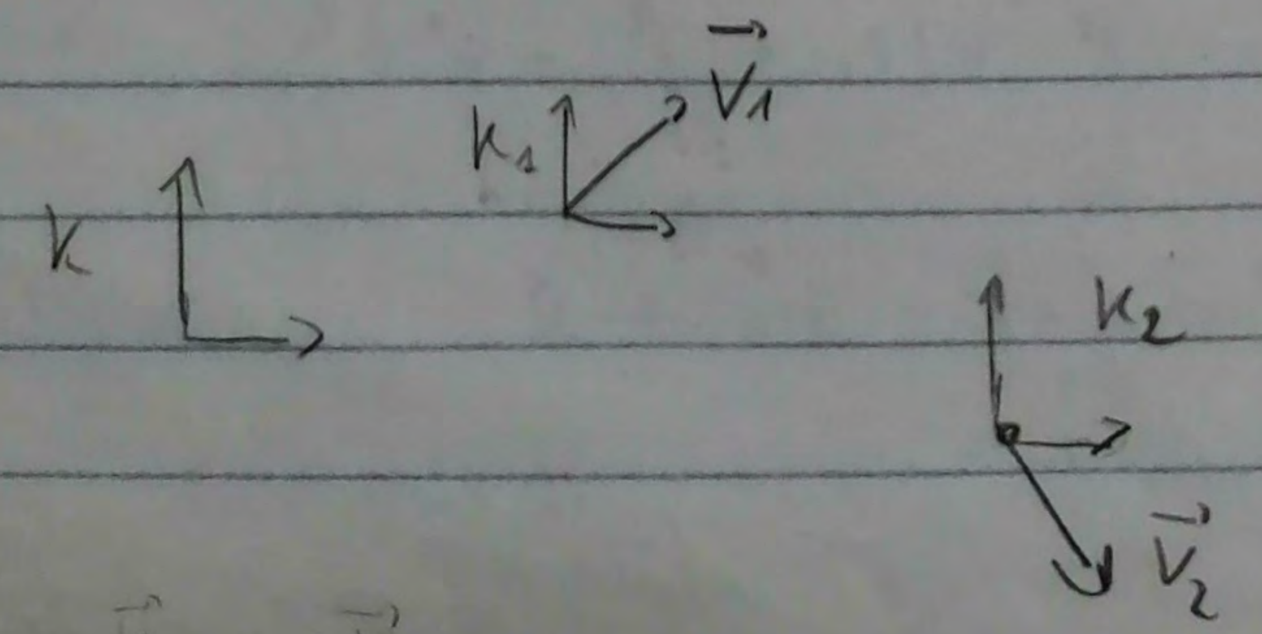
$\Rightarrow$ in general $ds^2 = a\, ds'^2$, where $a = a(|\vec{v}_{rel}|)$
↑ because they are infinitesimal of the same order.
↑ relative velocity.

From ~~uniformity~~ homogeneity of space and time $a(\cancel{\vec{x}}, \cancel{t}, |\vec{v}_{rel}|)$ can not depend on $\vec{x}, t$. (It means invariance: $t \to t + \text{const.}$, $x_i \to x_i + \text{const.}$)

From isotropy of space $a(|\vec{v}_{rel}|, \cancel{\theta})$ can not depend on the direction of the relative velocity.

What is "a"?

Let us consider 3 reference frames K, K_1, K_2



$\vec{V}_1, \vec{V}_2$ velocities of K_1, K_2 respect to K.

$\vec{v}_2 - \vec{v}_1 = \vec{v}_{12}$

$\vec{v}_2 - \vec{v}_1 = \vec{v}_{12}$ $\quad |\vec{v}_{12}|^2 = (\vec{v}_2 - \vec{v}_1)\cdot(\vec{v}_2 - \vec{v}_1) = |\vec{v}_2|^2 + |\vec{v}_1|^2 - 2(\vec{v}_1 \cdot \vec{v}_2)$

Then: $ds^2 = a(|\vec{v}_1|)\, ds_1^2$ ⊛

$ds^2 = a(|\vec{v}_2|)\, ds_2^2$ ⊛⊛

but also $ds_1^2 = a(|\vec{v}_{12}|)\, ds_2^2$, where

$\vec{v}_{12}$ velocity of K_2 respect to K_1.

Finally, comparing ⊛ and ⊛⊛ we get:

$ds^2 = a(|\vec{v}_1|)\, a(|\vec{v}_{12}|)\, ds_2^2$

$ds^2 = a(|\vec{v}_2|)\, ds_2^2$

the ratio is $\frac{ds^2}{ds^2} = 1 = \frac{a(v_1) \cdot a(v_{12})}{a(v_2)}$

$$\Rightarrow \boxed{\frac{a(v_2)}{a(v_1)} = a(v_{12})} \longleftarrow ⊛$$

where $v_1 = |\vec{v}_1|$, $v_2 = |\vec{v}_2|$, $v_{12} = |\vec{v}_{12}|$.

Now, $|\vec{v}_{12}|$ depends on $|\vec{v}_1|$, $|\vec{v}_2|$, and θ, where

$\frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1|\,|\vec{v}_2|} = \cos\theta$,

However θ is not in the left side of ⊛,
therefore ⊛ can be valid iff $a(|\vec{v}|) = 1$ or $a(|\vec{v}|) = \text{const}$.

Finally: $\boxed{ds^2 = ds'^2}$, because we can absorbe the constant in x_i, t.

Tachyon and causality violation (Tolman thought experiment)

Let me consider two points A and B in space and a signal propagation from A to B at the speed v_0,

$$\Rightarrow \Delta t = t_B - t_A = \frac{|\vec{x}_B - \vec{x}_A|}{v_0}.$$

Here A happens before B.

Let's consider now another reference frame moving with relative speed $\vec{V}$, the time of arrival in B is given according to the Lorentz transformation,

$$\Delta t' = t'_B - t'_A = \left(t_B - \frac{v x_B}{c^2}\right)\gamma(v) - \left(t_A - \frac{v x_A}{c^2}\right)\gamma(v)$$

$$= \gamma \cdot \Delta t - \frac{v}{c^2}\gamma \underbrace{(x_B - x_A)}_{v_0 \Delta t}$$

$$= \gamma(v)\,\Delta t\left(1 - \frac{v_0 v}{c^2}\right)$$

if $v_0 > c \Rightarrow \Delta t' < 0 \Rightarrow t'_B < t'_A \Rightarrow$

causality violation because the effect happen before the cause in this frame.

GENERAL RELATIVITY

1° AXIOM

The space-time is a 4-dimensional Manifold.

Def. Let us consider a separable Hausdorff space S and a family of open sets $\{A_i\}$, which cover S.

Let also us assume that exists "M" such that:

- any A_i is Homeomorphic (one to one correspondence) to an open of $\mathbb{R}^M$: this correspondence is called CHART and defines a coordinate system $\{x^\alpha\}$ in A_i.

The set of charts defines an ATLAS of S.

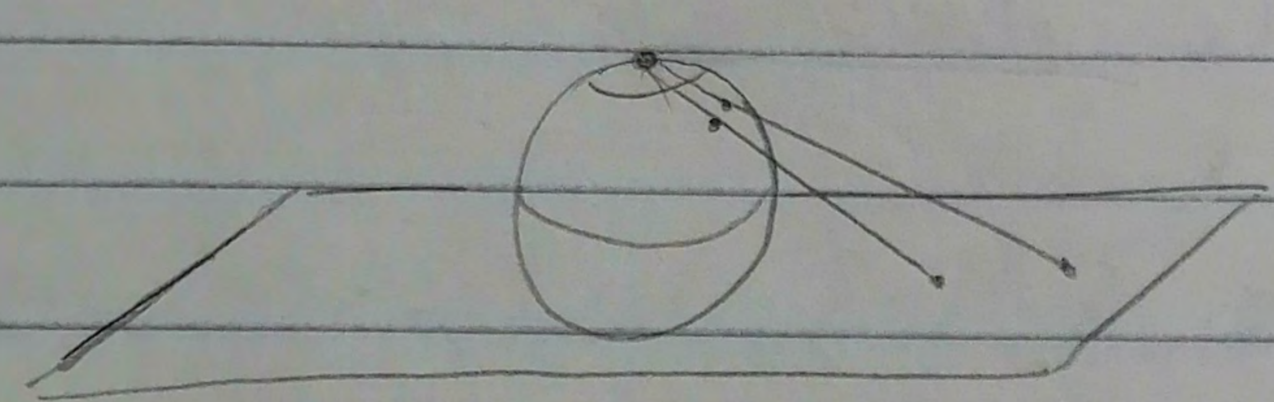
NAMELY: CHART : (A_i, φ_i)

ATLAS : $\{(A_i, \varphi_i)\}$ $\qquad \varphi_i : A_i \to \mathbb{R}^M$

- If $\{x^\alpha\}$, $\{y^\alpha\}$ are coordinate systems relative to TWO CHARTS (Also of TWO different ATLAS), in the intersection of the TWO CHARTS the coordinates $\{y^\alpha\}$ are C^∞ functions of the $\{x^\alpha\}$ invertible.

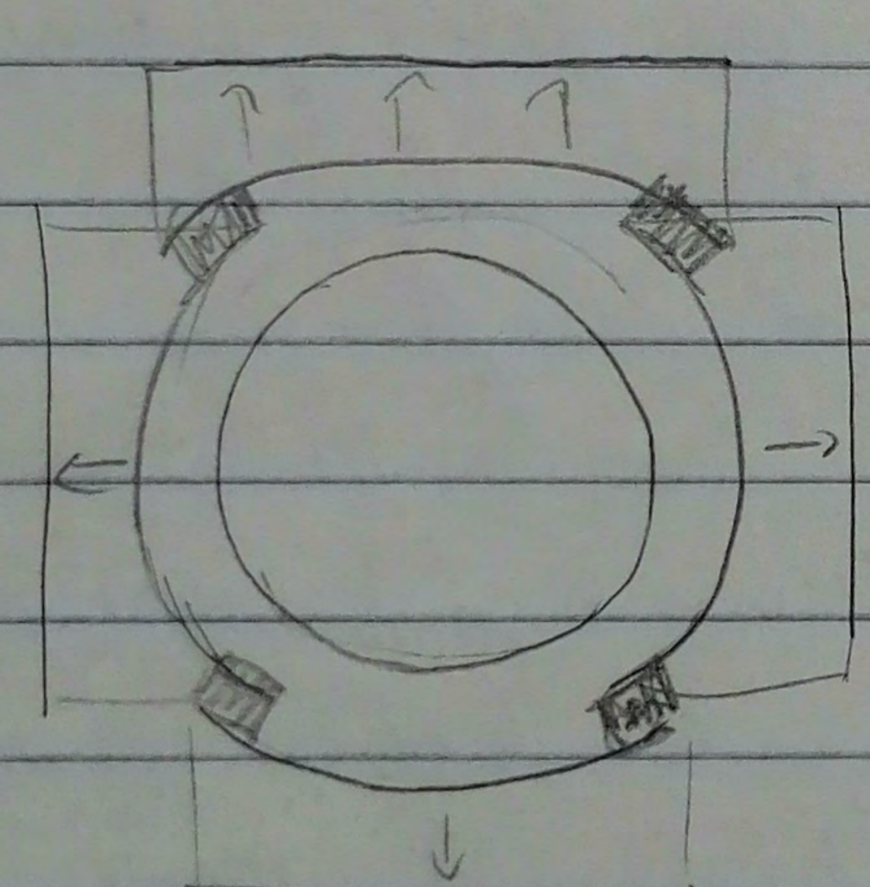
$\Rightarrow$ S is a MANIFOLD (C^∞) of dimension "M".

EXAMPLES : Plane, Sphere, torus are 2-dimensional manifolds, but not Homomorphic.
For the sphere we need 2-CHARTS.
For the torus 3-CHARTS.

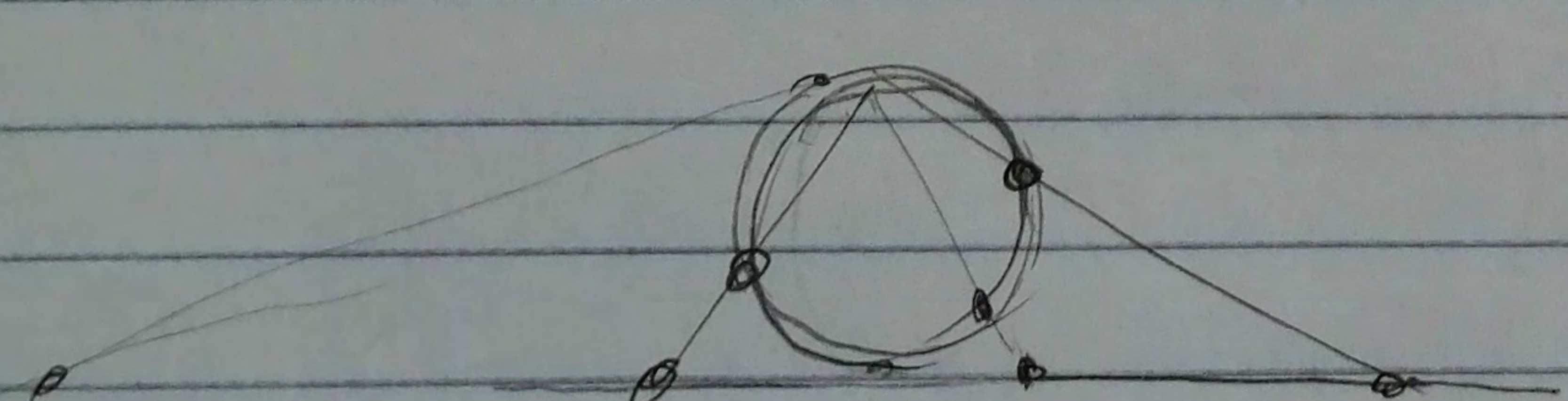


STEREOGRAPHIC PROJECTION.

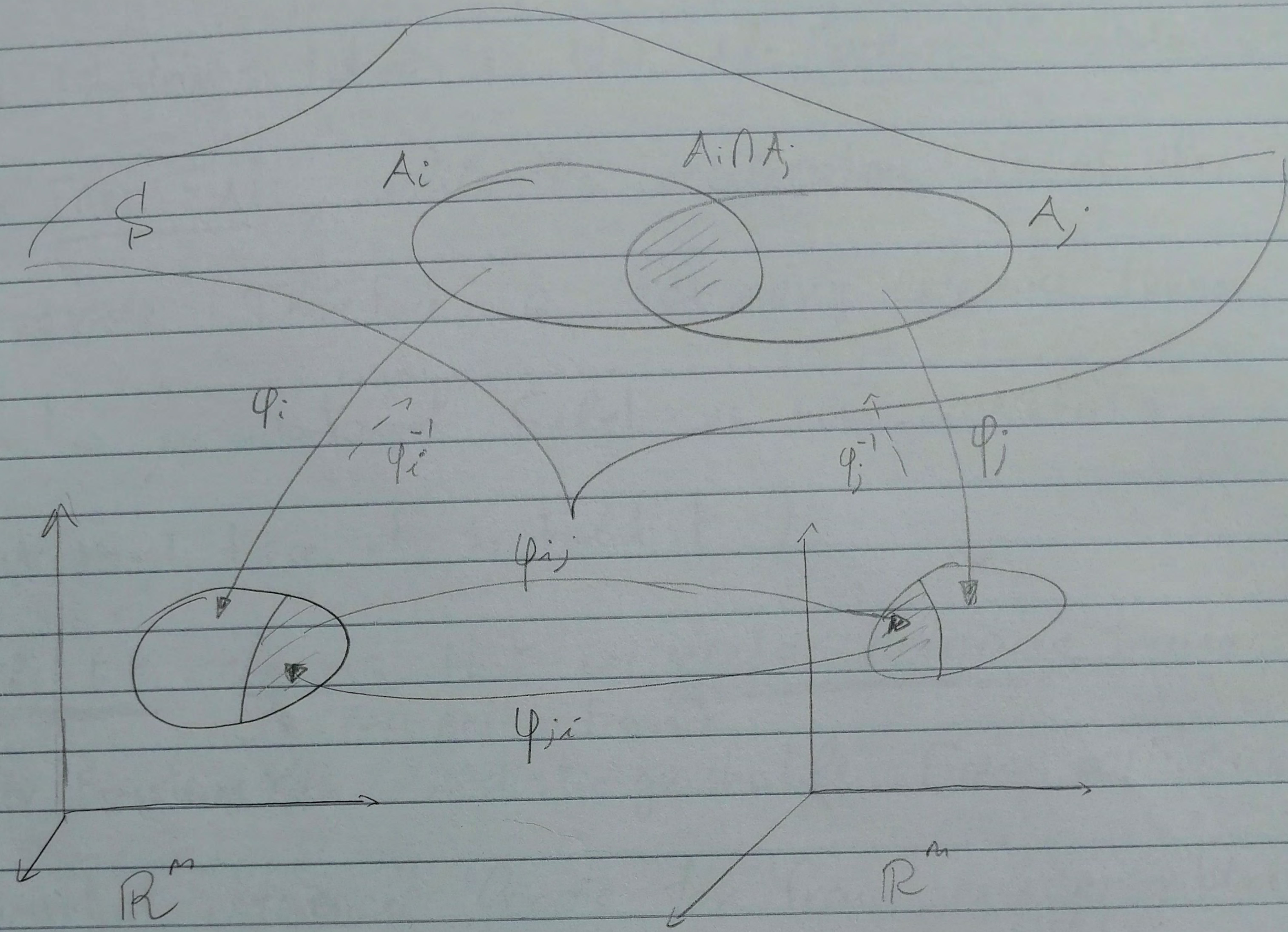
FOUR CHARTS COVERING OF THE CIRCLE.



We can not cover the circle with only one CHART.

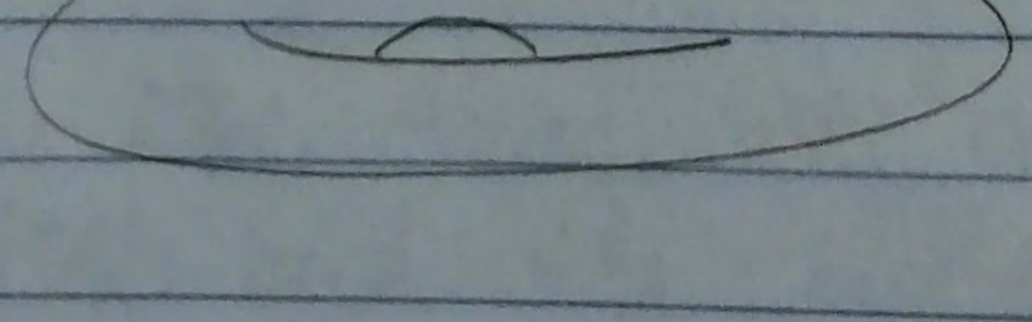


circle cover with two charts



$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}\Big|_{\varphi_i(A_i \cap A_j)} : \varphi_i(A_i \cap A_j) \longrightarrow \varphi_j(A_i \cap A_j)$$

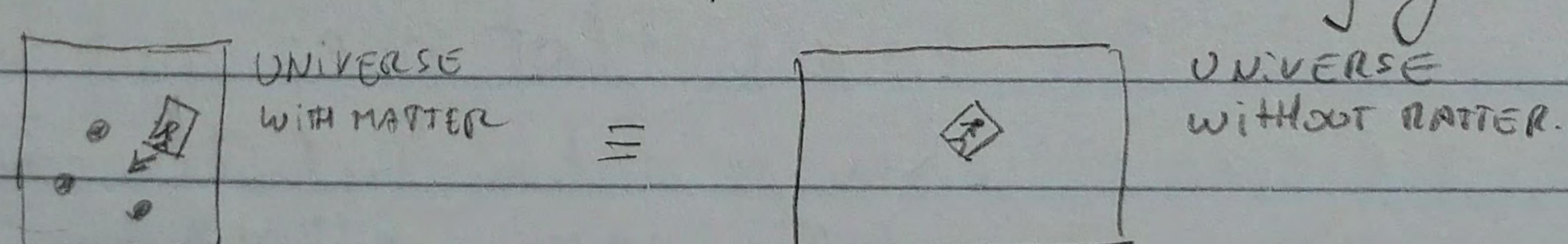
Torus



EQUIVALENCE PRINCIPLE (EP)

The Einstein's idea is that the "natural" state is the FREE FALL, while the acceleration is of the observer. (Therefore, a stationary reference frame in a gravitational field is non inertial: we must force it to hold it.)

The EP claims that an inertial reference frame (FREE FALL Reference Frame) for Einstein is indistinguishable from a Newtonian inertial reference locate far from any gravitational source.



Weak EP: it applays only to the motion of particles due to gravity.

STRONG EP: the equivalence is extended to all Physical phenomena.

The equivalence is only local because the gravitational fields and the accelerations can be non uniform, (Tidal forces).

SECOND 2° AXIOM

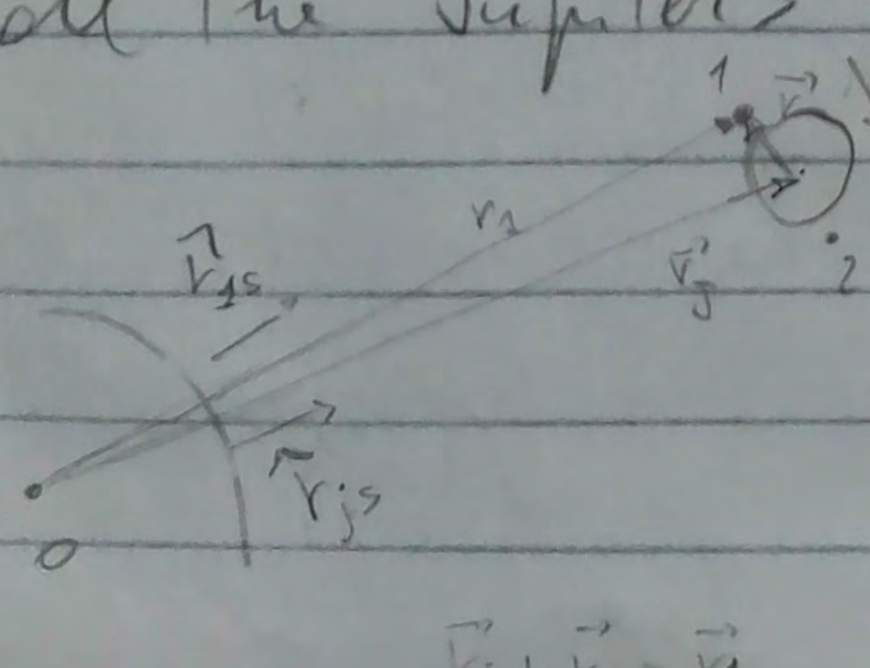
In the neighborhood of any point in the spacetime it is always possible to find a local Inertial reference frame (LIRF) with the following properties:

- In LIRF the Physics is Lorentzian (locally)
  special relativity locally.
- The LIRF is identified by the absence of gravity.

In Classic Physics a reference frame is "inertial" if a body is not subject to forces. Now we do not need to remove the forces that are universal (gravity). FOR EXAMPLE: proton or electron feel the same ~~interaction~~ and have the same acceleration.

The weak EP was well known to Galileo and in particular to Newton. Indeed, Newton based the equivalence between inertial and gravitational mass on the observation that all the Jupiter's satellites move as if there was not the sum.

$\hat{r}_{j1} = -\hat{r}_{1j}$



$\vec{r}_j + \vec{v} = \vec{r}_1$

$$\cancel{m_J}\,\ddot{\vec{r}}_j = -\frac{G\,\cancel{m_J}\,M_s\,\hat{r}_{js}}{|r_j - r_s|^2} - \frac{G\,m_1\,\cancel{m_J}\,\hat{r}_{j1}}{|r_j - r_1|^2}$$

$$\cancel{m_1}\,\ddot{\vec{r}}_1 = -\frac{G\,\cancel{m_1}\,M_s\,\hat{r}_{1s}}{|\vec{r}_1 - \vec{r}_s|^2} - \frac{G\,\cancel{m_1}\,M_J\,\hat{r}_{1J}}{|\vec{r}_1 - \vec{r}_j|^2}$$

$\vec{r}_j - \vec{r}_s \simeq \vec{r}_1 - \vec{r}_s$

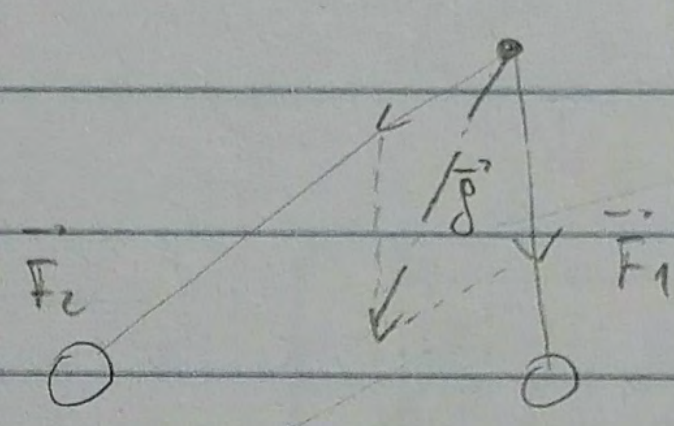
$\vec r_J + \vec r = \vec r_1$ $\quad\quad$ $\vec r_1 - \vec r_J = \vec r$ (Position of 1 respect to J)

$$\Rightarrow \ddot{\vec r} \simeq -\frac{G m_J \hat r_{1J}}{|\vec r_1 - \vec r_J|^2} + \frac{G m_1 \hat r_{J1}}{|\vec r_J - \vec r_1|^2} \quad\quad \hat r_{J1} = -\hat r_{1J}$$

$$= -\frac{G(m_J + m_1)\hat r_{1J}}{|\vec r|^2} \quad\quad m_J + m_1 \simeq m_J$$

$$\Rightarrow \ddot{\vec r} = -G\frac{m_J \hat r_{1J}}{|\vec r|^2} \quad\quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

System of bodies:



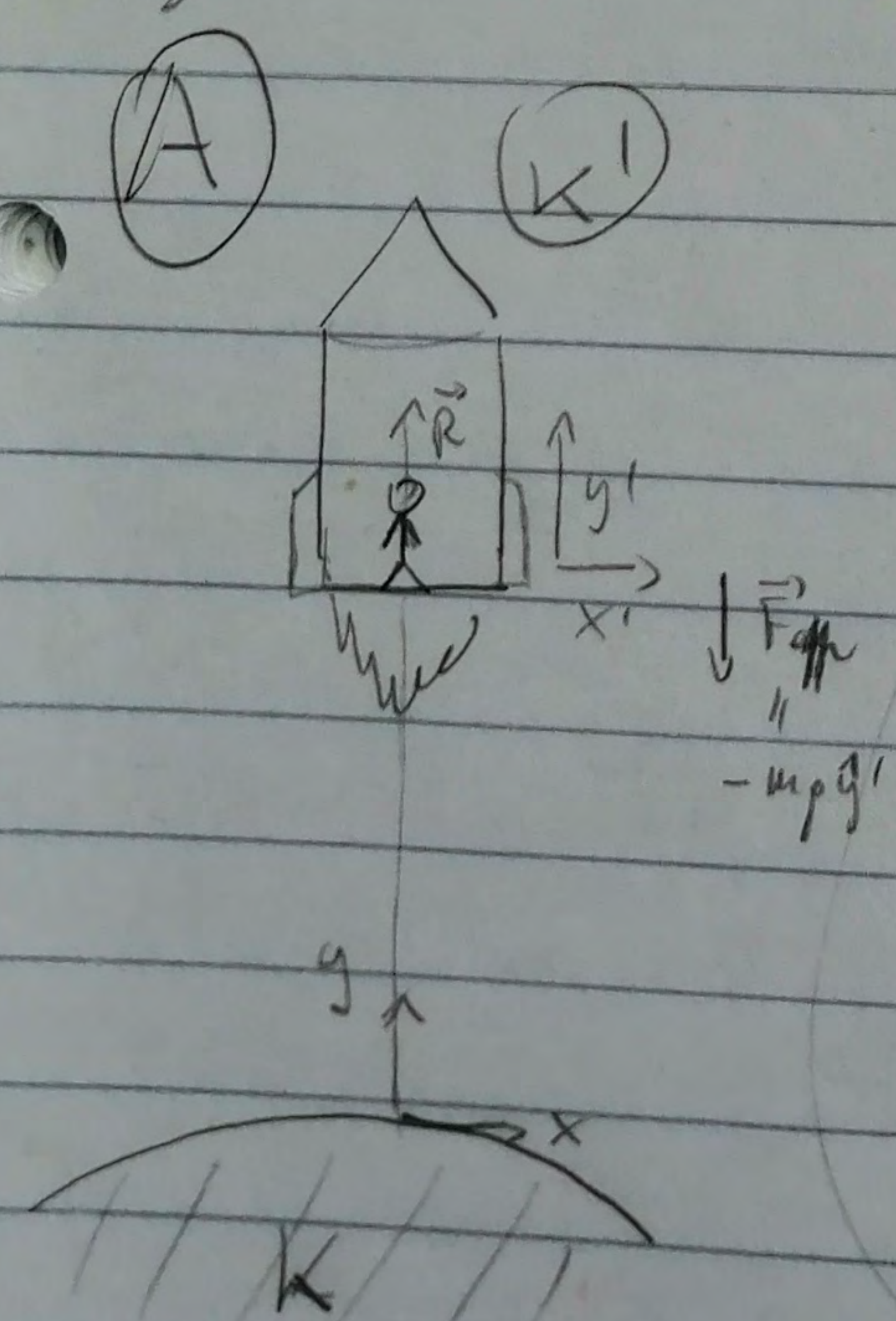
$$\vec F_1 = -\frac{m m_1}{r_1^2}\hat r_1$$

$$\vec F_2 = -\frac{m m_2}{r_2^2}\hat r_2$$

2º FORMULATION OF THE EP.

A non inertial reference system is equivalent to a gravitational field.

A $\quad$ K'



$\vec F_{eff} = -mg\hat y' = -mg\hat y$

$m\ddot y'\hat y = \vec R + \vec F_{eff} \quad\quad \vec F_{eff} = -m\vec a_{rel}$

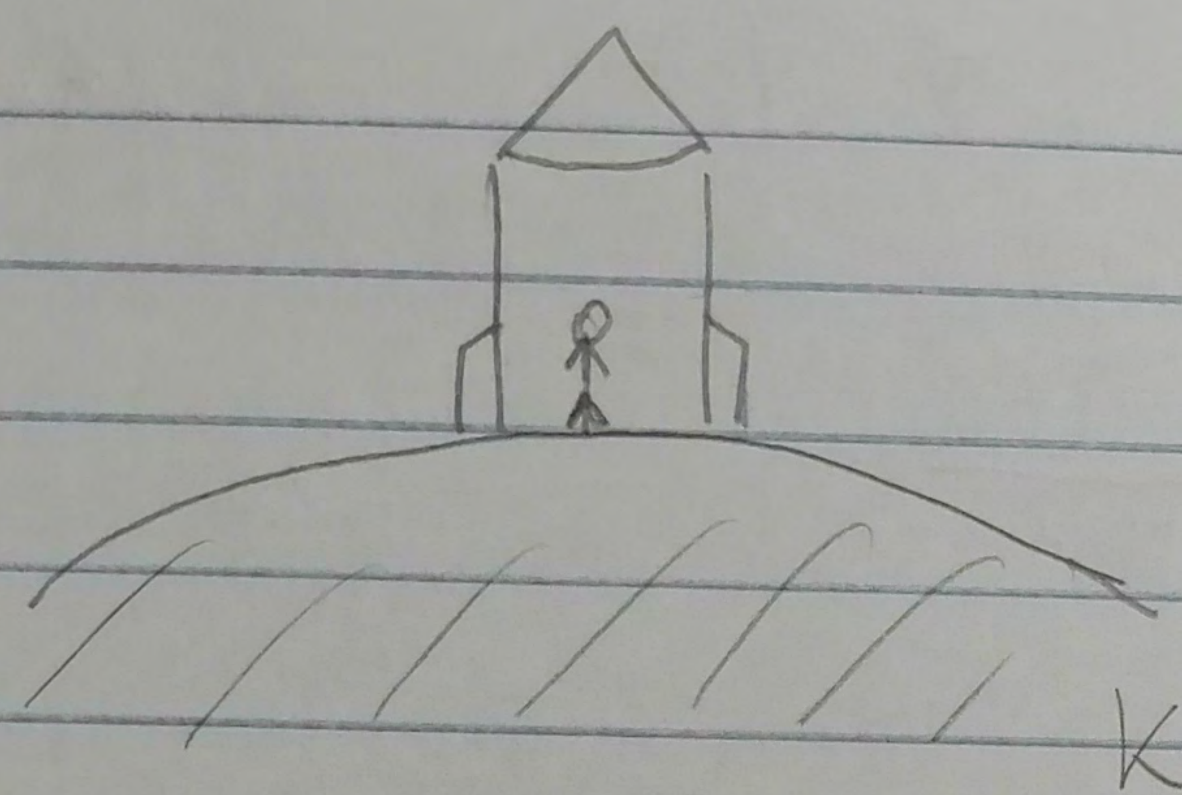
$\ddot y' = 0$ (He is at rest)

$\vec a = \vec a' + \vec a_{rel} \quad\quad \vec a' = \ddot{\vec y}' = 0$

$\vec a = g\hat y \quad\quad \Rightarrow \vec a_{rel} = \vec a = g\hat y$

$\Rightarrow 0 = R\hat y - mg\hat y \quad \Rightarrow \boxed{R = +mg}$

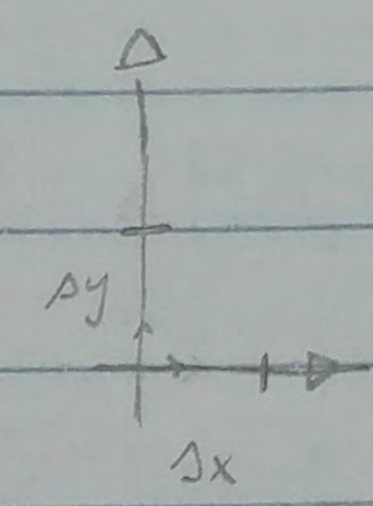
B



$$m\ddot{y}\hat{y} = R\hat{y} - mg\hat{y}$$

$$\ddot{y} = 0 \Rightarrow R = mg$$

A $\equiv$ B.



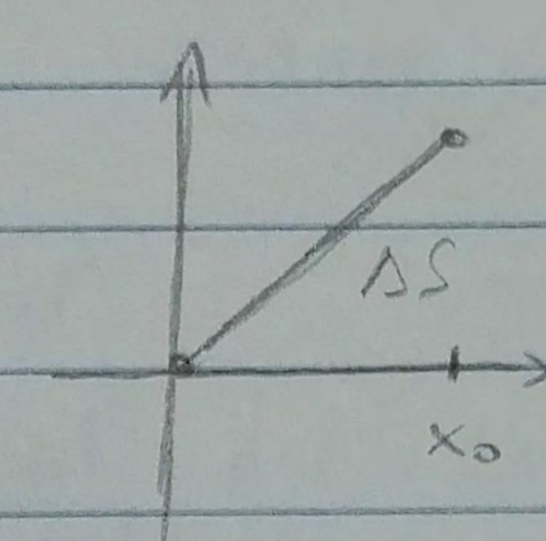
$$ds^2 = dx^2 + dy^2$$

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

$$\Delta x = \int dx$$

$$\Delta y = \int dy$$

$$\Delta S = \int ds$$



$$\Delta S = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}\, d\lambda$$

$$= \int_0^{x_0} dx \sqrt{1 + \frac{dy}{dx}}$$

$$= \int_0^{x_0} dx \sqrt{2}$$

$$= \sqrt{2}\, x_0 = \sqrt{x_0^2 + x_0^2} = \sqrt{x_0^2 + y_0^2}\Big|_{y_0 = x_0}$$

$$= x_0$$

Path integration.

$y = x$

$\frac{dy}{dx} = 1$

The EP asserts that the Physics locally is Lorentzian, but we do not expect the same at finite distance as evident for example from the Tidal Forces.

Anyway:
Locally $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.

Let us make a coordinate transformation and in general we find:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta,$$

$$\leftarrow \begin{cases} ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \\ = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\gamma} \frac{\partial x^\beta}{\partial x'^\delta} dx'^\gamma dx'^\delta \\ = g_{\alpha\beta}(x(x'))\, dx'^\alpha dx'^\beta. \end{cases}$$

where $g_{\alpha\beta}(x)$ is in general a function of the coordinates.

Def. A MANIFOLD in which is defined a metric $g_{\alpha\beta}$ is said: Riemanian MANIFOLD.

SemiRiemanian because the signature is "-2".

Examples. Rotating reference system.
Rindler spacetime

301 Room

4-6 p.m.

One example

Let us consider the metric :

$ds^2 = \eta^2 d\xi^2 - d\eta^2 - dy^2 - dz^2$ of

Coordinates : (ξ, η, y, z).

$\mathcal{M} = \mathcal{M}_2(\xi,\eta) \times E_2$ 2-dimensional euclidean plane of coordinates (y,z).

Locally: $\eta = x_0 + x$

$ds^2 = (x_0 + x)^2 d\xi^2 + - dx^2 - dy^2 - dz^2$

$\sim x_0^2 d\xi^2 - dx^2 - dy^2 - dz^2$

$= dt^2 - dx^2 - dy^2 - dz^2$

$\boxed{x_0 \xi = t}$

Properties :

1) It is diagonal ;

2) on a curve ξ = const. the distance from the point $\eta = 0$ is $|\eta|$;

3) On a curve η = const. we have $d\tau = |\eta| \, d\xi$,

3 $\Rightarrow$ that ξ does not measure the time of a clock. We need an adjustment factor η.

Light - geodesics

$ds^2 = 0$ locally, but ds^2 is invariant then $ds^2 = 0$ in any coordinate system.
Therefore, the light follows :

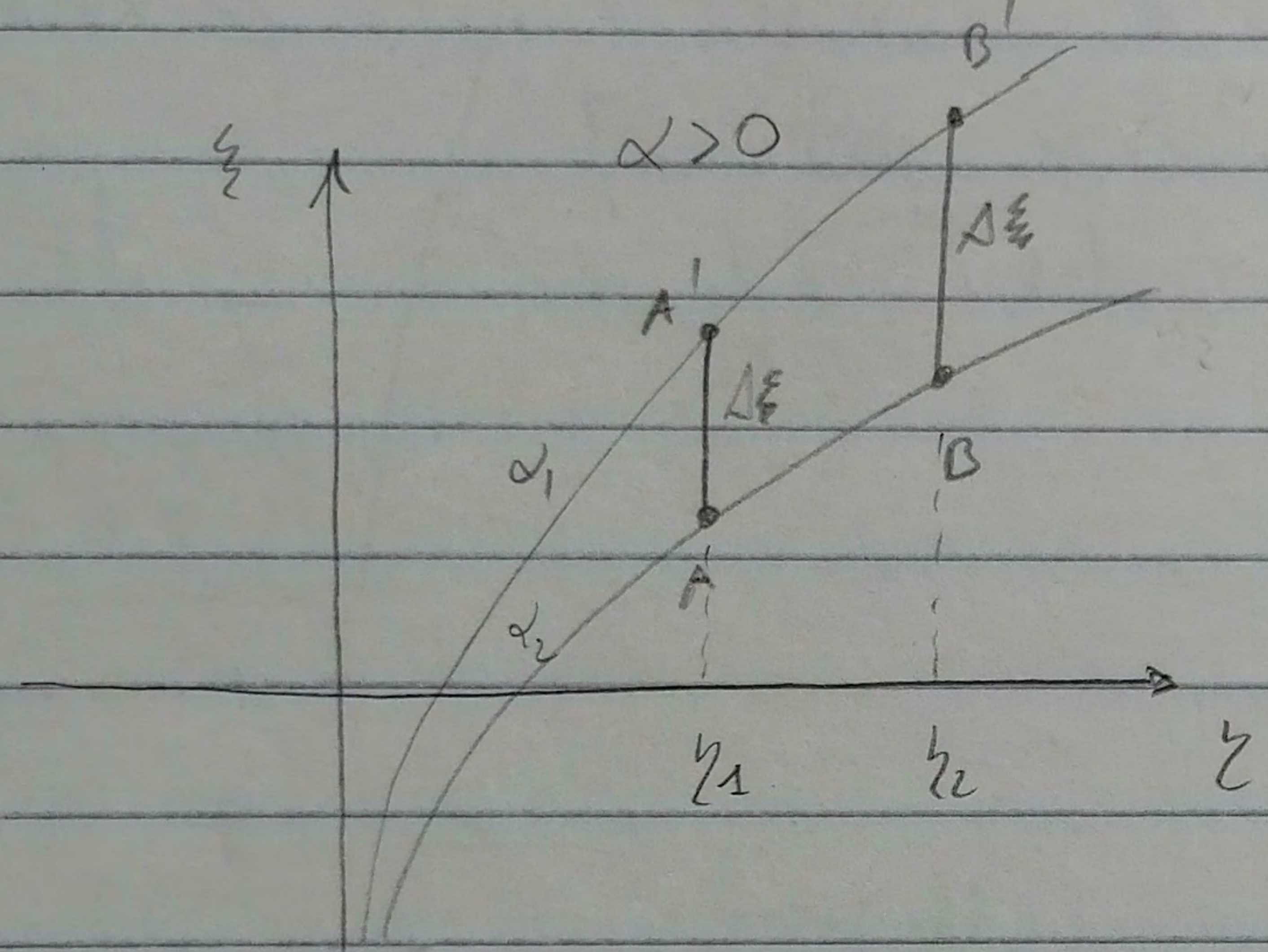
$$\eta \, d\xi = \pm \, d\eta \, ,$$

We can integrate e we find:

$$\eta = \alpha e^{\pm\xi} \quad \Leftarrow \quad \frac{d\eta}{d\xi} = \pm \alpha e^{\pm\xi} = \pm \eta \, ,$$

$$\lg\left(\frac{\eta}{\alpha}\right) = \pm \xi \quad \Rightarrow \quad \xi = \pm \lg \frac{\eta}{\alpha}$$

$\frac{\eta}{\alpha} > 0$



$\frac{\eta}{\alpha} > 0 \Rightarrow \eta > 0 \Leftrightarrow \alpha > 0$ $\Rightarrow$ the subspaces $\eta > 0$ e $\eta < 0$ are disconnected.
$\eta < 0 \Leftrightarrow \alpha < 0$

<u>Notice</u>
The metric is singular in $\eta = 0$. Is the singularity physical or only mathematical (namely it derives from an unsuitable choice of coordinates)?

This is a very important exercise.

Let us now consider emission of a light signal from A to B and another one from A' to B', then:

$$\left.\begin{aligned} \xi_{A'} - \xi_A &= \lg \frac{\eta_1}{\alpha_1} - \lg \frac{\eta_1}{\alpha_2} = -\lg \frac{\alpha_1}{\alpha_2} \\ \xi_{B'} - \xi_B &= \lg \frac{\eta_2}{\alpha_1} - \lg \frac{\eta_2}{\alpha_2} = -\lg \frac{\alpha_1}{\alpha_2} \end{aligned}\right\} \Rightarrow \xi_{A'} - \xi_A = \xi_{B'} - \xi_B \equiv \Delta\xi$$

Therefore, $\Delta\tau_1 = \eta_1 \Delta\xi$ and $\Delta\tau_2 = \eta_2 \Delta\xi$

$$\Rightarrow \Delta\tau_2 = \frac{\eta_2}{\eta_1}\Delta\tau_1 \qquad \Rightarrow \Delta\tau_2 > \Delta\tau_1$$

In a different way:

$\Delta\xi$ does not change from the departure (η_1, ξ_A) to the arrival (η_2, ξ_B) for the first photon. For the second photon, it departs from $(\eta_1, \xi_{A'})$ and arrives in $(\eta_2, \xi_{B'})$.

If $\eta_2 \gtrsim \eta_1 \Rightarrow$ $\Delta\tau_2 = \left(1 + \frac{\Delta\eta}{\eta}\right)\Delta\tau_1 > \Delta\tau_1$ (2.2)

$\eta_2 = \eta_1 + \Delta\eta$

$\eta_1 \equiv \eta$

proper time measured in η_2 — proper time measured in η_1.

(2.2) is called $\boxed{\text{"redshift"}}$.

Why this name!

Let us consider the emission of a train of monochromatic waves of frequency ν_1. The number of periods is $N_1 = \nu_1 \Delta\tau_1$ in η_1.

The number of periods in η_2 is $N_2 = \nu_2 \Delta\tau_2$, but $N_2 = N_1$ (it is just an integer), therefore:

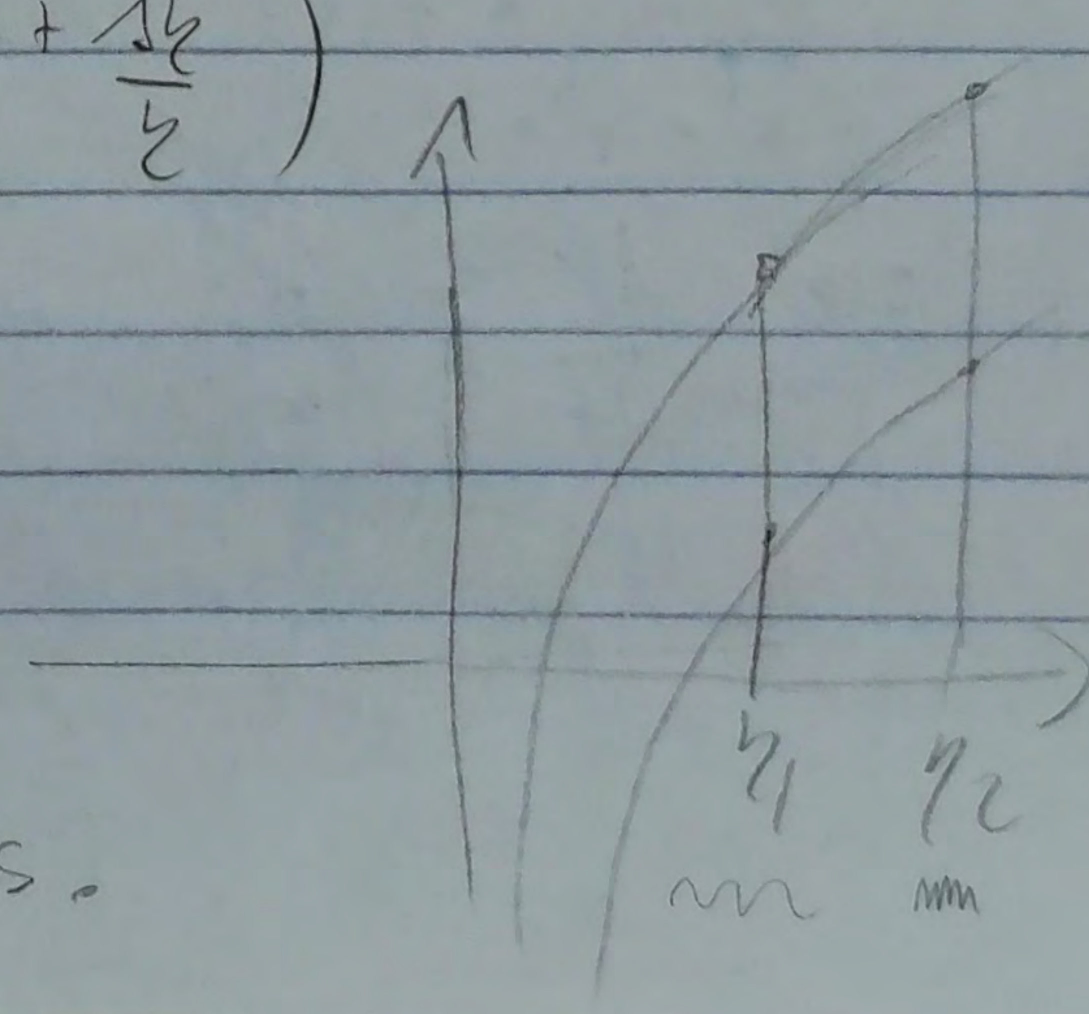
if there is no dispersion

$$\nu_1 \Delta\tau_1 = \nu_2 \Delta\tau_2 \quad \text{or} \quad \frac{\nu_1}{\nu_2} = \frac{\Delta\tau_2}{\Delta\tau_1} = \left(1 + \frac{\Delta\eta}{\eta}\right)$$

$$\frac{\nu_2}{\nu_1} \simeq \left(1 - \frac{\Delta\eta}{\eta}\right)$$

$$\nu_2 \cong \nu_1\left(1 - \frac{\Delta\eta}{\eta}\right) < \nu_1 .$$

When η increases the frequency decreases.



Singularity and accelerated reference frame.

Let us consider the following coordinate transformation:

$\boxed{\eta > 0}$

$x = \eta \cosh \xi \quad , \quad t = \eta \sinh \xi$

$x^2 - t^2 = \eta^2$

$\eta = \sqrt{x^2 - t^2} \quad , \quad \xi = \frac{1}{2} \lg \left(\frac{x+t}{x-t} \right)$ (2.5)$_{A,B}$

$\boxed{\eta < 0}$

$x = \eta \sinh \xi \quad , \quad t = \eta \cosh \xi$

$t^2 - x^2 = \eta^2$

$\eta^2 = t^2 - x^2$

$|\eta| = \sqrt{t^2 - x^2}$

$t^2 - x^2 > 0$

$|t| > |x|$

$x < 0 \Rightarrow -x < |t|$

$x < -|t|$

$x^2 - t^2 > 0 \Rightarrow |x| > |t|$

The metric becomes: $ds^2 = \eta^2 d\xi^2 - d\eta^2$

$= dt^2 - dx^2 .$

$\Downarrow$

No singularity.

What is the physical meaning of the metric?

For $\eta = \text{const}$ from $(2.5)_A$ we find:

$\eta^2 = x^2 - t^2 \quad , \quad 0 = x\,dx - t\,dt \Rightarrow x\,dx = t\,dt$

$x^2 = \eta^2 + t^2$

$\Rightarrow \frac{dx}{dt} = \frac{t}{x} \Rightarrow v = \frac{t}{\sqrt{\eta^2 + t^2}} .$

$\frac{dx}{dt} \equiv v$

$= \frac{t/|\eta|}{\sqrt{1 + \frac{t^2}{|\eta|^2}}}$

I remind the unifor motion for $\vec{F} = \text{const}$, namely

$$v = \frac{F/m \cdot t}{\sqrt{1 + \left(\frac{F}{m}\right)^2 t^2}} \qquad (c=1)$$

Comparing we find $\eta = \frac{m}{F}$.

We discovered that the metric $ds^2 = \eta^2 d\xi^2 - d\eta^2$ is the metric seen by an observer at rest in an accelerating Reference system.

because $\eta = \text{const}$.

Do we have gravity?

Horizon (2.5 A,B)

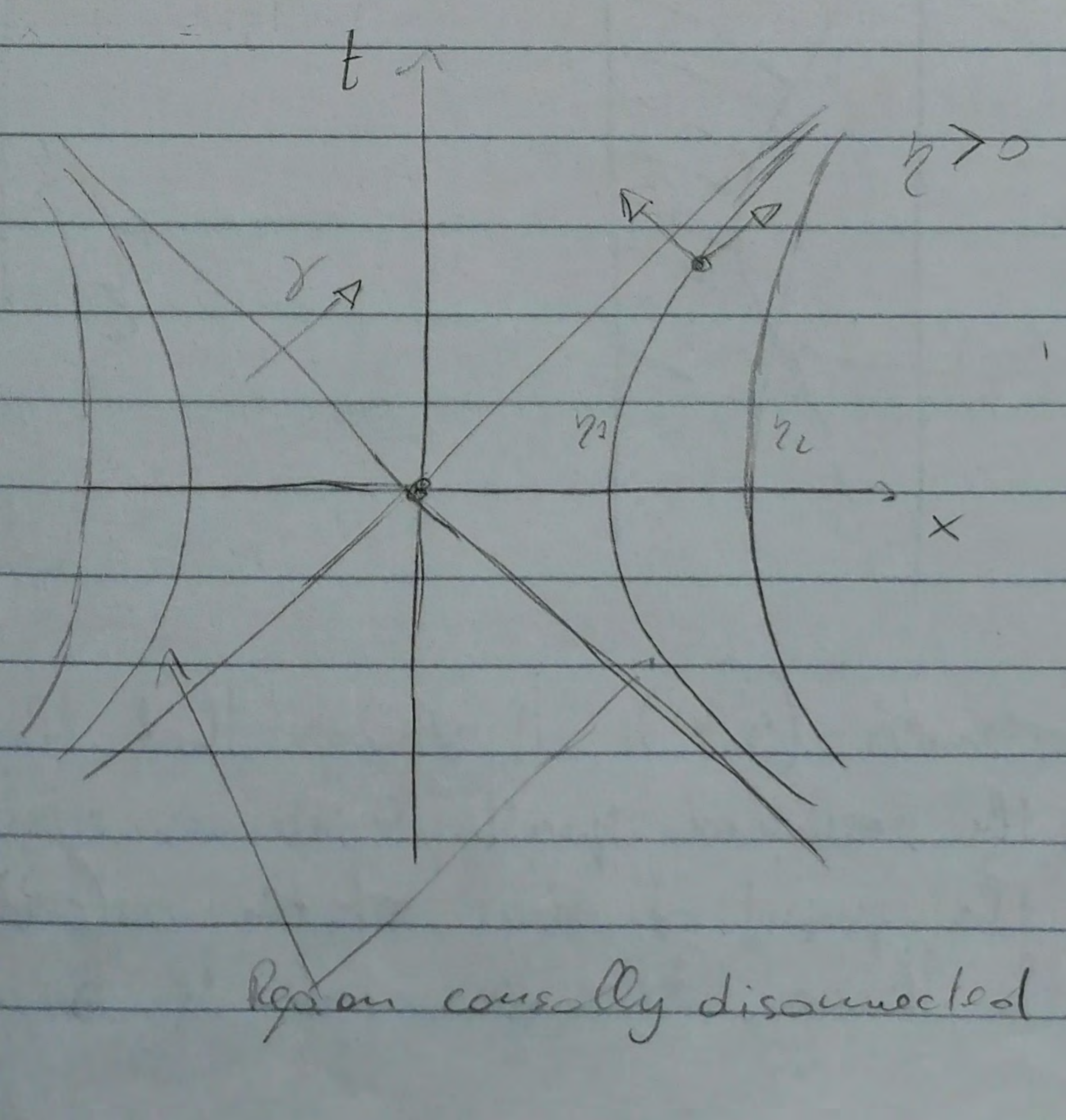
$\eta > 0 \Rightarrow x > |t|$

$\eta < 0 \Rightarrow x < -|t|$

$x^2 - t^2 = \eta^2$, $x^2 = t^2 + \eta^2$

$x^2 - \eta^2 = t^2$

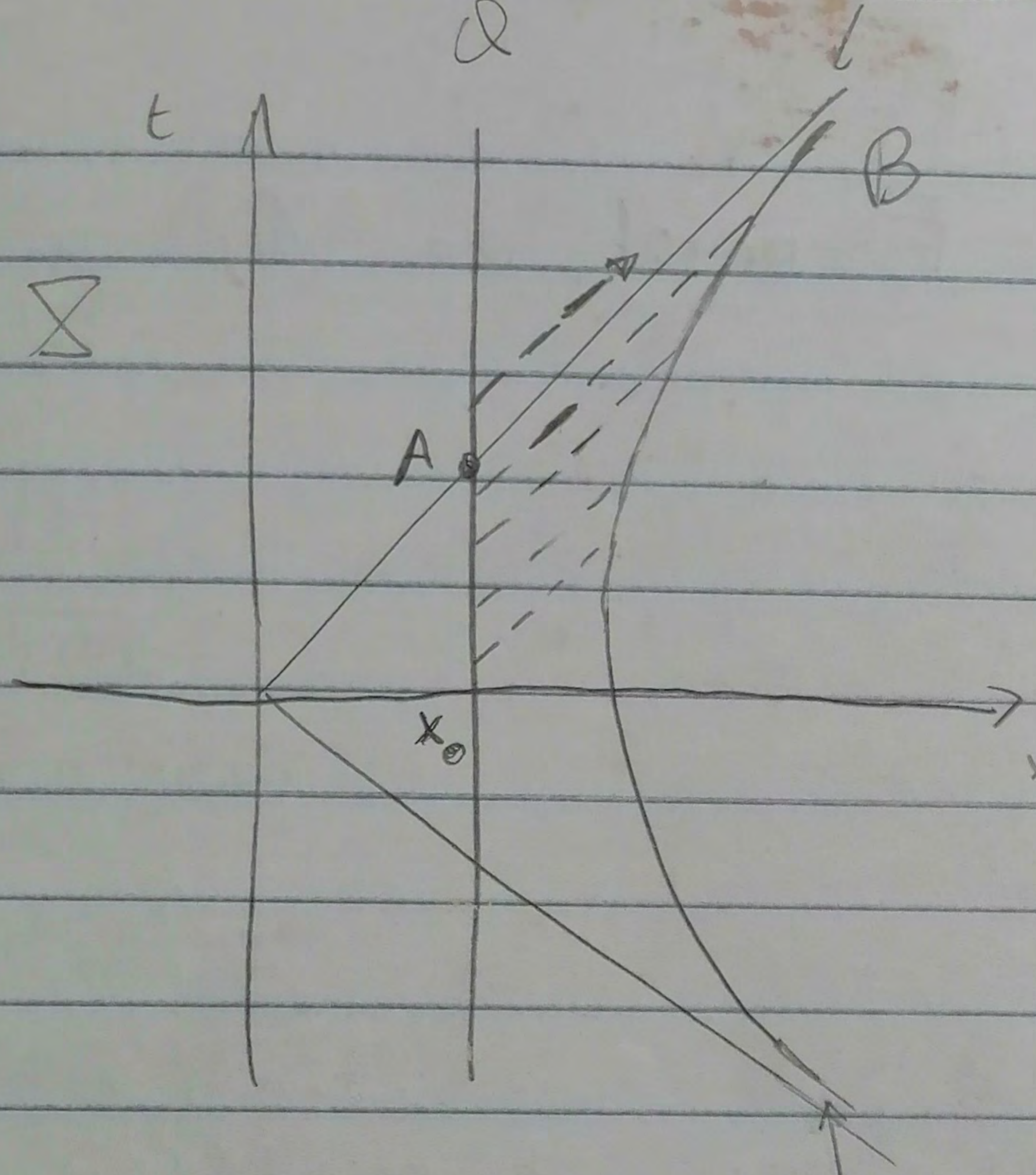
$|t| = \sqrt{x^2 - \eta^2}$ $\quad |x| > |\eta|$



$\eta = 0$ ⇒ Horizon singularity.

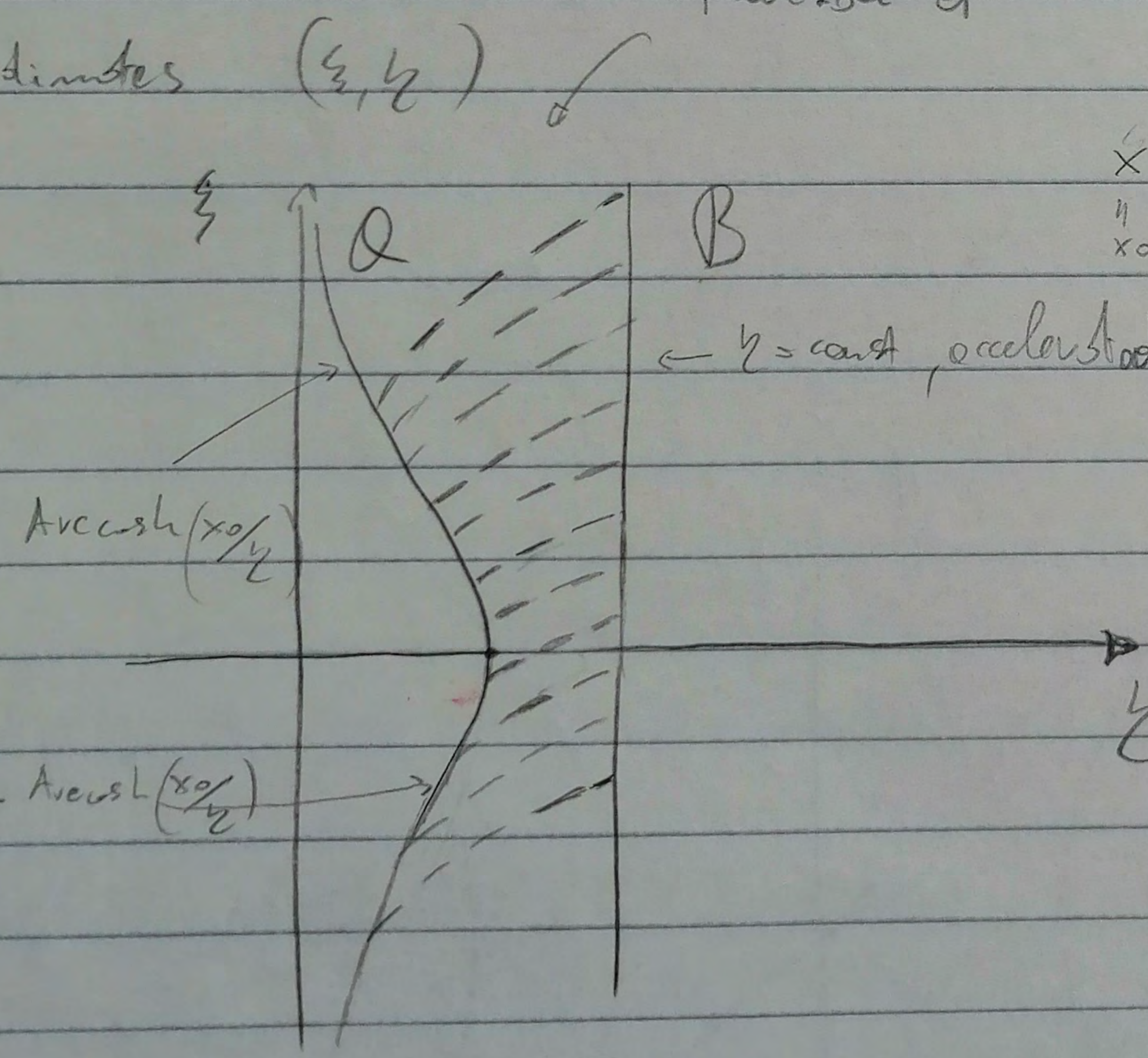
Region causally disconnected

Coordinates (x,t)



Horizon of ∞ in these coordinates.

Coordinates (ξ, η)



$$\underset{x_0}{x} = \text{const} = \eta \cosh \xi$$

$$\Rightarrow \cosh \xi = \frac{x_0}{\eta}$$

$$\xi = \text{arccosh}\, x_0/\eta$$

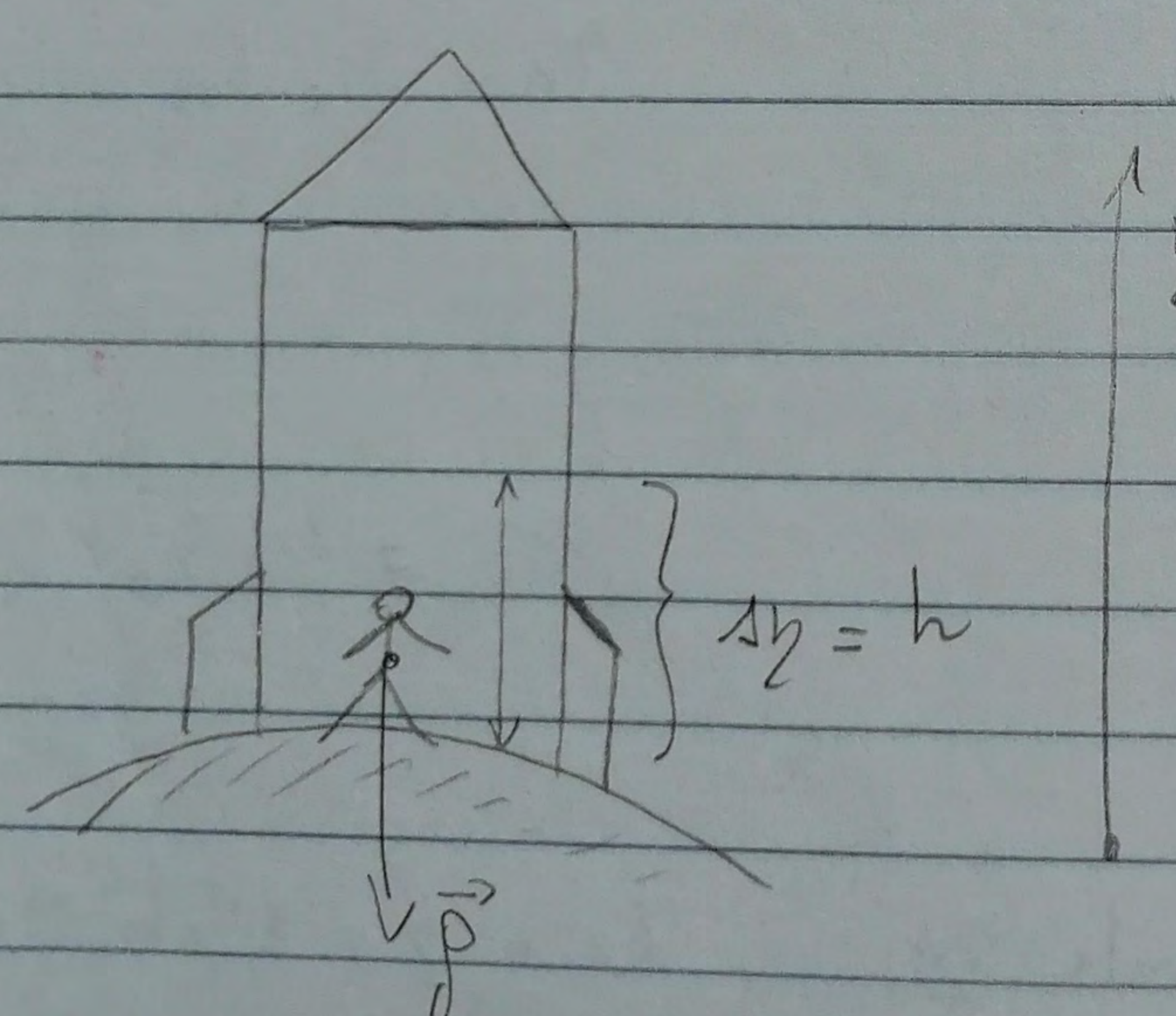
$$\xi > 0$$

In coordinates (ξ, η) it seems that the event A never happens.
Indeed, the accelerated spaceship B receives signals forever.
From the point of view of the accelerated spaceship the fall towards the Horizon takes an infinite amount of time.

GRAVITATIONAL REDSHIFT

We can now implement the equivalence principle in its second formulation.

Since an accelerating reference frame is equivalent to a gravitational field we can just leave the spaceship on the launching pad (without to swich on the engines):



We can apply the formula:

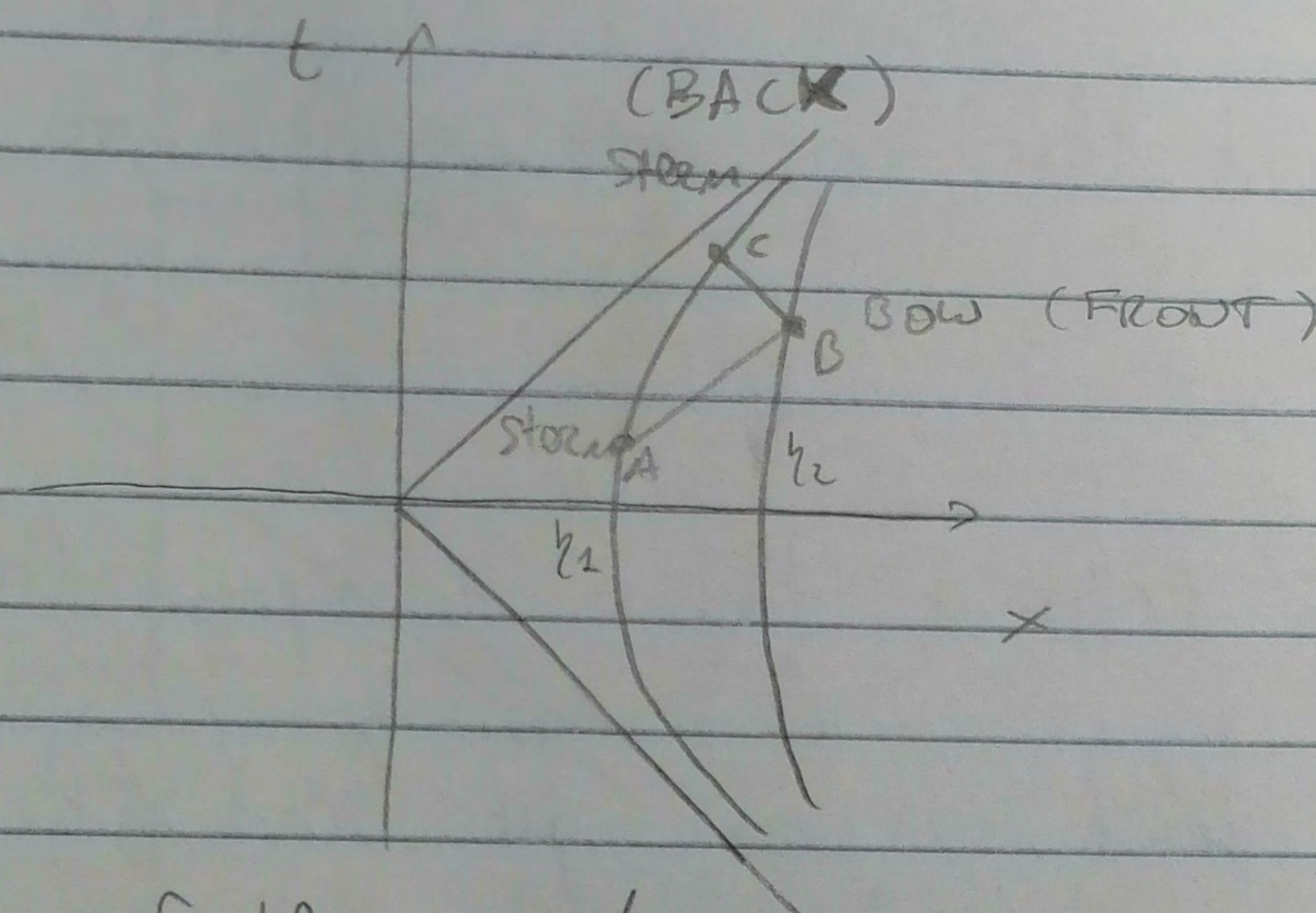
$\nu_2 = \nu_1\left(1 - \frac{\Delta z}{z}\right)$ with $\frac{1}{z} = g$, $\Delta z = h$

$\nu_2 = \nu_1\left(1 - \frac{gh}{c^2}\right) < \nu_1$

$\Delta\tau_2 > \Delta\tau_1$.

The accelerated R.F. is RIGID
We use the RADAR MEASURAMENT.

Propagation of light in the spaceship.



The points of the spaceship have different acceleration. We have to prove that it is Rigid!

~~Coordinates of the events:~~

$$x_A = \eta_1 \cosh \xi_A$$

$$\Delta\tau = \eta \Delta\xi \Rightarrow \tau_A = \eta_1 \xi_A$$

$$= \eta_1 \cosh \frac{\tau_A}{\eta_1}$$

$$\tau_B = \eta_2 \xi_B$$

$$x_B = \eta_2 \cosh \frac{\tau_B}{\eta_2}$$

$$x_C = \eta_1 \cosh \frac{\tau_C}{\eta_2}$$

$$\tau_C = \eta_1 \xi_C$$

And

$$t_A = \eta_1 \sinh \frac{\tau_A}{\eta_1}, \quad t_B = \eta_2 \sinh \frac{\tau_B}{\eta_2}, \quad t_C = \eta_1 \sinh \frac{\tau_C}{\eta_1}.$$

The light in coordinate (x,t) propagates such that $ds^2 = 0$, but $ds^2 = dt^2 - dx^2 \Rightarrow$

$$(x_B - x_A)^2 = (t_B - t_A)^2 \Rightarrow x_B - x_A = t_B - t_A \quad \begin{pmatrix} x_B > x_A \\ t_B > t_A \end{pmatrix}$$

$$(x_C - x_B)^2 = (t_C - t_B)^2 \qquad x_B - x_C = t_C - t_B$$

$|x_C - x_B| = |t_C - t_B| = t_C - t_B$ because $t_C > t_B$

$x_B - x_C = t_C - t_B$ because $x_C < x_B$

$$\Rightarrow \begin{cases} x_B - t_B = x_A - t_A \\ x_B + t_B = x_C + t_C \end{cases}$$

Solving we find: $\begin{cases} \eta_2 e^{-\tau_B/\eta_2} = \eta_1 e^{-\tau_A/\eta_1} \\ \eta_2 . e^{\tau_B/\eta_2} = \eta_1 e^{-\tau_C/\eta_1} \end{cases}$

$$\Rightarrow \eta_2^2 = \eta_1^2 \, e^{\frac{\tau_C - \tau_A}{\eta_1}}$$

$\lg(\quad) = \lg(\quad)$

$$\Rightarrow \tau_C - \tau_A = 2\eta_1 \lg \frac{\eta_2}{\eta_1}$$

proper time to go back and forward.

Finally the length is: $l_{12} = \frac{1}{2} \cdot \overset{1}{\underset{\shortparallel}{C}} \cdot (\tau_C - \tau_A)$

$$= \frac{1}{2} 2 \eta_1 \lg \eta_2/\eta_1$$

$$\cong \eta_1 \lg \frac{\eta_1 + (\eta_2 - \eta_1)}{\eta_1}$$

$$\cong \eta_1 \lg \left(1 + \frac{\eta_2 - \eta_1}{\eta_1}\right)$$

$$\overset{\eta_2 \sim \eta_1}{\cong} \eta_1 \left(\frac{\eta_2 - \eta_1}{\eta_1}\right) = \eta_2 - \eta_1$$

* It is not exactly $\eta_2 - \eta_1$ because of the gravitational redshift. The correct computation must be done infinitesimally in "τ" and integrating afterwards.
τ: depends on η.

* However, l_{12} is independent on time.

* Note that the spaceship is rigid, but the acceleration of its points are different. This is because of the "length contraction".

$ds^2 = dt^2 - dx^2$

$x = \eta \cosh \xi$ $\qquad\qquad$ $\cosh^2 - \sinh^2 = 1$

$t = \eta \sinh \xi$ $\qquad\qquad$ $d\cosh = \sinh$

$d\sinh = \cosh$

$dx = d\eta (\cosh \xi) + \eta (\sinh \xi) d\xi$

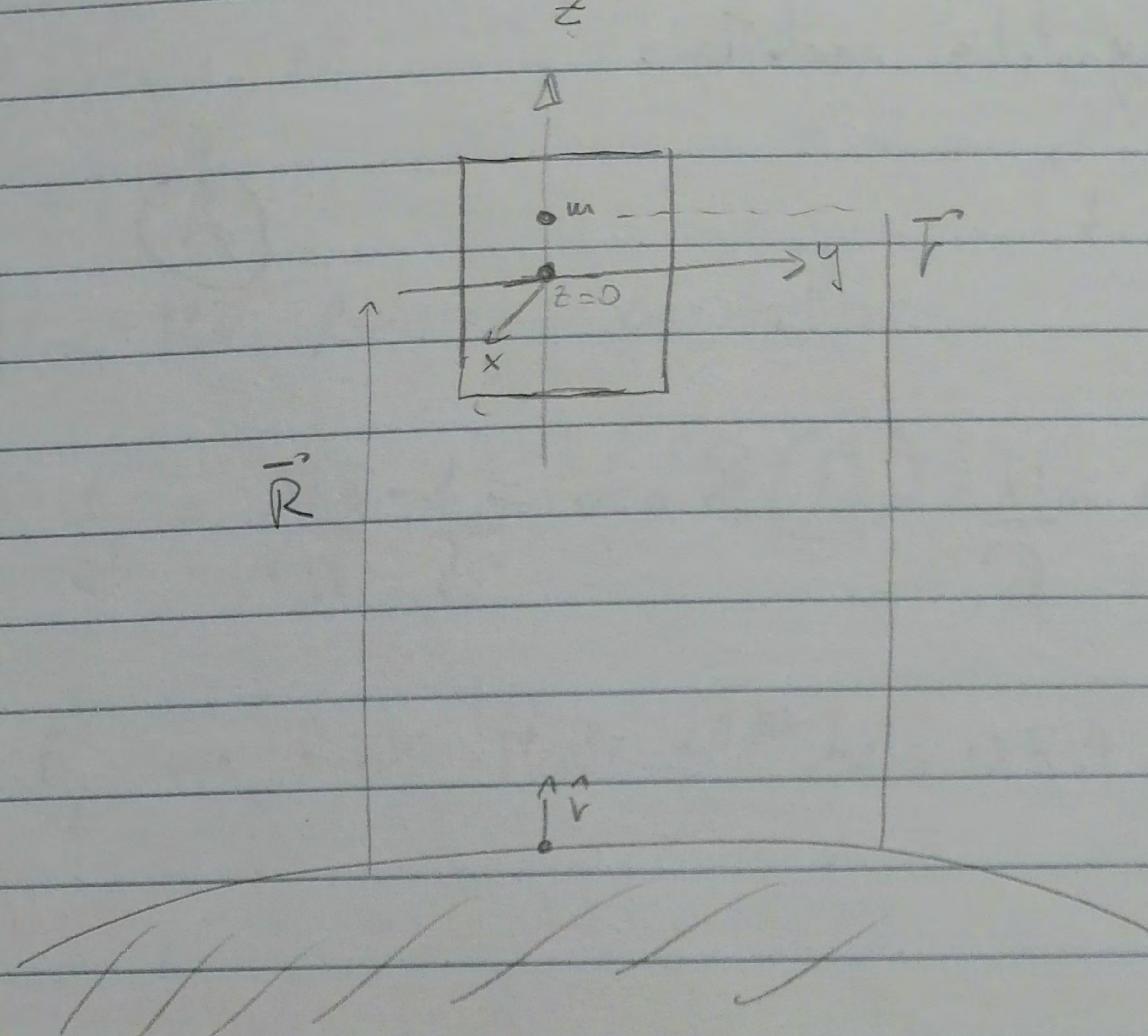
$dt = d\eta (\sinh \xi) + \eta (\cosh \xi) d\xi$

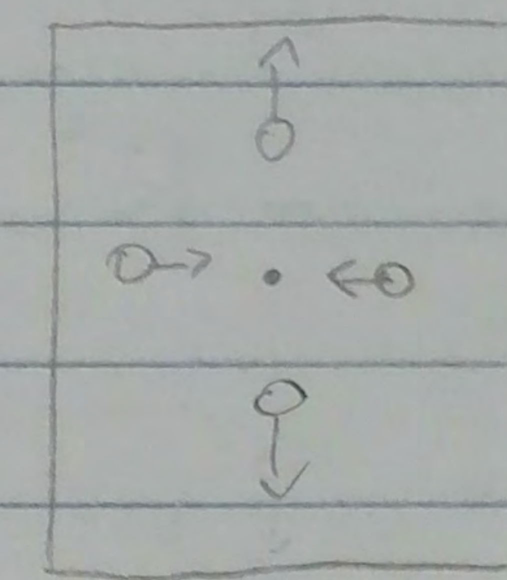
$$dt^2 - dx^2 = d\eta^2 \sinh^2 \xi + \eta^2 \cosh^2 \xi \, d\xi^2 + \cancel{2 \, d\eta \, d\xi \, \eta \sinh \xi \cosh \xi}$$

$$- \left[d\eta^2 \cosh^2 \xi + \eta^2 \sinh^2 \xi + \cancel{2 \, d\eta \, d\xi \, \eta \sinh \xi \cosh \xi} \right]$$

$$= - d\eta^2 + \eta^2 d\xi^2$$

Tidal Forces





• center of M.

$$m\ddot{\vec{r}} = -\frac{GmM}{r^2}\hat{r} \qquad r = R + z$$

$$= -\frac{GmM}{(R+z)^2}\hat{r}$$

$$= \frac{-GmM}{R\left(1+\frac{z}{R}\right)^2}\hat{r} \overset{z \ll R}{\cong} -\frac{GMm}{R}\left(1 - 2\frac{z}{R}\right)\hat{r}$$

$\vec{R} = R\hat{r}$

$$= -\frac{GMm}{R} + \frac{2GMm}{R^3}z$$

$$\cancel{m}(\ddot{R} + \ddot{z}) = -\frac{GMm}{R} + \frac{2GMmz}{R^3} + \dots$$

$$\ddot{z} = \frac{2GM}{R^3}z.$$

Analog effect we have in the x and y directions:

$$\frac{d^2y}{dt^2} = -\frac{M}{R^3}y \quad , \quad \frac{d^2x}{dt^2} = -\frac{M}{R^3}x$$

We introduce the syntetic notation:

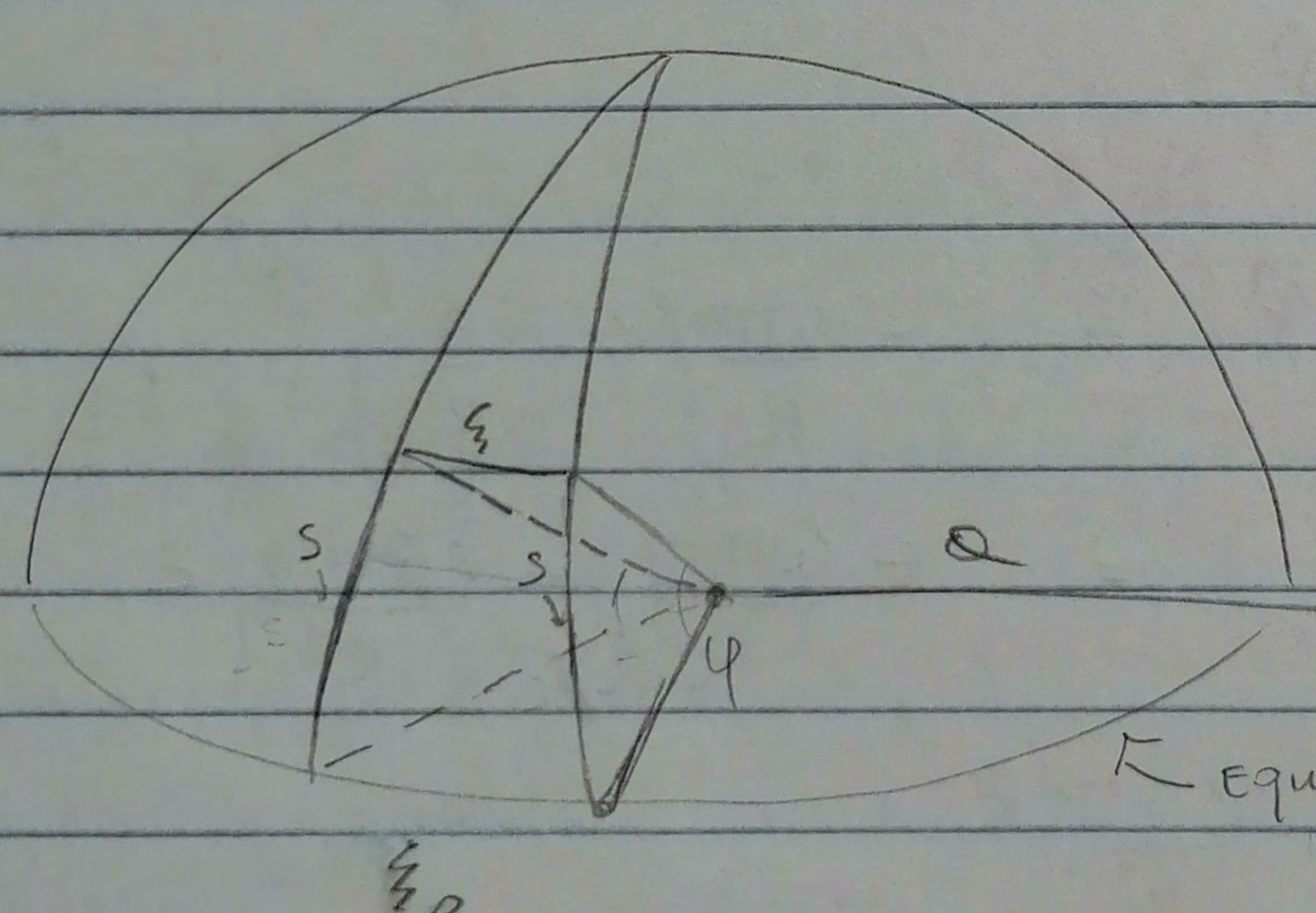
$$\frac{d^2\xi^i}{dt^2} = -R^i{}_{0k0}\,\xi^k \qquad \text{A}$$

$$R^1{}_{010} = R^2{}_{020} = \frac{M}{R^3}, \quad R^3{}_{030} = \frac{-2M}{R^3}$$

$$\frac{M}{R^3} = \frac{0.444}{6.38^3 \cdot 10^{24}} = 1.71 \times 10^{-27}\ cm^{-2}$$

Consider now:

Geometry of a sphere



a = Radius

$s = a\varphi$

$$\sin\frac{\xi}{2a} = \sin\frac{\xi_0}{2a}\cos\varphi \qquad (\leftarrow \text{Exercice})$$

for ξ_0 small $\Rightarrow$ $\xi = \xi_0 \cos\varphi = \xi_0 \cos\frac{s}{a}$

$$\Rightarrow \frac{d^2\xi}{ds^2} = -\frac{1}{\varrho^2}\xi$$

We define the poussion curvature:

$$\frac{1}{\varrho^2} = -\frac{1}{\xi}\frac{d^2\xi}{ds^2} . \quad B$$

Operatively we look How the distance ξ between the Two circles changes.

By analogy between B and A,

$R^i{}_{jkl}$ = CURVATURE OF THE SPACE-TIME

$$\underline{u} = u_\alpha \underline{\omega}^\alpha \quad , \underline{v} = v^\alpha \underline{e}_\alpha$$

$$\underline{u}(\underline{v}) = (u_\alpha \underline{\omega}^\alpha, v^\beta \underline{e}_\beta) = u_\alpha v^\beta \delta^\alpha_\beta = u_\alpha v^\alpha$$

Expansion in $\frac{1}{c}$ and linear in $g^{\mu\nu}$

$$R_{\sigma\nu} = \partial_\mu \Gamma^\mu{}_{\nu\sigma} - \partial_\nu \Gamma^\mu{}_{\mu\sigma} + \Gamma^\mu{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\mu{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma}$$

$$R_{00} = \underset{①}{\partial_\mu \Gamma^\mu{}_{00}} - \underset{\frac{1}{c}\frac{\partial}{\partial t}\ ②}{\partial_0 \Gamma^\mu{}_{\mu 0}} + \underset{③}{\underbrace{\Gamma^\mu{}_{\mu\lambda}}_{\frac{1}{2} g^{\tau\delta} \frac{\partial g_{\tau\delta}}{\partial x^\lambda}} \Gamma^\lambda{}_{00}} - \underset{④}{\Gamma^\mu{}_{0\lambda} \Gamma^\lambda{}_{\mu 0}}$$

$$\Gamma^i{}_{00} = \frac{1}{2} g^{im} \left(\underset{\frac{1}{c}\frac{\partial}{\partial t}}{\frac{\partial g_{m0}}{\partial x^0}} + \underset{\frac{1}{c}\frac{\partial}{\partial t}}{\frac{\partial g_{m0}}{\partial x^0}} - \frac{\partial g_{00}}{\partial x^m} \right) \Rightarrow ①\ \underset{\frac{1}{c}\frac{\partial}{\partial t}\left(\ll \frac{\partial}{\partial x^\alpha}\right)}{\partial_0 \Gamma^0{}_{00}} + \underset{\frac{\partial}{\partial x^\alpha}}{\partial_\alpha \Gamma^\alpha{}_{00}}$$

$$\approx \partial_\alpha \Gamma^\alpha{}_{00} \quad (\alpha = 1,2,3)$$

$\forall \alpha = 1,2,3$

$$\Gamma^\alpha{}_{00} \approx -\frac{1}{2} g^{\alpha\beta} \frac{\partial g_{00}}{\partial x^\beta}$$

$$\Gamma^\mu{}_{\mu 0} = \frac{1}{2} g^{\tau\delta} \frac{\partial g_{\tau\delta}}{\partial x^0} \sim \frac{1}{c}\frac{\partial}{\partial t} \quad \Rightarrow ② \sim \frac{1}{c^2}$$

③ $R_{\mu\nu} \sim \partial^2 g + (\partial g)(\partial g) = \partial^2 g + O(g^2) \quad \Rightarrow \Gamma\Gamma \approx O(g^2)$

④ $\Gamma\Gamma \sim O(g^2)$

① $$\underset{\frac{1}{c}}{\partial_0 \Gamma^0{}_{00}} + \partial_{\underset{\alpha=1,2,3}{\alpha}} \Gamma^\alpha{}_{00} \sim \partial_\alpha \Gamma^\alpha{}_{00} = -\frac{1}{2} \partial_\alpha \left(\overset{-\delta^{\alpha\beta}}{g^{\alpha\beta}} \frac{\partial g_{00}}{\partial x^\beta} \right)$$

$$\Rightarrow R_{00} \sim R^0{}_0 \sim \partial_\alpha \Gamma^\alpha{}_{00} \simeq \frac{1}{2} \Delta g_{00}$$

$$\approx +\frac{1}{2} \partial_\alpha \partial^\alpha g_{00}$$
$$= +\frac{1}{2} \Delta g_{00}$$

Energy tensor for point particles

$$S_p = -mc \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, d\lambda \,, \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda} \qquad \underset{\text{density}}{\mathcal{T}_p^{\mu\nu}} = \sqrt{-g}\, T_p^{\mu\nu}$$

$$= -mc \int d\lambda \int d^4x \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, \delta^4(x^\mu - x^\mu(\lambda)) = \underbrace{\int d^4x \int (-mc)\, d\lambda \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, \delta^4(x^\mu - x^\mu(\lambda))}_{= \int d^4x\, L \quad (\text{there is no } \sqrt{g})}$$

$$T_p^{\mu\nu} = \frac{2}{\sqrt{-g}} \left\{ \frac{\partial(\overbrace{\sqrt{-g}\mathcal{L}}^{L})}{\partial g^{\mu\nu}} - \cancel{\frac{\partial}{\partial x^\ell} \frac{\partial(\sqrt{-g}\,\mathcal{L})}{\partial\left(\frac{\partial g^{\mu\nu}}{\partial x^\ell}\right)}} \right\}$$

$$= \frac{2}{\sqrt{-g}} \frac{\partial L}{\partial g^{\mu\nu}} = \frac{2}{\sqrt{g}} (-mc) \int d\lambda \frac{\dot{x}^\mu \dot{x}^\nu}{2\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}\, \delta^4(x^\mu - x^\mu(\lambda))$$

$$\mathcal{T}_p^{\mu\nu} = \frac{\cancel{\sqrt{g}}\, \cancel{2}}{\cancel{\sqrt{g}}} (-mc) \int d\lambda \frac{\dot{x}^\mu \dot{x}^\nu}{\cancel{2}\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}\, \delta^4(x^\mu - x^\mu(\lambda)) \quad , \; ds = \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$$

$$s = \lambda \Rightarrow ds = 1$$

$$\underset{s=\tau}{=} - \int ds\, mc\, \dot{x}^\mu \dot{x}^\nu\, \delta^4(x^\mu - x^\mu(s)) \Rightarrow T_p^{\mu\nu} = -\frac{1}{\sqrt{g}} \int ds\, mc\, \dot{x}^\mu \dot{x}^\nu\, \delta^4(x - x(s))$$

$$\delta(x-y) = \lim_{a \to 0} \frac{1}{\sqrt{\pi}\cdot a} e^{-(x-y)^2/a^2}$$

$$\underbrace{\frac{d}{dx}\delta(x-y)}_{\frac{1}{\sqrt{\pi}a} e^{-(x-y)^2/a^2} \left(-\frac{1}{a^2}\right) 2(x-y)} = \underbrace{-\frac{d}{dy}\delta(x-y)}_{-\left[\frac{1}{\sqrt{\pi}a} e^{-(x-y)^2/a^2} \left(-\frac{1}{a^2}\right) 2(x-y)\cdot(-1)\right]}$$

Point-particle energy tensor:

$$T^{ab}(\bar{x}) = \frac{m}{\sqrt{|g|}} \int d\tau \, \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \, \delta^4(\bar{x} - x(\tau))$$

$\neq 0$ if $\bar{x} = x(\tau)$ particle trajectory.

Let us compute:

$$\nabla_a T^{ab} = \partial_a T^{ab} + \Gamma^a{}_{ca} T^{cb} + \Gamma^b{}_{ca} T^{ac}$$

$$= \frac{1}{\sqrt{|g|}} \partial_a \left(\sqrt{|g|} \, T^{ab} \right) + \Gamma^b{}_{ca} T^{ac} = 0$$

$$\Rightarrow \underbrace{\nabla_a}_{\nabla_{\bar{x}^a}} T^{ab}(\bar{x}) = \frac{1}{\sqrt{|g|}} \overbrace{\partial_a}^{\frac{\partial}{\partial \bar{x}^a}} \left\{ \cancel{\sqrt{|g|}} \, \frac{m}{\cancel{\sqrt{|g|}}} \int d\tau \, \dot{x}^a \dot{x}^b \, \delta^4(\bar{x} - x(\tau)) \right\} +$$

$$+ \Gamma^b{}_{ca} \frac{m}{\sqrt{|g|}} \int d\tau \, \dot{x}^a \dot{x}^c \, \delta^4(\bar{x} - x(\tau)) = 0$$

$$\Longrightarrow \int d\tau \, \dot{x}^a \dot{x}^b \, \frac{\partial}{\partial \bar{x}^a} \delta^4(\bar{x} - x(\tau)) +$$

$$+ \Gamma^b{}_{ca} \int d\tau \, \dot{x}^a \dot{x}^c \, \delta^4(\bar{x} - x(\tau)) = 0.$$

Note:

$$\frac{\partial}{\partial \bar{x}^a} \delta^4(\bar{x} - x(\tau)) = - \frac{\partial}{\partial x^a} \delta^4(\bar{x} - x(\tau))$$

and $$\underbrace{\dot{x}^a}_{\frac{dx^a}{d\tau}} \frac{\partial}{\partial x^a} \delta^4(\bar{x} - x(\tau)) = \frac{d}{d\tau} \delta^4(\bar{x} - x(\tau))$$

Finally:

$$-\int d\tau\, \dot{x}^b \underbrace{\frac{d}{d\tau}}\, \delta^4(\bar{x}-x(\tau)) + \Gamma^b{}_{ca}\int d\tau\, \dot{x}^c\, \dot{x}^a\, \delta^4(\bar{x}-x(\tau)) = 0$$

$$\int d\tau \left(\ddot{x}^b + \Gamma^b{}_{ca}\, \dot{x}^c\, \dot{x}^a\right) \delta^4(\bar{x}-x(\tau)) = 0$$

$$0 = 0$$

Einstein eq.s $\Rightarrow$ the equations of motion for matter.

In contrast, in Maxwell theory the Lorentz law has to be postulated separately.

$$\int_A^B d\left(\dot{x}^b_{(\tau)}\, \delta^4(\bar{x}-x(\tau))\right) = \dot{x}^b_{(\tau)}\, \delta^4(\bar{x}-x(\tau))\Big|_{\tau_A}^{\tau_B}$$

Scalar Matter

$$\mathcal{L} = \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \quad , \quad \leftarrow \text{Klein-Gordon field}$$

$$S_{KG} = \int \sqrt{|g|}\, \mathcal{L}\, d^4x \quad .$$

$$\delta S_{KG} = \frac{1}{2} \int d^4x \left[(\delta \sqrt{-g})\, \mathcal{L} + \sqrt{-g}\, \delta \mathcal{L} \right]$$

$$= \int d^4x \left(-\frac{1}{2} \sqrt{-g}\, g_{ab}\, \delta g^{ab} \mathcal{L} + \sqrt{-g} \frac{1}{2} \delta g^{ab} \overset{\partial_a}{\overset{\parallel}{\nabla_a}} \phi \overset{\partial_b}{\overset{\parallel}{\nabla_b}} \phi \right)$$

$$= \int d^4x \sqrt{-g}\, \delta g^{ab} \left[\frac{1}{2} \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \mathcal{L} \right] =$$

$$= \int d^4x \frac{1}{2} \sqrt{-g}\, T_{ab}\, \delta g^{ab} .$$

$$T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{ab}} = \nabla_a \phi \nabla_b \phi - g_{ab} \mathcal{L} .$$

Electromagnetic-field

$$S_{em} = -\frac{1}{4} \int d^4x \sqrt{|g|}\, F_{ab} F^{ab} = -\frac{1}{4} \int d^4x \sqrt{|g|}\, g^{ac} g^{bd} F_{ab} F_{cd} .$$

$F_{ab}(\cancel{g})$, F_{ab} does not depend on g_{ab}.

$$\delta S_{em} = -\frac{1}{4} \int d^4x \left[(\delta \sqrt{|g|}) F_{ab} F^{ab} + \sqrt{|g|} (\delta g^{ac}) g^{bd} F_{ab} F_{cd} + \right.$$

$$\left. + \sqrt{|g|}\, g^{ac} (\delta g^{bd}) F_{ac} F_{bd} \right]$$

$$= -\frac{1}{4} \int d^4x \left[-\frac{1}{2} \sqrt{-g}\, g_{ac} (\delta g^{ac}) + 2 \sqrt{|g|} (\delta g^{ac}) g^{bd} F_{ab} F_{cd} \right]$$

$$= -\frac{1}{4}\int d^4x \sqrt{-g}\, \delta g^{ab}\left(-\frac{1}{2} g_{ab} F_{cd}F^{cd} + 2 g^{cd} F_{ac}F_{bd}\right)$$

$$= \int d^4x \frac{1}{2}\sqrt{g}\, T_{ab} \Rightarrow ?$$

$$T_{ab} = -F_{ac}F_b{}^c + \frac{1}{4} g_{ab} F_{cd}F^{cd}.$$

EOM. SCALAR

$$S = \int d^4x \sqrt{g}\left(\frac{1}{2} g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right)$$

$$\frac{\delta S}{\delta\phi} = 0 \Rightarrow \int d^4x \sqrt{g}\left(-\delta\phi\,\nabla_a(g^{ab}\nabla_b\phi) - \frac{\partial V}{\partial\phi}\delta\phi\right) = 0,$$

$$\Rightarrow g^{ab}\nabla_a\nabla_b\phi = -\frac{\partial V}{\partial\phi},$$

$$\Box\phi = -\frac{\partial V}{\partial\phi}.$$

$$T_{ab} = \nabla_a\phi\nabla_b\phi - g_{ab}\left[\frac{1}{2}g^{cd}\nabla_c\phi\nabla_d\phi - V(\phi)\right].$$

$$\nabla^a T_{ab} = \Box\phi(\nabla_b\phi) + \nabla_a\phi\nabla^a\nabla_b\phi - \frac{1}{2}\underbrace{\nabla_b\left[(\nabla_c\phi)(\nabla_d\phi)g^{cd}\right]}_{\substack{(\nabla_b\nabla_c\phi)(\nabla^c\phi) + \\ + (\nabla_c\phi)(\nabla_b\nabla^c\phi) \\ \parallel \\ 2(\nabla_c\phi)(\nabla_b\nabla^c\phi)}}$$

$$+ \underbrace{g_{ab}\nabla^a V(\phi)}_{\nabla_b V = \frac{\partial V}{\partial\phi}\nabla_b\phi} = 0$$

$$\nabla^a(\nabla_a\phi\nabla_b\phi) = \Box\phi + \nabla_a\phi\nabla^a\nabla_a\phi$$

Some Γ-Symbols for Schwarzschild

$$\Gamma^1_{\ kl} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

g^{ik} is diagonal $\Rightarrow$ $m = 1$

$$\Gamma^1_{\ kl} = \frac{1}{2} g^{11} \left(\frac{\partial g_{1k}}{\partial x^l} + \frac{\partial g_{1l}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^l} + \frac{\partial g_{11}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^1} \right)$$

$$\Gamma^1_{\ 11} = \frac{1}{2} \frac{1}{A} \left(A' + \cancel{A'} - \cancel{A'} \right) = \frac{A'}{2A}$$

$$\Gamma^1_{\ 22} = \frac{1}{2} \frac{1}{A} \left(\cancel{\frac{\partial g_{11}}{\partial x^2}} + \cancel{\frac{\partial g_{11}}{\partial x^2}} - \underbrace{\frac{\partial g_{22}}{\partial r}}_{-2r} \right) = -\frac{r}{A}$$

$\vdots$

$$S = -m\int ds = -m\int \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\lambda$$

$$x'(\lambda'(\lambda)) = x(\lambda)$$

$x'(\lambda') = x(\lambda)$ (scalar in λ)

$\lambda' = f(\lambda)$

$$\frac{dx'(\lambda')}{d\lambda'} = \frac{dx'}{d\lambda}\cdot\left(\frac{d\lambda}{d\lambda'}\right) = \left(\frac{d\lambda}{d\lambda'}\right)\cdot\left(\frac{dx}{d\lambda}\right)$$

$$\dot{x}'^\mu(\lambda') = \left(\frac{d\lambda}{d\lambda'}\right)\dot{x}^\mu(\lambda)$$

$$S' = -m\int \sqrt{g_{\mu\nu}(\underbrace{x'(\lambda')}_{x})\frac{dx'^\mu}{d\lambda'}\cdot\frac{dx'^\nu}{d\lambda'}}\, d\lambda'$$

$$= -m\int \sqrt{g_{\mu\nu}(x(\lambda))\,\dot{x}^\mu\dot{x}^\nu\left(\frac{d\lambda}{\cancel{d\lambda'}}\right)\left(\frac{d\lambda}{d\lambda'}\right)}\,\cancel{d\lambda'}$$

$$= -m\int \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\, d\lambda$$

gauge choice $\lambda' = \tau \Rightarrow \sqrt{\dot{x}^2} = \sqrt{\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}} = 1$

Massive particle in the Schwartzschild metric (radial motion)

$$S = -mc^2 \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, d\tau \qquad (+---)$$

ALTERNATIVE

$du^\mu = 0 \Rightarrow Du^\mu = 0$

$\frac{du^\mu}{d\lambda} + \Gamma^\mu{}_{\lambda\nu} u^\lambda \frac{dx^\nu}{d\lambda} = 0$

2° order d.e.

$$\mathcal{L} = -m\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -m \frac{1}{\cancel{2}\sqrt{\dot{x}^2}} \cancel{2} g_{\mu\nu} \dot{x}^\nu$$

$$= -m \frac{g_{\mu\nu} \dot{x}^\nu}{\sqrt{\dot{x}^2}}$$

$\mathcal{L}(\cancel{t}) \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{t}} = -m \frac{g_{00}\dot{t}}{\sqrt{\dot{x}^2}} = \frac{m^2 g_{00}\dot{t}}{\mathcal{L}} = \text{const} \equiv -E$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0$

$$\frac{m g_{00}\dot{t}}{\underbrace{\sqrt{\dot{x}^2}}_{=1}} = E \Rightarrow \dot{t} = \underbrace{\frac{E}{m}}_{e} \frac{1}{g_{tt}} = \frac{e}{1-\frac{r_g}{r}}$$

$$\frac{ds^2}{d\tau^2} = 1 \Rightarrow g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 = 1$$

$$\cancel{\left(1-\frac{r_g}{r}\right)} \frac{e^2}{\left(1-\frac{r_g}{r}\right)^{\cancel{2}}} - \frac{1}{1-\frac{r_g}{r}}\dot{r}^2 = 1$$

$$e^2 - \dot{r}^2 = \left(1-\frac{r_g}{r}\right)$$

$$\left(\dot{r}^2 + \cancel{1} - \frac{2mr^2}{r^3+\alpha L^2} = \cancel{e^2}\right)$$

$$e^2 = \dot{r}^2 + \left(1-\frac{r_g}{r}\right)$$

$\dot{r}^2 = \frac{2mr^2}{r^3+\alpha L^2} \simeq r^2$

$|\dot{r}| = |r|$

$-\dot{r} = r$

$$\dot{r}^2 + \left(1-\frac{r_g}{r}\right) = e^2$$

$\frac{dr}{d\tau} = -r \Rightarrow \lg r = -\tau + c$

$\tau = 0 \Rightarrow \lg r_0 = c$

Boundary condition:

for $r \to \infty$, $\dot{r} \to 0 \Rightarrow e = 1 \Rightarrow \dot{r}^2 = \frac{r_g}{r}$

$\lg r_0 - \lg r = \tau$

$\tau = \lg \frac{r_0}{r}$

$$|\dot{r}| = \sqrt{\frac{r_g}{r}} \qquad |\dot{r}| = \begin{cases} -\dot{r} & \dot{r} < 0 \\ \dot{r} & \dot{r} > 0 \end{cases} \qquad \Rightarrow -\dot{r} = \sqrt{\frac{r_g}{r}}$$

$$\frac{dr}{d\tau} = \pm\sqrt{\frac{r_g}{r}}$$

$$\frac{dr}{d\tau} = -\sqrt{\frac{r_g}{r}}$$

$$\sqrt{\frac{r}{r_g}}\, dr = -d\tau$$

$$\frac{1}{\sqrt{r_g}} \frac{r^{\frac{1}{2}+1}}{\frac{1}{2}+1} = -\tau + \text{const}$$

$$\frac{2 r^{3/2}}{3\sqrt{r_g}} = \text{const} - \tau$$

$$r(\tau = 0) = r_0$$

$$\boxed{-\frac{2}{3}\frac{1}{\sqrt{r_g}}\left(r^{3/2} - r_0^{3/2}\right) = +\tau}$$

$$\tau = \frac{2}{3}\left(\frac{r_0^{3/2}}{r_g^{\frac{1}{2}}} - \frac{r^{3/2}}{r_g^{\frac{1}{2}}}\right)$$

For $r_0 = r_g = \frac{2MG_N}{c^2}$ and $r = 0$,

$$\tau_{sing} = \frac{2}{3}\frac{r_g}{c} = \frac{2}{3}\frac{1}{c}\frac{2MG_N}{c^2} = \frac{4}{3}\frac{MG_N}{c^3}$$

$$= \frac{4}{3} M$$

For $M = M_\odot \Rightarrow r_{g\odot} = 3\text{ km}$, $\tau_{sing} = \frac{2}{3}\cdot\frac{3}{c}\text{ km}$

$$= \frac{2}{c}\text{ km} = \frac{2\cdot\text{km}}{3\cdot 10^5\,\frac{\text{km}}{\text{s}}} \sim 10^{-6}\text{ sec}$$

$M_p = \sqrt{\frac{\hbar}{G_N}}$, $l_p = \sqrt{G_N \hbar}$

Light's propagation

$(+ - - -)$

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2m}{r}} - r^2 d\Omega^2$$

$\dot{r} = \frac{dr}{d\lambda}$

$$ds^2 = 0 \Rightarrow \left(\frac{1-2m}{r}\right)\dot{t} - \frac{\dot{r}}{1-\frac{2m}{r}} - r^2\dot{\varphi} = 0$$

(equatorial plane)

$$\boxed{S = \int d\lambda \mathcal{L}}$$

$[\lambda] = L$

$$\mathcal{L} = -\frac{1}{2}e^{-1}(\lambda)\, g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

$[e] = \frac{1}{M}$

$$\frac{\delta S}{\delta e} = 0 \Rightarrow ds^2 = 0$$

$$+\frac{1}{e^2}\delta e\, \dot{x}^2 = 0$$

$$e'^{-1}(\lambda') = A e^{-1}(\lambda) = \left(\frac{d\lambda'}{d\lambda}\right)^{+1} e^{-1}(\lambda)$$

$$S = -\frac{1}{2}\int d\lambda' \frac{1}{e'(\lambda')} g_{\mu\nu}\frac{dx^\mu}{d\lambda'}\frac{dx^\nu}{d\lambda'} =$$

$$= -\frac{1}{2}\int d\lambda \cancel{\left(\frac{d\lambda'}{d\lambda}\right)} A\, e^{-1}(\lambda)\, g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\cancel{\left(\frac{d\lambda}{d\lambda'}\right)^2} =$$

$$= -\frac{1}{2}\int d\lambda\, e^{-1}(\lambda)\, g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}.$$

$e(\tau) = \frac{1}{M_p}$ (gauge fixing) ← for example.

$$\mathcal{L} = -\frac{1}{e}\left\{\frac{1}{2}\left(\frac{1-2m}{r}\right)\dot{t}^2 - \frac{1}{2}\frac{\dot{r}^2}{1-\frac{2m}{r}} - \frac{1}{2}r^2\dot{\varphi}^2\right\}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = \text{const.} = -\frac{1}{e}\left(\frac{1-2m}{r}\right)\dot{t} \equiv -E,$$

$$+\left(1-\frac{2m}{r}\right)\dot{t} = Ee \equiv \frac{E}{M_p} \equiv \epsilon \qquad [\dot{\varphi}] = 1/L$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \text{const.} = \frac{1}{e}r^2\dot{\varphi} \equiv \tilde{J} \Rightarrow r^2\dot{\varphi} = \frac{\tilde{J}}{M_p} \equiv J$$

$M \frac{1}{M^2} M \Rightarrow [\tilde{J}] = M^0$ $\quad [J] = \frac{1}{M} = L$

$$\mathcal{L} = -\frac{1}{e}\left\{\frac{1}{2}\cancel{\left(\frac{1-2m}{r}\right)}\frac{\epsilon^2}{\left(\frac{1-2m}{r}\right)^{\cancel{2}}} - \frac{1}{2}\frac{\dot{r}^2}{1-\frac{2m}{r}} - \frac{1}{2}\cancel{r^2}\frac{J^2}{r^{\cancel{4}2}}\right\} = 0$$

e $ds^2 = 0$ (photons).

$$\Rightarrow -\dot{r}^2 + \epsilon^2 - \left(\frac{1-2m}{r}\right)\frac{J^2}{r^2} = 0.$$

$g \frac{dx}{d\lambda'}\frac{dx}{d\lambda'} = g'$

Again reparametrizations $\lambda \to \lambda' = \alpha \cdot \lambda + \beta$ $\Rightarrow$

$$\left(1-\frac{2m}{r}\right)\frac{dt}{d\lambda'} = \left(1-\frac{2m}{r}\right)\frac{dt}{d\lambda}\cdot\underbrace{\frac{d\lambda}{d\lambda'}} = \left(1-\frac{2m}{r}\right)\frac{dt}{d\lambda}\cdot\frac{1}{\alpha} = E$$

$\alpha \equiv \frac{1}{E}$ $\Rightarrow$ I can fix $E = 1$.

We also define $J \equiv b$.

Finally, $\dot{r}^2 - 1 + \frac{b^2}{r^2}\left(1-\frac{2m}{r}\right) = 0$

(A) $$\boxed{\dot{r}^2 + b^2\left(\frac{1}{r^2} - \frac{2m}{r^3}\right) = 1}$$

and

(B) $\left(1-\frac{2m}{r}\right)\frac{dt}{d\lambda} = E = 1$, (C) $r^2\frac{d\varphi}{d\lambda} = b$

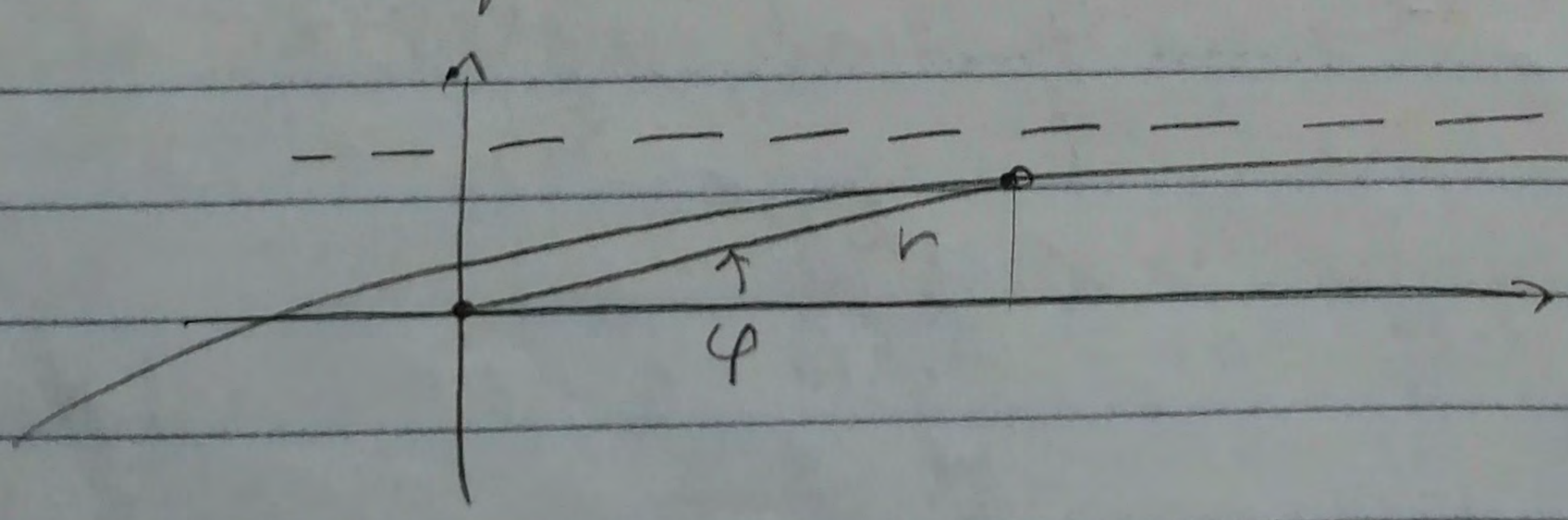
Notice.

For $r \to \infty$, $\overset{(A)}{\Rightarrow}$ $\dot{r}^2 = 1$ $\Rightarrow$ $|\dot{r}| = 1$ $\Rightarrow$ $\dot{r} = -1$ for photons going towards $r = 0$.

(B) $\Rightarrow$ $\frac{dt}{d\lambda} = 1$, (C) $\Rightarrow$ $r^2\frac{d\varphi}{dr} = -b$,

(integrating: $d\varphi = -b\frac{dr}{r^2}$, $\varphi = -b\frac{r^{-2+1}}{-2+1} = \frac{b}{r}$

therefore b = impact factor:



$b = r\sin\varphi$

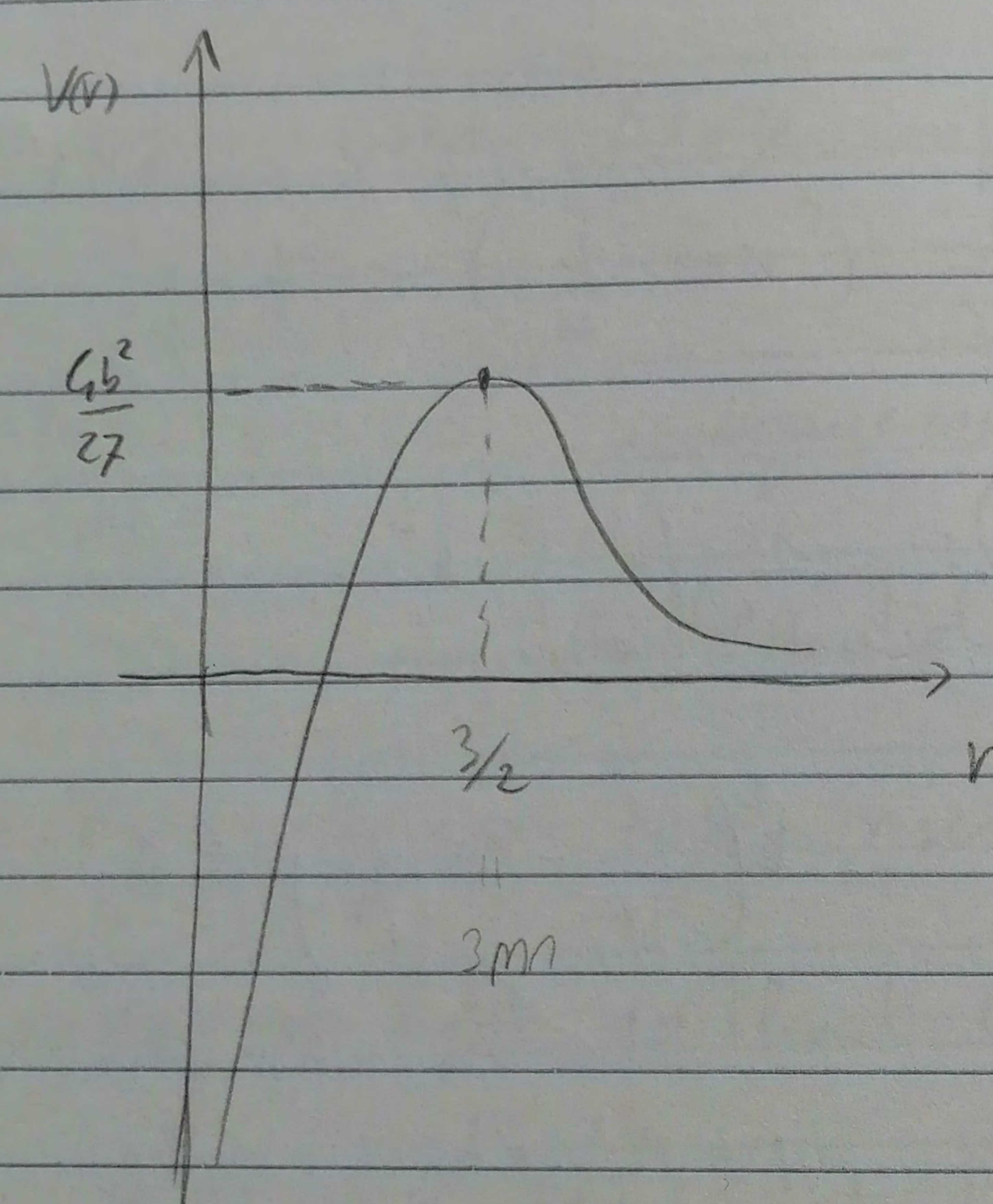
$b \to \lim(r\sin\varphi) \approx r\varphi$

large r $\Rightarrow$ small φ.

$b = r\varphi$

Analysis of the eom:

$$\dot{r}^2 + V(r) = 1 \qquad V(r) = b^2\left(\frac{1}{r^2} - \frac{1}{r^3}\right) \qquad \left(m = \frac{1}{2}\right)$$



$$\dot{r}^2 = 1 - V(r)$$

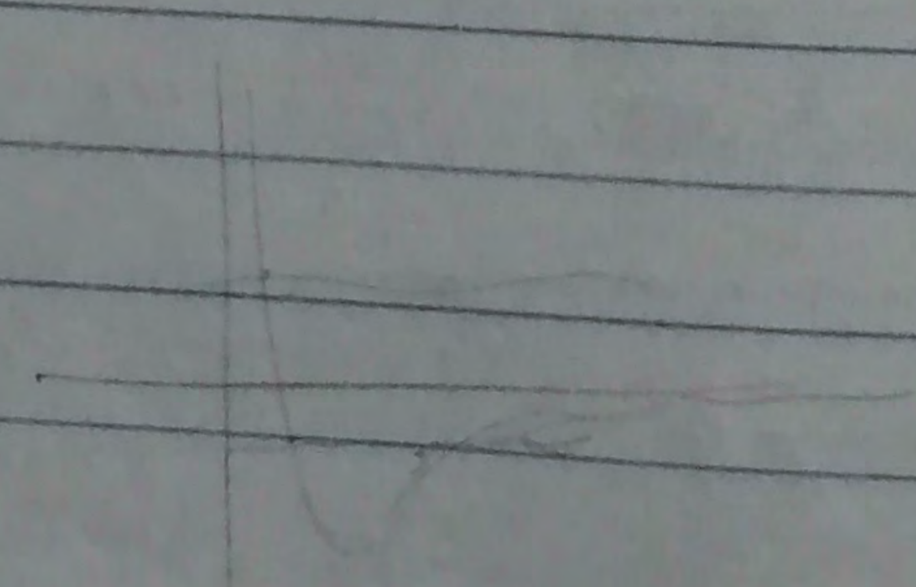
$$1)\ \dot{r}^2 = 1 - V(r) = 1 - b^2\left(\frac{1}{r^2} - \frac{1}{r^3}\right) > 0$$

$$1 - \frac{27}{4}\left(\frac{1}{\left(\frac{3}{2}\right)^2} - \frac{1}{\left(\frac{3}{2}\right)^3}\right) = 1 - \frac{27}{4}\left(\frac{2}{3}\right)^2\left(1 - \frac{2}{3}\right) = 1 - \left(\frac{3-2}{3}\right)^3 = 0$$

1) $b > \sqrt{\frac{27}{4}}\left(\frac{1}{2m}\right)$ the light is confined in $r < \frac{3}{2}$, or it reaches a minimum r and bounces back.

2) In the first case the light falls in $r = 0$ following a spiral trajectory.

3) For $b = \sqrt{\frac{27}{4}}$ we can have an instable circular trajectory.



$$E = V + T$$
$$T = E - V > 0$$

Gravitational deflection of light.

We remind $\dot{r}^2 + b^2\left(\frac{1}{r^2} - \frac{2m}{r^3}\right) = 1$ (A)

we introduce the variable: $u = \frac{b}{r}$ $\left(r = \frac{b}{u}\right)$ $\frac{1}{r} = \frac{u}{b}$

$\Rightarrow \frac{dr}{d\lambda} = -\frac{1}{u^2}\frac{du}{d\lambda}\cdot b$ $\Rightarrow$ (A): $\frac{1}{u^4}\left(\frac{du}{d\lambda}\right)^2 b^2 +$

$+ b^2\left(\frac{u^2}{b^2} - 2m\frac{u^3}{b^3}\right) = 1$

(B) $r^2\frac{d\varphi}{d\lambda} = b \rightarrow \frac{b^2}{u^2}\frac{d\varphi}{d\lambda} = \cancel{b}$ $d\varphi = \frac{u^2}{b}d\lambda$

$d\lambda = \frac{b}{u^2}d\varphi$

(A): $\frac{1}{\cancel{u^4}}\left(\frac{du}{\frac{b}{\cancel{u^2}}d\varphi}\right)^2 \cancel{b^2} + b^2u^2\left(\frac{1}{b^2} - 2m\frac{u}{b^3}\right) = 1$

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 - 2m\frac{u^3}{b} = 1$$

I derive respect to φ:

$$2\left(\frac{du}{d\varphi}\right)\left(\frac{d^2u}{d\varphi^2}\right) + 2\left(\frac{du}{d\varphi}\right)\cdot u - 2m\frac{3u^2}{b}\frac{du}{d\varphi} = 0$$

$$\cancel{2\left(\frac{du}{d\varphi}\right)}\frac{d^2u}{d\varphi^2} + \cancel{2}\cancel{\left(\frac{du}{d\varphi}\right)}\left(u - 2m\frac{3}{2}\frac{u^2}{b}\right) = 0$$

$$\frac{d^2u}{d\varphi^2} + u = \frac{3}{2}\frac{u^2}{b}(2m) \quad \text{D}$$

We fix $2m = 1$ and we look for a perturbative solution in $\frac{1}{b}$

$\frac{1}{b} \ll 1$, $b \gg 1$ (large impact factor)

$\left(\frac{d^2u}{d\varphi^2}\right) + u = 0$ homogeneous eq. has solution $u = \sin\varphi$,

with initial conditions $u(0) = 0$ $u'(0) = 1$.

But $u = \frac{b}{r} \Rightarrow b = r\sin\varphi$ that is the straight trajectory of light without GR corrections. In Newton Physics the light propagate in a straight line.

We replace the solution $\sin\varphi$ in D:

$$\frac{d^2u}{d\varphi^2} + u = (2m)\frac{3}{2}\frac{1}{b}\sin^2\varphi = (2m)\frac{3}{2}\frac{1}{b}(1 - \cos 2\varphi). \quad \text{E}$$

We now look for a solution of the form:

$$u = \alpha\cos\varphi + \beta\sin\varphi + \gamma + \delta\cos 2\varphi + \epsilon\sin 2\varphi$$

we replace in E and we find: $\gamma = \frac{3}{4b}$, $\delta = \frac{1}{4b}$, $\epsilon = 0$.

Therefore: $u = \alpha\cos\varphi + \beta\sin\varphi + \frac{3}{4b} + \frac{1}{4b}\cos 2\varphi$,

using the initial conditions we have $\alpha = -\frac{1}{b}$, $\beta = 1$.

Finally:

$$u = -\frac{1}{b}\cos\varphi + \sin\varphi + \frac{3}{4b} + \frac{1}{4b}\cos 2\varphi . \quad \boxed{f}$$

We define $\varphi_{min} = \frac{\pi}{2} + \eta$ corresponding at the minimum of r (max of u).

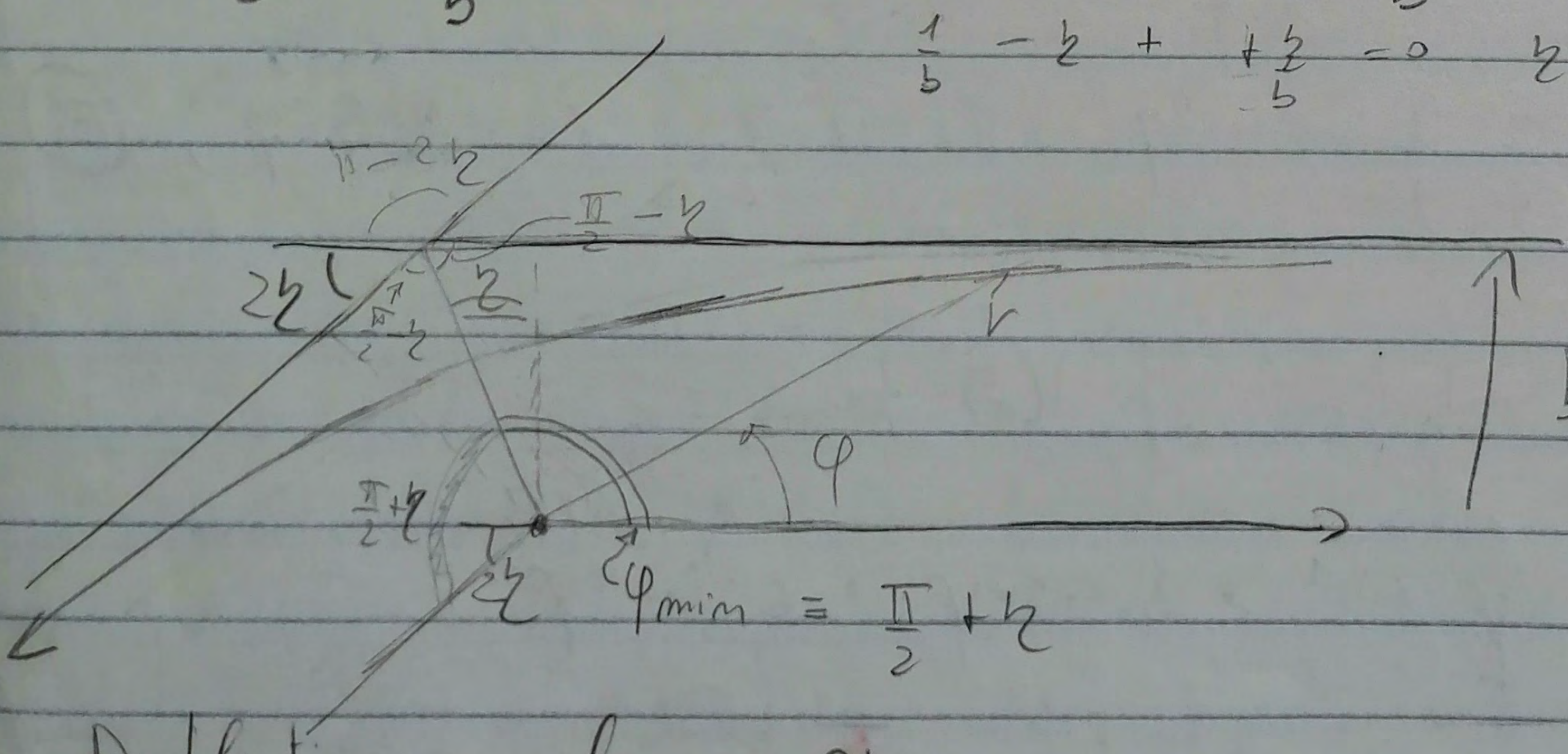
Deriving $\boxed{f}$: $\frac{1}{b}\sin\varphi + \cos\varphi - \frac{1}{2b}\sin 2\varphi = 0$

$$\frac{1}{b}\cos\eta - \sin\eta + \frac{1}{2b}\underbrace{\sin 2\eta}_{2\cos\eta\sin\eta} = 0$$

$\Rightarrow \eta \simeq \frac{1}{b}$

$$\frac{1}{b}\cos\eta - \sin\eta + \frac{1}{b}\cos\eta\sin\eta$$

$$\frac{1}{b} - \eta + \frac{\eta}{b} = 0 \quad \eta \simeq \frac{1}{b} \text{ for large } b.$$



Deflection angle: 2η

$$2\eta = \frac{4GM}{c^2 b}$$

For the Sun $b = 7 \cdot 10^5\ \text{km}$, $\frac{GM}{c^2} \sim 1.5\ \text{km}$ $\Rightarrow$ $2\eta \approx 1.75''$.

Another derivation. (Hartle pag. 210)

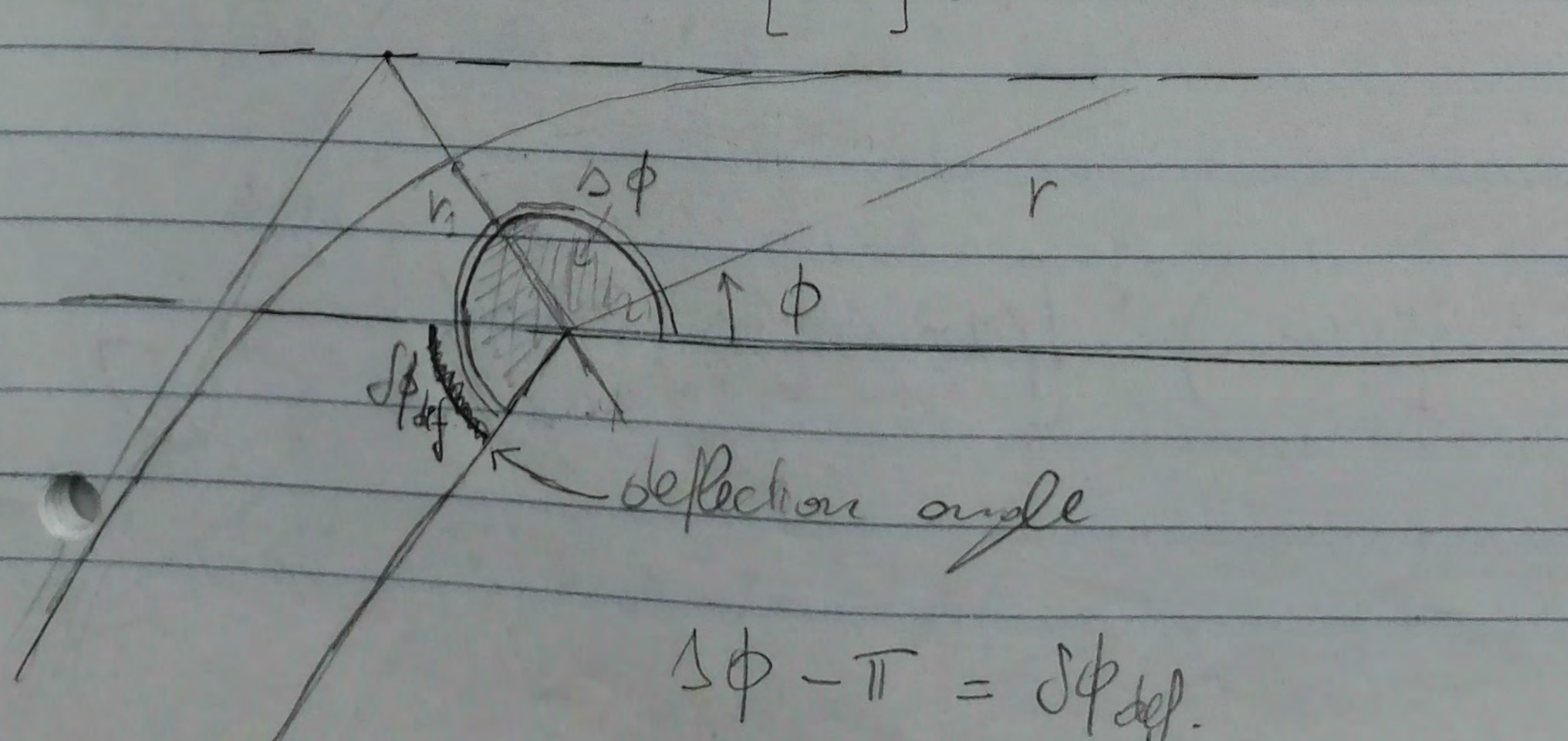
$$\dot{r}^2 = 1 - b^2\left(\frac{1}{r^2} - \frac{2m}{r^3}\right)$$

$$r^2 \frac{d\varphi}{d\lambda} = b \Rightarrow d\lambda = \frac{r^2}{b} d\varphi \Rightarrow \frac{dr}{d\lambda} = \frac{dr}{d\varphi} \cdot \frac{b}{r^2}$$

$$\frac{b^2}{r^4} \cdot \left(\frac{dr}{d\varphi}\right)^2 + \frac{b^2}{r^2}\left(1 - \frac{2m}{r}\right) = 1$$

$$\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)$$

$$\frac{d\varphi}{dr} = \pm \frac{1}{r^2}\underbrace{\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)\right]^{-\frac{1}{2}}}_{[\ \ ]^{\frac{1}{2}}}$$



$$\Delta\phi - \pi = \delta\phi_{def}.$$

r_1 is the radius where $\frac{d\varphi}{dr} = 0$, $[\ ] = 0 \Rightarrow r_1$

$$\Rightarrow \Delta\phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)\right]^{-\frac{1}{2}}$$

we introduce the new variable ω, such that $r = \frac{b}{\omega}$

$dr = -\frac{1}{\omega^2} d\omega\, b$ $\qquad r \to \infty \Rightarrow \omega \Rightarrow 0$

$$\Delta\phi = 2\int_{\omega_1}^{0} -\frac{d\omega}{\cancel{\omega^2}} \cancel{b} \frac{\cancel{\omega^2}}{b^{\cancel{2}}}\left[\frac{1}{b^2} - \frac{\omega^2}{b^2}\left(1 - \frac{2m\omega}{b}\right)\right]^{-\frac{1}{2}}$$

$$= 2\int_{0}^{\omega_1} \frac{d\omega}{\cancel{b}}\left[\frac{1}{\cancel{b^2}} - \frac{\omega^2}{\cancel{b^2}}\left(1 - \frac{2m}{b}\omega\right)\right]^{-\frac{1}{2}}$$

$$= 2\int_{0}^{\omega_1} d\omega\left[1 - \omega^2\left(1 - \frac{2m}{b}\omega\right)\right]^{-\frac{1}{2}}$$

For the sun $b \approx R_\odot \simeq 6.96 \times 10^5$ km

$M_\odot \simeq 1.47$ km

$\Rightarrow \frac{2M}{b} \approx 10^{-6}$

$$\Delta\phi = 2\int_{0}^{\omega_1} d\omega \left(1 - \frac{2m}{b}\omega\right)^{-\frac{1}{2}}\left[\left(1 - \frac{2m\omega}{b}\right)^{-1} - \omega^2\right]^{-\frac{1}{2}}$$

$\circledast$

Massive point particle

$$S = -\int \left(-\frac{1}{2}\cdot\frac{\dot{x}^2}{e} + \frac{1}{2} m^2 e \right) d\tau \qquad e = e(\tau) \qquad [e] = -1 \sim M^{-1}$$

$+ \leftarrow (+ - - -)$ $\qquad \dot{x}^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$

$$\frac{\delta S}{\delta e} = 0 \Rightarrow -\frac{1}{2}\cdot -\frac{1}{e^2}\cdot \delta e\, \dot{x}^2 + \frac{1}{2}m^2 \delta e = 0$$

No derivatives of e $\Rightarrow$ no dynamics

$$\delta S = -\int \left(\frac{1}{2}\frac{\dot{x}^2}{e^2} + \frac{1}{2} m^2\right)\delta e\, d\tau$$

$$\Rightarrow -\frac{\dot{x}^2}{e^2} + m^2 = 0 \quad (*)$$

$$e^2 = -\frac{\dot{x}^2}{m^2} \Rightarrow -\frac{1}{2}\frac{\dot{x}^2}{\sqrt{-\dot{x}^2/m^2}} + \frac{1}{2} m^2\sqrt{-\dot{x}^2/m^2} = \frac{1}{2} m\sqrt{-\dot{x}^2} + \frac{1}{2} m\sqrt{-\dot{x}^2} = m\sqrt{-\dot{x}^2}$$

$e = \frac{1}{m}$ $\Rightarrow$ proper time gauge

$$\mathcal{L} = -\left[-\frac{1}{2e}\left(g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2\right) + \frac{1}{2} m^2 e\right] \qquad \left(\frac{\partial\mathcal{L}}{\partial t} = 0\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = +\frac{g_{tt}\dot{t}}{e} = E, \quad g_{tt}\dot{t} = +E e = +\frac{E}{m} \equiv \epsilon$$

$[\epsilon] = M^0$

$$\boxed{g_{tt}\dot{t} = \epsilon}$$

I replace in $(*)$ $\quad m^2\dot{x}^2 + m^2 = 0$

$$\dot{x}^2 = -1$$

$$g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 = -1$$

$$g_{tt}\frac{\epsilon^2}{(g_{tt})^2} + \dot{r}^2 g_{rr} = -1$$

$$\epsilon^2 + \dot{r}^2 \underbrace{g_{tt}\, g_{rr}}_{-S^2(r)} = -g_{tt}$$

$-\left(1-\frac{2M}{r}\right)$

$-S^2_{(r)}\dot{r}^2 + g_{tt} = -\epsilon^2$

$S^2 \dot{r}^2 + S\left(1-\frac{2M}{r}\right) = +\epsilon^2$

$(+---) \Rightarrow S \equiv -\int ds$

$= -\int \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$

$(-+++) \Rightarrow S = -\int \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$

Conformally coupled particles

Now $\dot{x}^2 \equiv \hat{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$

$[e] = \frac{1}{M}$ $\quad [\phi] = M$

$$S = -\int \left(-\frac{1}{2} f \hat{g}_{\mu\nu}\frac{\dot{x}^\mu \dot{x}^\nu}{e} + \frac{1}{2}\phi^2 e\right) d\tau$$

$$S = -m\int \sqrt{-\underbrace{K_4^2}_{=\frac{1}{M_p^2}}\phi^2 g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\, d\lambda \qquad \frac{1}{M_p^2} \Rightarrow \frac{m^2}{M_p^2} = f$$

$$= -\int \sqrt{-f\phi^2 g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

$$\frac{\delta S}{\delta e} = 0 \Rightarrow f\frac{\dot{x}^2}{e^2} + \phi^2 = 0 \Rightarrow e^2 = -f\frac{\dot{x}^2}{\phi^2}$$

$$\Rightarrow S = -\int \left(-\frac{1}{2}\frac{f\dot{x}^2}{\sqrt{-\frac{\dot{x}^2}{\phi^2}f}} + \frac{1}{2}\phi^2\sqrt{-\frac{\dot{x}^2 f}{\phi^2}}\right) d\tau$$

$$= -\int \left(\frac{1}{2}\sqrt{-f\phi^2\dot{x}^2} + \frac{1}{2}\sqrt{-\phi^2\dot{x}^2}\right) d\tau$$

$\leftarrow$ 2º ordine

$$= -\int \sqrt{-f\phi^2\dot{x}^2}\, d\tau = -\int \sqrt{-\underbrace{\frac{m^2}{M_p^2}}_{f} M_p^2 \dot{x}^2}\, d\tau = -m\int \sqrt{-\dot{x}^2}\, d\tau$$

for $\phi = M_p \Rightarrow S$

$(-(-dt^2 + dx^2))^{\frac{1}{2}}$

$= (dt^2 - dx^2)^{\frac{1}{2}}$

$= (dt^2)^{\frac{1}{2}}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = +f\frac{\hat{g}_{tt}}{e}\dot{t} = +E \qquad \dot{t} = \frac{E e}{f\hat{g}_{tt}}$$

$$\frac{1}{e^2}f\left(\hat{g}_{tt}\dot{t}^2 + \hat{g}_{rr}\dot{r}^2\right) + \phi^2 = 0$$

$$\boxed{e^2 = \frac{f}{\phi^2}} \Rightarrow \hat{g}_{tt}\dot{t}^2 + \hat{g}_{rr}\dot{r}^2 = -1$$

proper time gauge

$$\cancel{\hat{g}_{tt}}\frac{E^2 e^2}{f\hat{g}_{tt}^{\cancel{2}}} + \hat{g}_{rr}\dot{r}^2 = -1$$

$$\frac{E^2}{\phi^2} + \hat{g}_{rr}\hat{g}_{tt}\dot{r}^2 = -g_{tt}$$

$(-+++) \Rightarrow \quad ds^2 = -d\tau^2$

$$E^2 \frac{\cancel{\delta}}{M_p^2} - S^{\cancel{2}} \dot{r}^2 = + \cancel{\delta}\left(1 - \frac{2M}{r}\right)$$

$$\frac{E^2}{M_p^2} - S\dot{r}^2 - \left(1 - \frac{2M}{r}\right) = 0$$

$E = M_p \Rightarrow \quad -S\dot{r}^2 + \frac{2M}{r} = 0$

$$S\dot{r}^2 = \frac{2M}{r}$$

$$S = \int \left(-\frac{1}{2} \frac{\cancel{\phi^2}\, \hat{\rho}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\cancel{M_p^2}\, e} + \frac{1}{2}\phi^2 e \right) d\tau$$

$$\left(e' = \Omega^2 e \qquad \phi'^2 e' = \Omega^{-2} \Omega^2 \phi e \right)$$

$$e^2 = \frac{f}{\phi^2} = \frac{f}{S^{-1} M_p^2}$$

Ok? Sì perché $\phi(x(\tau))$!

Alternativa: $S = -\int \left(-\frac{1}{2} \frac{f\, \phi^2 \hat{\rho}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} + \frac{1}{2} e \right) d\tau$

$$\frac{\delta S}{\delta e} = \frac{f\, \phi^2 \dot{x}^2}{e^2} \delta e + \delta e = 0 \qquad \frac{M^2}{M} = M \qquad [e] = M$$

$$e^2 = -f\phi^2 \dot{x}^2$$

rimpiazzo nella S: $S = -\int \left(\frac{1}{2} \frac{f\phi^2 \dot{x}^2}{\sqrt{-f\phi^2\dot{x}^2}} + \frac{1}{2}\sqrt{-f\phi^2\dot{x}^2} \right) d\tau$

$$= -\int \sqrt{-f\phi^2\dot{x}^2}\, d\tau \qquad \text{Ok}$$

(*) We expand in $\frac{2M\omega}{b}$, $\omega_1 = \frac{b}{r_1}$

$$\Delta\phi = 2\int_0^{\omega_1} d\omega \frac{\left(1 + \frac{1}{2}\frac{2m}{b}\omega\right)}{\left[1 + \frac{2m}{b}\omega - \omega^2\right]^{\frac{1}{2}}}$$

Finally: $\Delta\phi \simeq \pi + \frac{4M}{b}$

$$\delta\phi_{def.} = \Delta\phi - \pi = \frac{4M}{b} \quad \text{for small } \frac{M}{b}.$$

$$\delta\phi_{def.} = \frac{4GM}{c^2 b} \approx 1{,}7'' \text{ for Sun.}$$

$$E^2 - \frac{J^2 r'^2}{r^4} - \frac{J^2}{r^2}\left(1 - \frac{2M}{r}\right) = 1 - \frac{2M}{r}$$

$$\frac{J^2 r'^2}{r^4} = E^2 - \left(1 - \frac{2M}{r}\right) - \frac{J^2}{r^2}\left(1 - \frac{2M}{r}\right)$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{J^2} E^2 - \frac{r^4}{J^2}\left(1 - \frac{2M}{r}\right) - r^2\left(1 - \frac{2M}{r}\right)$$

$$= \frac{r^4}{J^2}\Big(E^2 -$$

THE KERR SPACETIME (1963)

This is the geometry of an axially-symmetric black-hole in vacuum, namely $R_{\mu\nu}=0$ (Ricci flat).

The symmetry $\varphi \to \varphi + \text{const} \Rightarrow g_{\mu\nu}(\cancel{\varphi})$.

The rotation at large distance implies $g_{03} \neq 0$, and actually:

$\boxed{g_{03} = \frac{2kM}{r} \sin^2\theta}$, where M is the angular momentum.

The metric is very difficult to derive solving the Einstein EqOM.

$$ds^2 = \left(1 - \frac{r_g r}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2\theta\right) \times$$

$$\times \sin^2\theta \, d\varphi^2 + \frac{2 r_g r a}{\rho^2} \sin^2\theta \, d\varphi \, dt,$$

where: $\Delta = r^2 - r r_g + a^2$, $\rho^2 = r^2 + a^2 \cos^2\theta$.

$r_g = 2mk \quad (k = G_N)$.

For large "r" $\quad g_{03} \simeq \frac{r_g \cancel{r} a}{r^{\cancel{2}}} \sin^2\theta = \frac{2mka \sin^2\theta}{r}$

$\Rightarrow M = ma$

For $t \to -t$ $(a \to -a)$ $\Rightarrow ds^2 \to ds^2$ (invariant for time inversion).

For $m = 0$ $\quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. Indeed, the metric

$M = 0$ but $a \neq 0$

$$ds^2 = dt^2 - \frac{\rho^2}{r^2 + a^2} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2) \sin^2\theta \, d\varphi^2,$$

can be transformed in the Minkowski metric with the following coordinate transformation:

$$\begin{cases} x = \sqrt{r^2+a^2}\,\sin\theta\,\cos\varphi \\ y = \sqrt{r^2+a^2}\,\sin\theta\,\sin\varphi \\ z = r\cos\theta \end{cases} \quad (*)$$

Notice, the surfaces $r = \text{const.}$ are ellipsoids crushed:

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1.$$

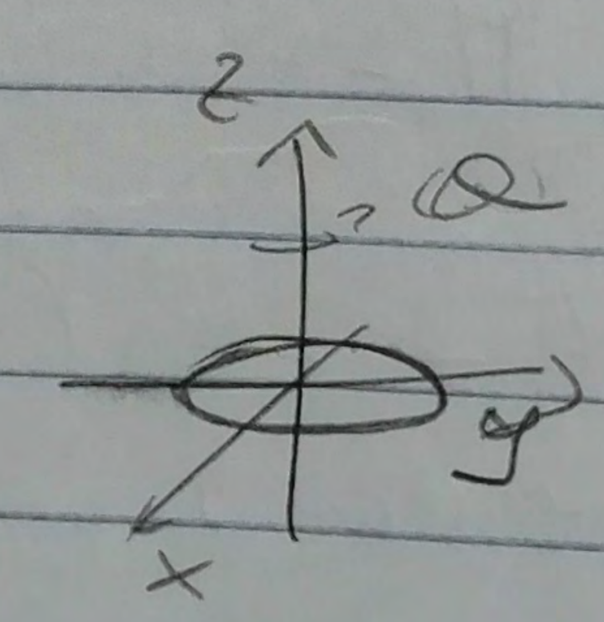
The new metric has potential singularities in: $\rho = 0$ and $\Delta = 0$

The singularity in $\rho = 0$ is unavoidable and it is a one dimensional circle. Indeed,

$$\rho^2 = r^2 + a^2\cos^2\theta = 0 \Rightarrow r = 0 \text{ and } \theta = \frac{\pi}{2}.$$

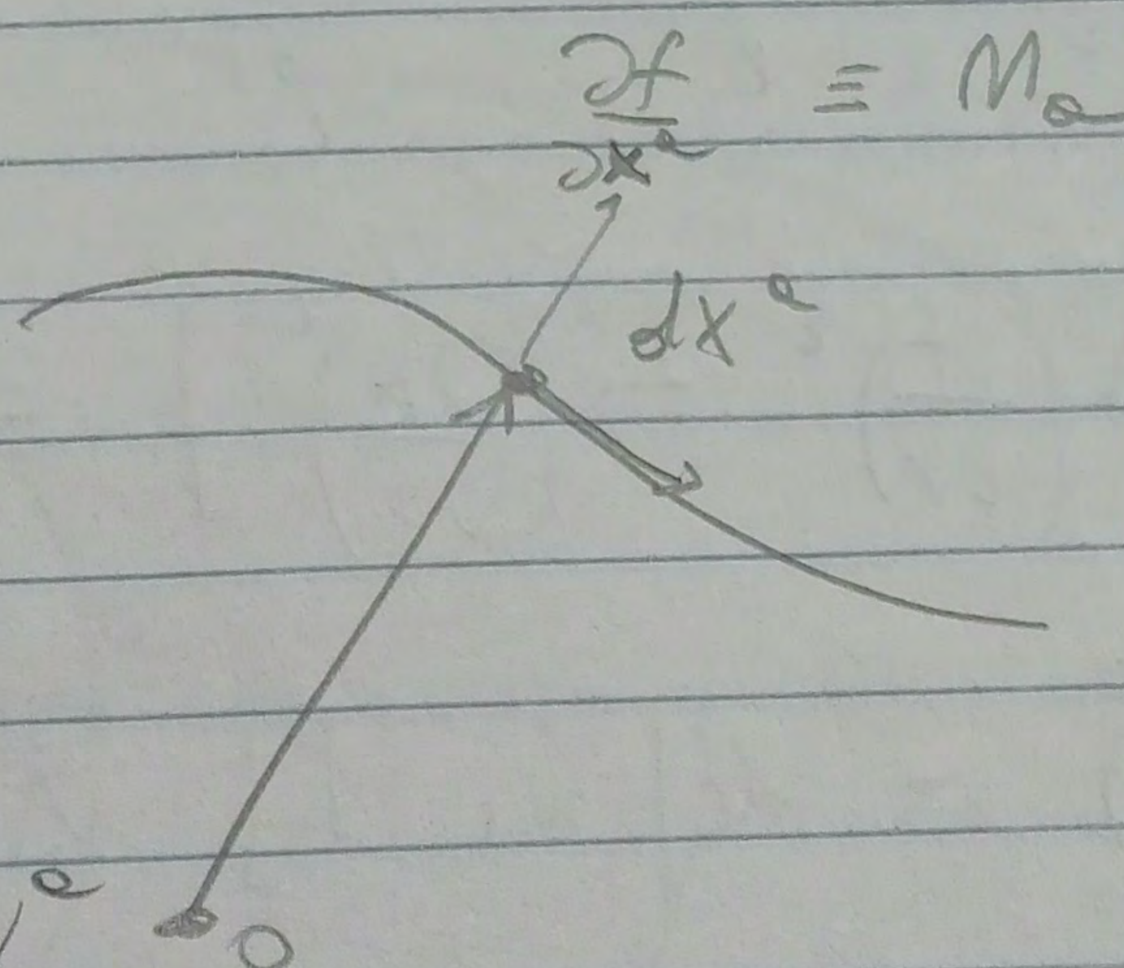
If we replace $r = 0$, $\theta = $ Pi/2 in $(*)$ we get:

$$\begin{cases} x = a \times \cos\varphi \\ y = a\,\sin\varphi \\ z = 0 \end{cases} \Rightarrow x^2+y^2 = a^2$$



Null-hypersurface

$f(x^a) = \text{const.}$



$$df = \underbrace{\frac{\partial f}{\partial x^a}}_{\equiv n_a} dx^a = 0$$

$$\Rightarrow \frac{\partial f}{\partial x^a} \perp dx^a, \text{ or } n_a \perp dx^a.$$

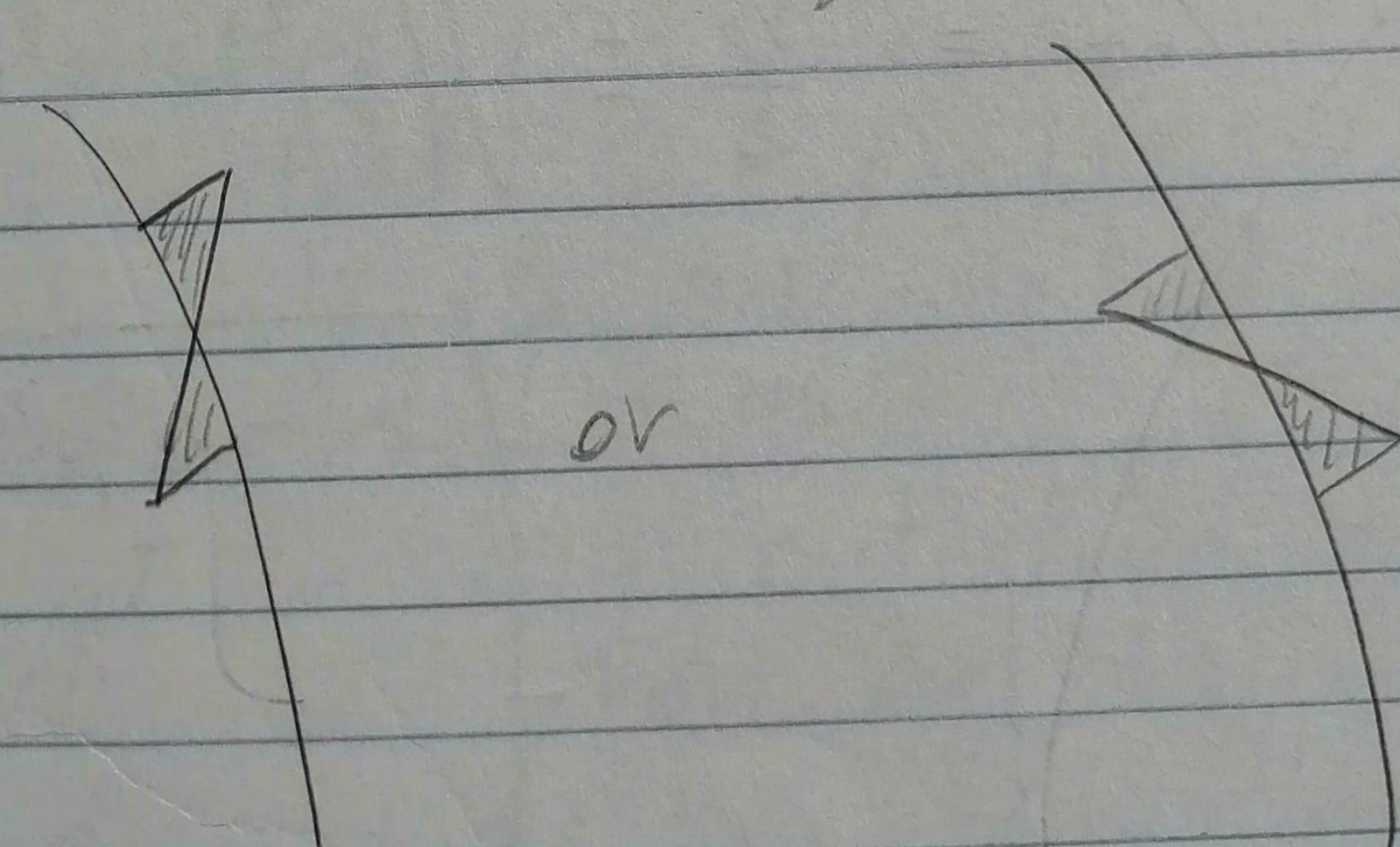
n_a is null vector ⇒

The surface is null ⇒ $n_a n^a = 0$

$\Rightarrow n^a \parallel dx^a$. $\quad n_a = g_{ab} dx^b = \frac{\partial f}{\partial x^a}$

Moreover, $n_a n^a = dx_a dx^a = ds^2 = 0$.

Therefore, the surface touch the lightcone:



For Kerr the condition $n_a n^a = 0$ tells us:

$$g^{ab} n_a n_b = 0, \quad g^{11}\left(\frac{\partial f}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial f}{\partial \theta}\right)^2 = 0,$$

We look for $f(r,\theta)$ because of the symmetry $\varphi \to \varphi + \text{const.}$

$$g^{\mu\nu}$$

$$\shortparallel$$

$$g^{\theta\theta} = -1/\rho^2 \qquad g^{11} = -\frac{\Delta}{\rho^2}$$

$$\Rightarrow \frac{1}{\rho^2}\left[\Delta\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2\right] = 0$$

$\Rightarrow f(r,\theta)$ is define by $\Delta = 0$ and $\frac{\partial f}{\partial \theta} = 0$.

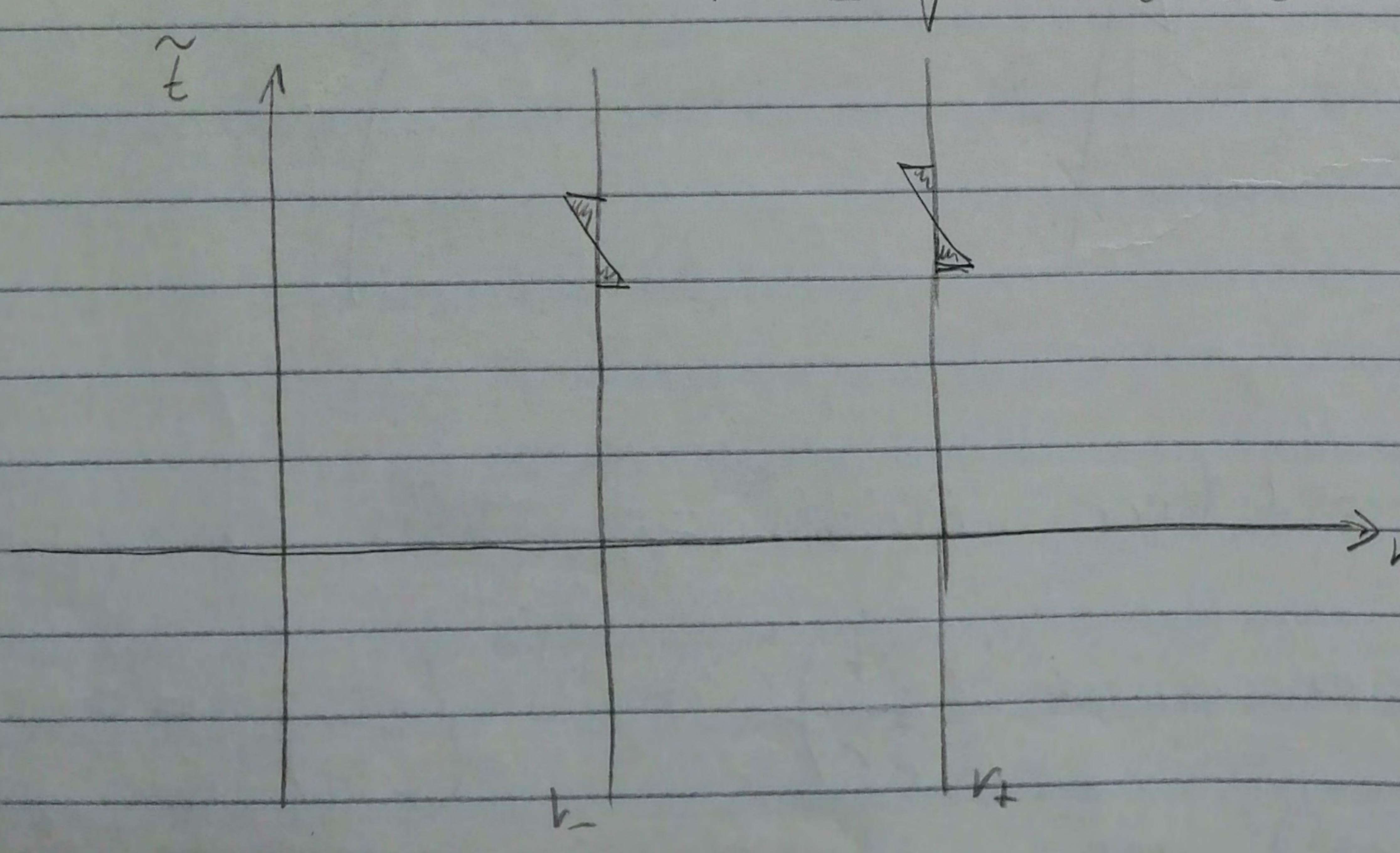
$$\Delta = 0 \Rightarrow r_{H\pm} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2} \qquad (g_{11} = \infty)$$

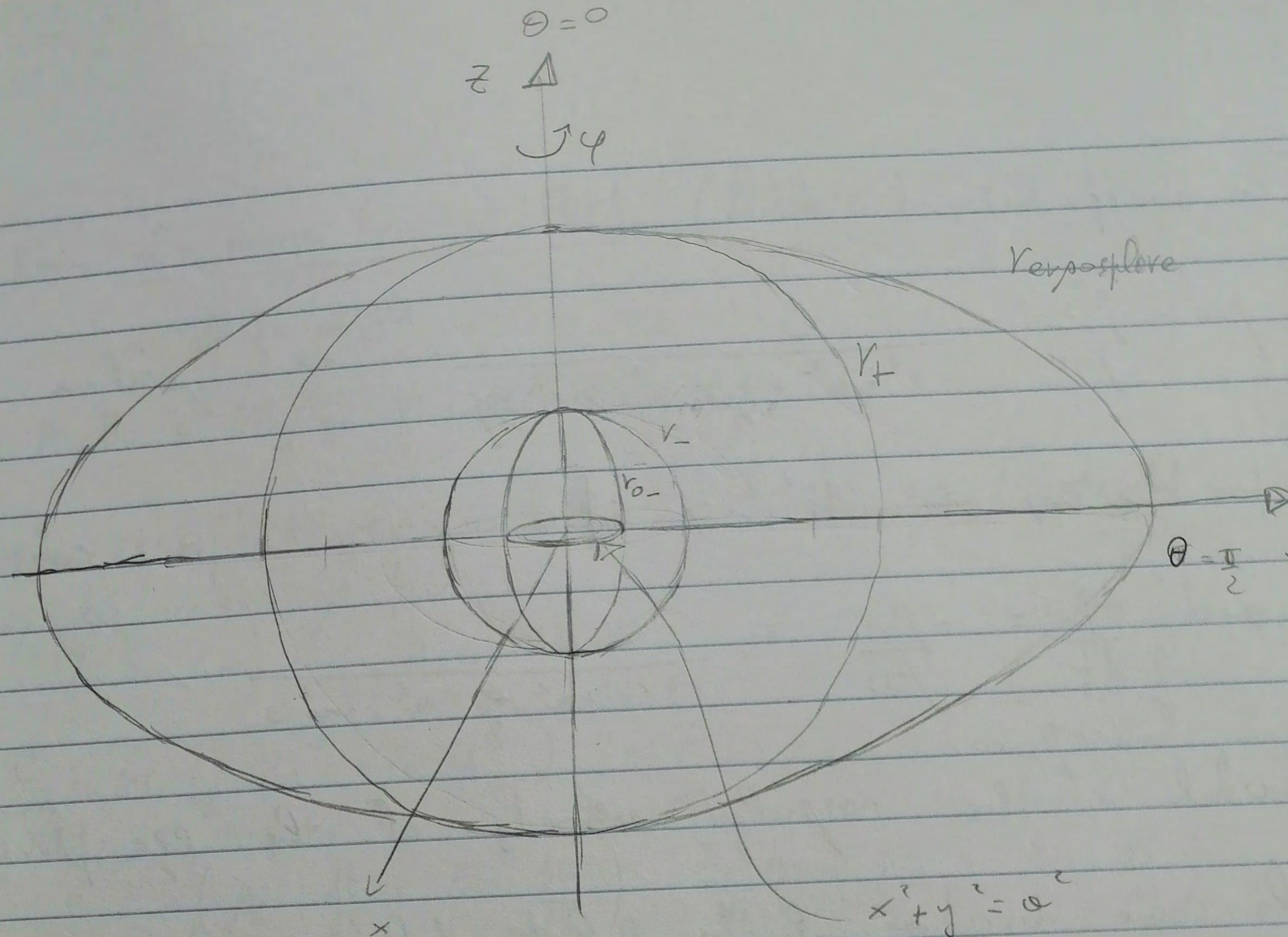
We have two horizons.

For Schwarzschild g_{00} change sign when $g_{11} \to \infty$. Here, instead,

$$g_{00} = 0 \Rightarrow r_{0\pm} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2\cos^2\theta}.$$

$$m \pm \sqrt{m^2 - a^2\cos^2\theta}$$





In the ergosphere for $r, \theta, \varphi = \text{const.}$ $ds^2 < 0$ $\Rightarrow$
no particle can stay at rest.
It is impossible $\varphi = \text{constant}$, but possible $r = \text{const.}$
(Remember the similarity with the Schwartzchild where for $r < 2m$
is impossible to stay in $r = \text{const.}$)
Particles and light can get in and out from the ergosphere
contrary to the event horizon. For an observer at infinity
a particle can cross r_{ergo} in finite time, but $\Delta t \to \infty$ for $\theta \sim 0, \pi$,
where r_{ergo} touches r_+.

The natural form of the metric in the ergosphere is:

$$ds^2 = \left(g_{00} - \frac{g_{03}^2}{g_{33}}\right) dt^2 + g_{11}\, dr^2 + g_{22}\, d\theta^2 + g_{33}\left(d\varphi + \frac{g_{03}}{g_{33}}\, dt\right)^2.$$

The coefficient of dt^2:

$$g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{\Delta}{r^2+a^2+r_s r a^2 \sin^2\theta/\rho^2} \quad \text{is positive}$$

for $r > r_+$, $\Rightarrow$ $ds^2 > 0$ if $r = \text{const.}$, $\theta = \text{const.}$,

and $\frac{d\varphi}{dt} = -\frac{g_{03}}{g_{33}} = \frac{r_s a r}{\rho^2(r^2+a^2)+r_s r a^2 \sin^2\theta}$,

(because $g_{33} < 0$)

which is the angular velocity of the ergosphere in the same direction of the Black Hole angular momentum.

Extraction of energy from a BH. (Penrose 1969)

mass of the particle

$$P_0 = m_0 u_0 = m_0 g_{0i} u^i \equiv \mathcal{E}_0$$

In the ergosphere t is not a time coordinate that implies an interesting fact.

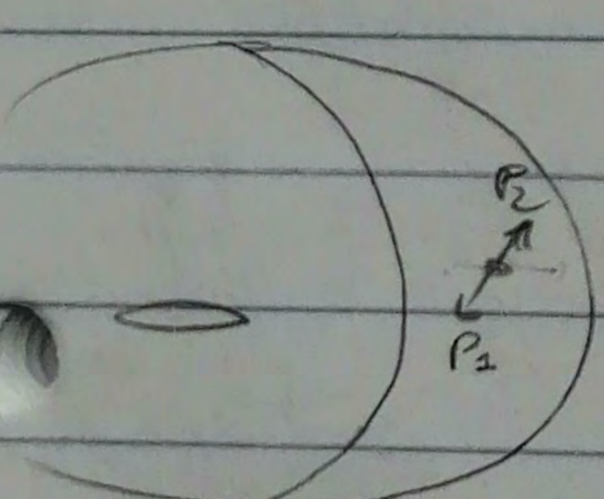
$$\mathcal{E}_0 = m_0 \left(g_{00} u^0 + g_{03} u^3\right) =$$

$$= m_0 \left(g_{00} \frac{dt}{ds} + g_{03} \frac{d\varphi}{ds}\right) = \frac{dt}{ds} m_0 \left(g_{00} + g_{03} \frac{d\varphi}{dt}\right)$$

$$= m_0 \frac{dt}{ds} \left(g_{00} + g_{03}\left(-\frac{g_{03}}{g_{33}}\right)\right)$$

<0

$g_{00}<0$ in the ergosphere $\Rightarrow$ $\mathcal{E}_0$ can be negative.
$g_{03}>0$



$$E = E_1 + E_2$$

Ex. E_1 has $\frac{dt}{ds} > 0$,
E_2 has $\frac{dt}{ds} < 0$.

If $E_1 < 0$, $E_2 > 0$ $\Rightarrow$ $E = -|E_1| + E_2$ $\Rightarrow$ $E_2 = E + |E_1| > E$

P_1 has negative energy and then can not get out the ergosphere.
P_2 " positive " " " " escapes outside the ".

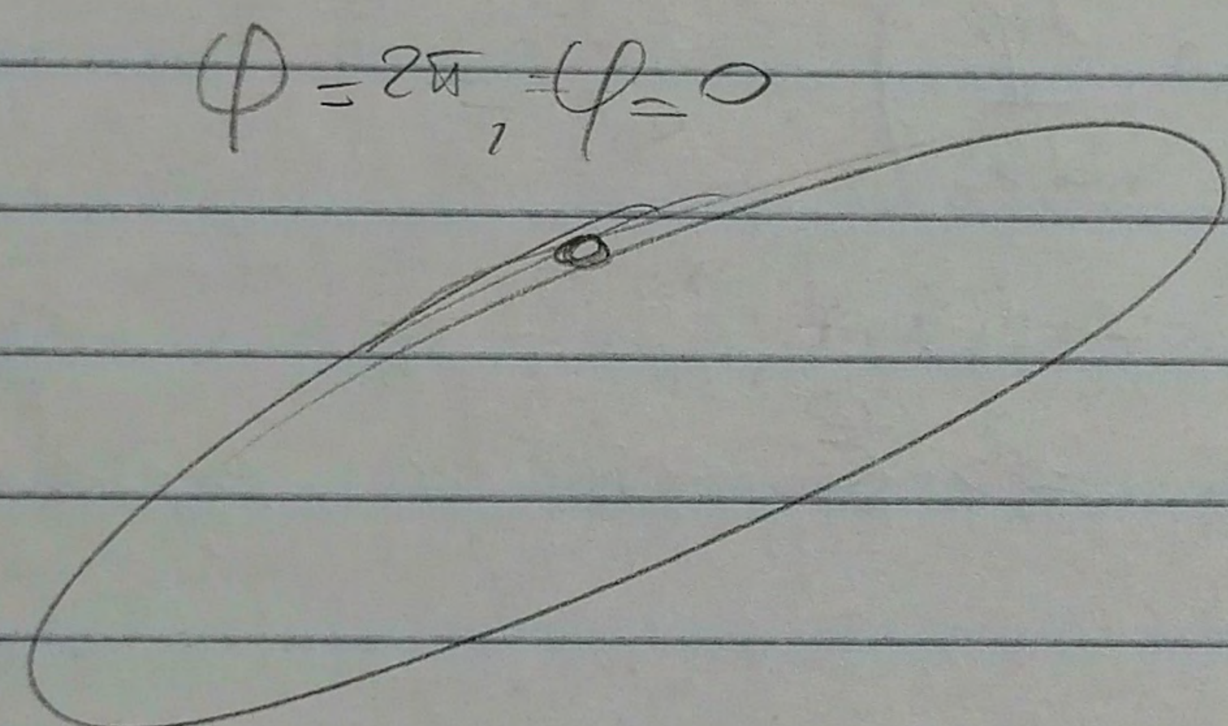
CLOSE TIME-LIKE CURVES

for $r = \text{const.} < 0$, $\theta = \frac{\pi}{2}$ (equatorial plane), $t = \text{const}$,

$$ds^2 = -\left(r^2 + a^2 + \frac{2ma^2}{r}\right) d\varphi^2 \simeq -\frac{2ma^2}{r} d\varphi^2 \underset{r \lesssim 0}{>} 0$$

$\Rightarrow \varphi$ is timelike direction, but φ is periodic

$\varphi \in [0, 2\pi] \Rightarrow$ CTC.



Penrose diagrams

$I^+ \equiv$ "future timelike infinity",

$t \to \infty$, r finite.

$I^- \equiv$ "past timelike infinity",

$t \to -\infty$, r finite.

$I^0 \equiv$ "spacelike infinity",

$r \to \infty$, t finite.

$\mathscr{I}^+ \equiv$ "future Null infinity",

$t + r \to +\infty$, $t - r$ finite.

($t = r + \text{const}$)

$\mathscr{I}^- \equiv$ "past Null infinity",

$t - r \to -\infty$, $t + r$ finite.

($t = -r + \text{const}$)

Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$(t, r, \theta, \varphi) \to (\psi, \xi, \theta, \varphi)$:

$$t + r = \tan \frac{1}{2}(\psi + \xi)$$

$$t - r = \tan \frac{1}{2}(\psi - \xi)$$

$$ds^2 = \frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{1}{2}(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)} + r^2 d\Omega^{(2)}$$

I^+ : $t \to \infty$, r const

$$\begin{cases} \tan\frac{1}{2}(\psi+\xi) = \infty \\ \tan\frac{1}{2}(\psi-\xi) = \infty \end{cases} \quad \begin{cases} \frac{1}{2}(\psi+\xi) = \frac{\pi}{2} \Rightarrow \psi = \pi - \xi \\ \frac{1}{2}(\psi-\xi) = \frac{\pi}{2} \Rightarrow \psi = \pi + \xi \end{cases}$$

$$\begin{cases} \boxed{\xi = 0} \\ \boxed{\psi = \pi} \end{cases}$$

I^- : $t \to -\infty$, r const

$$\begin{cases} \tan\frac{1}{2}(\psi+\xi) = -\infty \\ \tan\frac{1}{2}(\psi-\xi) = -\infty \end{cases} \quad \begin{cases} \psi+\xi = -\pi \\ \psi-\xi = -\pi \end{cases} \Rightarrow \begin{cases} \boxed{\xi = 0} \\ \boxed{\psi = -\pi} \end{cases}$$

I^0 : $r \to \infty$, t const

$$\begin{cases} \tan\frac{1}{2}(\psi+\xi) = \overset{+\infty}{r} \Rightarrow \frac{1}{2}(\psi+\xi) = \frac{\pi}{2} \qquad \psi = \pi - \xi \\ \tan\frac{1}{2}(\psi-\xi) = \underset{-\infty}{-r} \Rightarrow \frac{1}{2}(\psi-\xi) = -\frac{\pi}{2} \qquad \psi = \xi - \pi \end{cases}$$

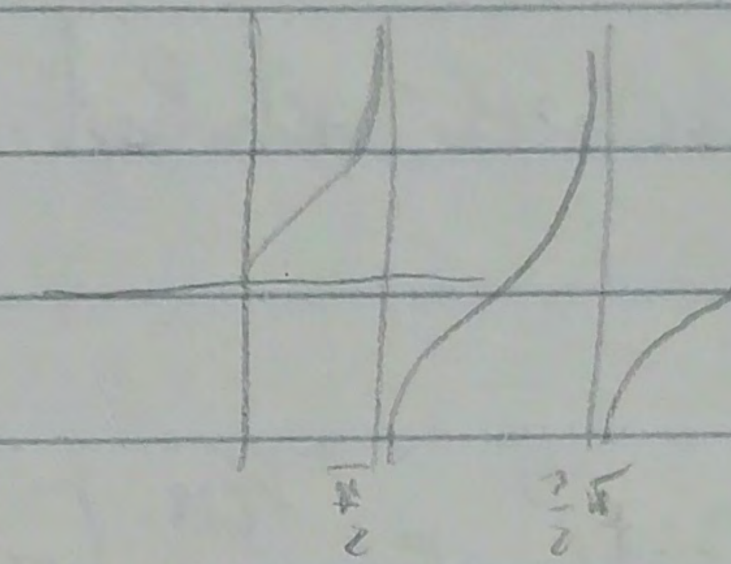
$$\begin{cases} \psi = \pi - \xi \\ \psi = \xi - \pi \end{cases} \qquad \pi - \xi = \xi - \pi \Rightarrow 2\xi = 2\pi \quad \xi = \pi \quad \psi = 0$$

$$\begin{cases} \boxed{\xi = \pi} \\ \boxed{\psi = 0} \end{cases}$$

$\mathcal{J}^+$ $\quad t+r\to\infty$, $\quad t-r=\text{const}$.

$$\begin{cases}\tan\frac{1}{2}(\psi+\xi)=\infty\\ \tan\frac{1}{2}(\psi-\xi)=\text{const}\end{cases}$$

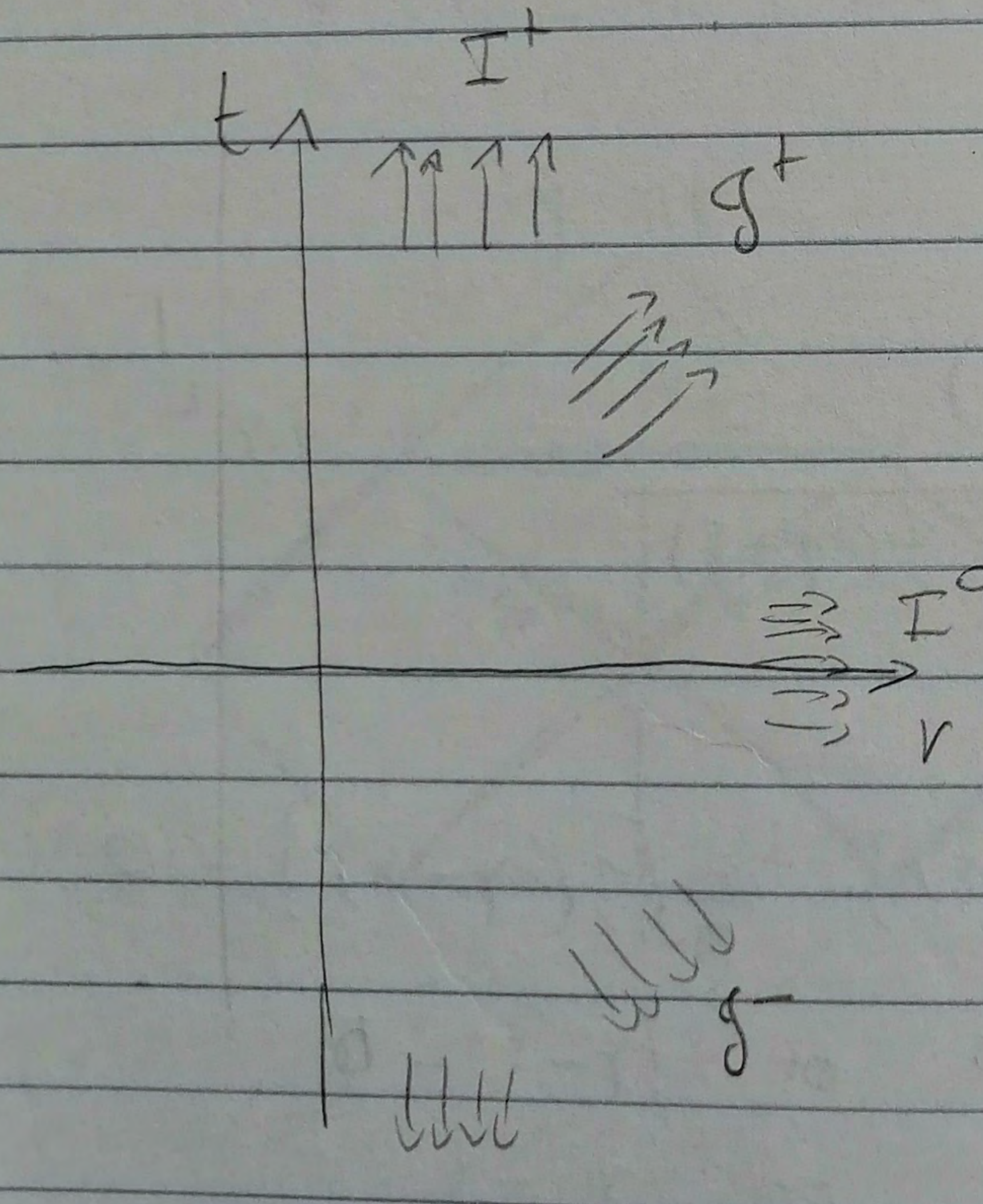
$\tan\left[\frac{1}{2}(\pi-2\xi)\right]=\text{const}$

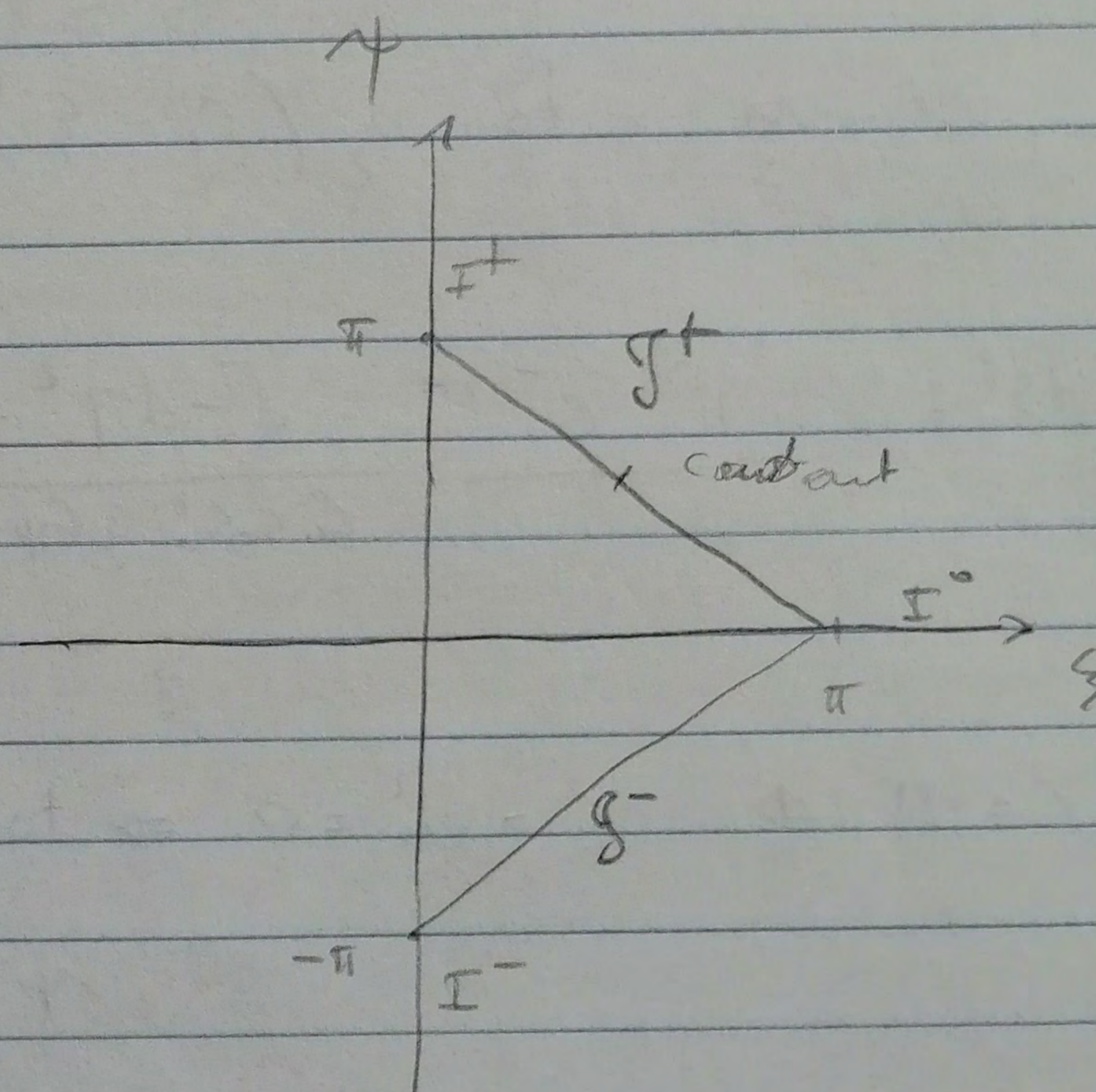


$t=-r+\infty$

$\psi+\xi=\pi \qquad \psi=\pi-\xi$

$\mathcal{J}^-$ $\quad t-r\to-\infty$, $\quad t+r\to\text{const}$ $\qquad \psi-\xi=-\pi \qquad \psi=\xi-\pi$





$\xi=\frac{\pi}{2}, \quad \psi=\frac{\pi}{2}$

$\tan\left(\frac{1}{2}0\right)=0$

$t-r=\text{const}=0$

Schwarzschild   pag 918-920 Gravitation MTW.

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} (-dv^2 + du^2) + r^2 d\Omega^{(2)}$$

↳ Kruskal-Szekeres coordinates (v, u, θ, φ).

$$\left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}} = v^2 - u^2 = \tan\left[\tfrac{1}{2}(\psi + \xi)\right] \tan\left[\tfrac{1}{2}(\psi - \xi)\right].$$

$$v + u = \tan \tfrac{1}{2}(\psi + \xi)$$

$$v - u = \tan \tfrac{1}{2}(\psi - \xi)$$

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r} \frac{(-d\psi^2 + d\xi^2)}{4\cos^2 \frac{1}{2}(\psi+\xi) \cdot \cos^2 \frac{1}{2}(\psi - \xi)} + r^2 d\Omega^2.$$

$$r = 2M \Rightarrow v^2 - u^2 = 0 \Rightarrow \tan\left[\tfrac{1}{2}(\psi+\xi)\right] \tan\left[\tfrac{1}{2}(\psi - \xi)\right] = 0$$

$$\psi + \xi = 0 \quad \text{or} \quad \psi - \xi = 0$$

$$\psi = -\xi \quad \text{or} \quad \psi = \xi$$

$$r = 0 \Rightarrow \tan \tfrac{1}{2}(\psi + \xi) \tan \tfrac{1}{2}(\psi - \xi) = 1 \Rightarrow \psi = \tfrac{\pi}{2} \ \forall \xi$$

or

$$\psi = -\tfrac{\pi}{2} \ \forall \xi$$

$$\tan \tfrac{1}{2}\left(\tfrac{\pi}{2} + \xi\right) \cdot \tan \tfrac{1}{2}\left(\tfrac{\pi}{2} - \xi\right) = 1$$

verify with mathematica.

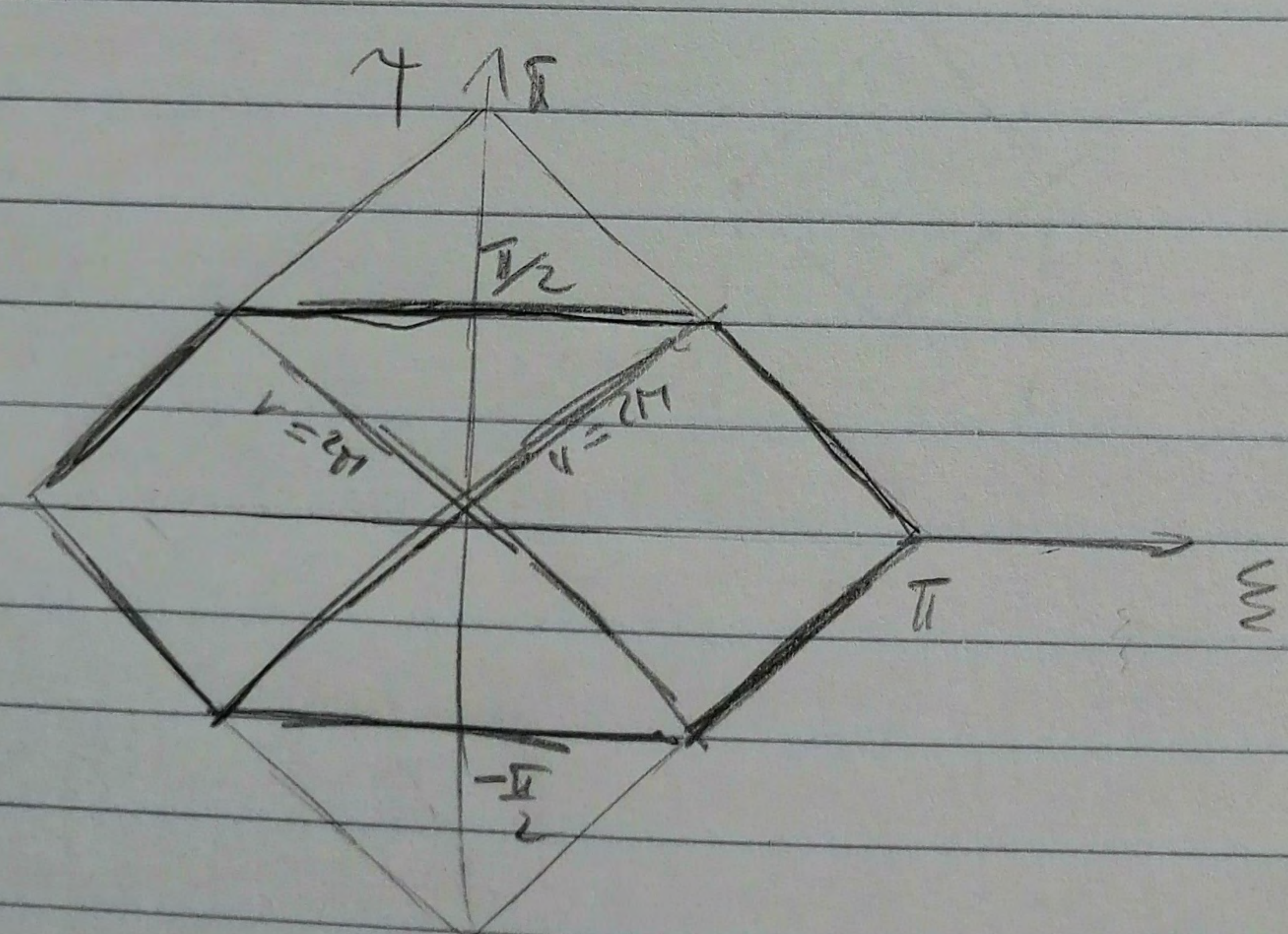
I^+ : $\psi = \pi, \ \xi = 0$.

I^- : $\psi = -\pi, \ \xi = 0$.

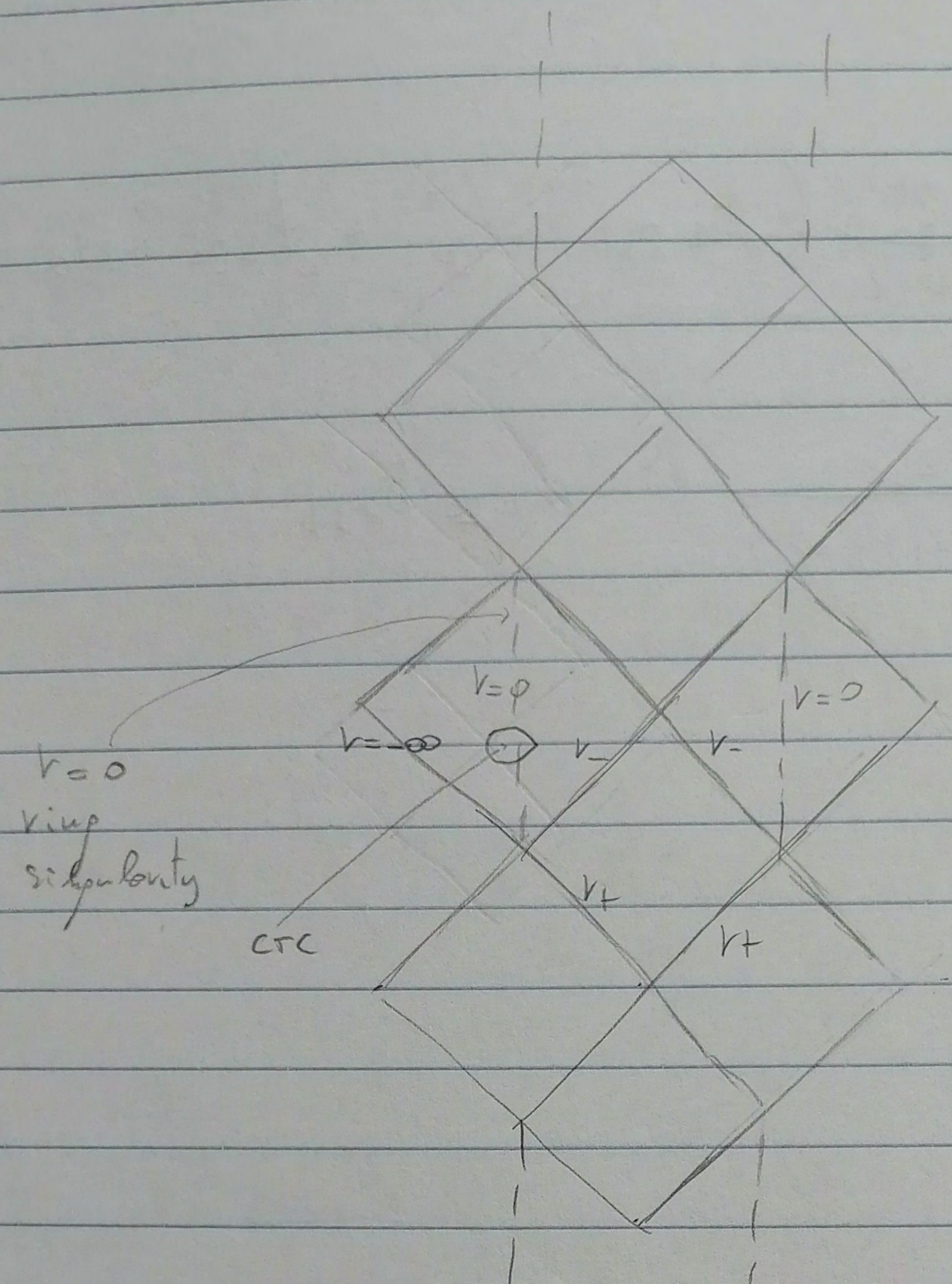
I^0 : $\psi = 0, \ \xi = \pi$.

$\mathscr{I}^+$: $\psi + \xi = \pi, \ -\pi < \psi - \xi < \pi$.

$\mathscr{I}^-$: $\psi - \xi = -\pi, \ -\pi < \psi + \xi < \pi$



Penrose diagram for Kerr-spacetime



Weyl-tensor

$$C_{abcd} = R_{abcd} + \frac{2}{D-2}\left(g_{a[d}R_{c]b} + g_{b[c}R_{d]a}\right)$$

$$+ \frac{2}{(D-1)(D-2)} R\, g_{a[c}\, g_{d]b}$$

$$\tilde{g}_{ab} = \Omega^2(x)\, g_{ab}$$

$$\tilde{C}^a{}_{bcd} = C^a{}_{bcd}$$

Weyl Lagrangian:

$$\mathcal{L} = -\alpha_g \sqrt{-g}\, C^{abcd} C_{abcd} .$$

Einstein conformal gravity:

$$\mathcal{L} = -2\sqrt{g}\left(\phi^2 R + 6 g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)\right).$$

CHARGED BLACK-HOLES (REISSNER-NORDSTROM)

Now $R_{\mu\nu} \neq 0$

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T_{\mu\nu} = 2\left(F_{\mu\sigma} F^{\sigma}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right) \\ \nabla_\mu F^{\mu\nu} = 0 \end{cases}$$

$$S = \int -2\kappa_4 \sqrt{-g}\, R \, d^4x + \frac{1}{4}\int d^4x \sqrt{-g}\, g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$$

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^{(2)}$$

$$E_r = F_{rt} = \frac{Q_e}{r^2}$$

If we have magnetic charge, $B_r = F_{\theta\varphi} = \frac{Q_m}{r^2}$.

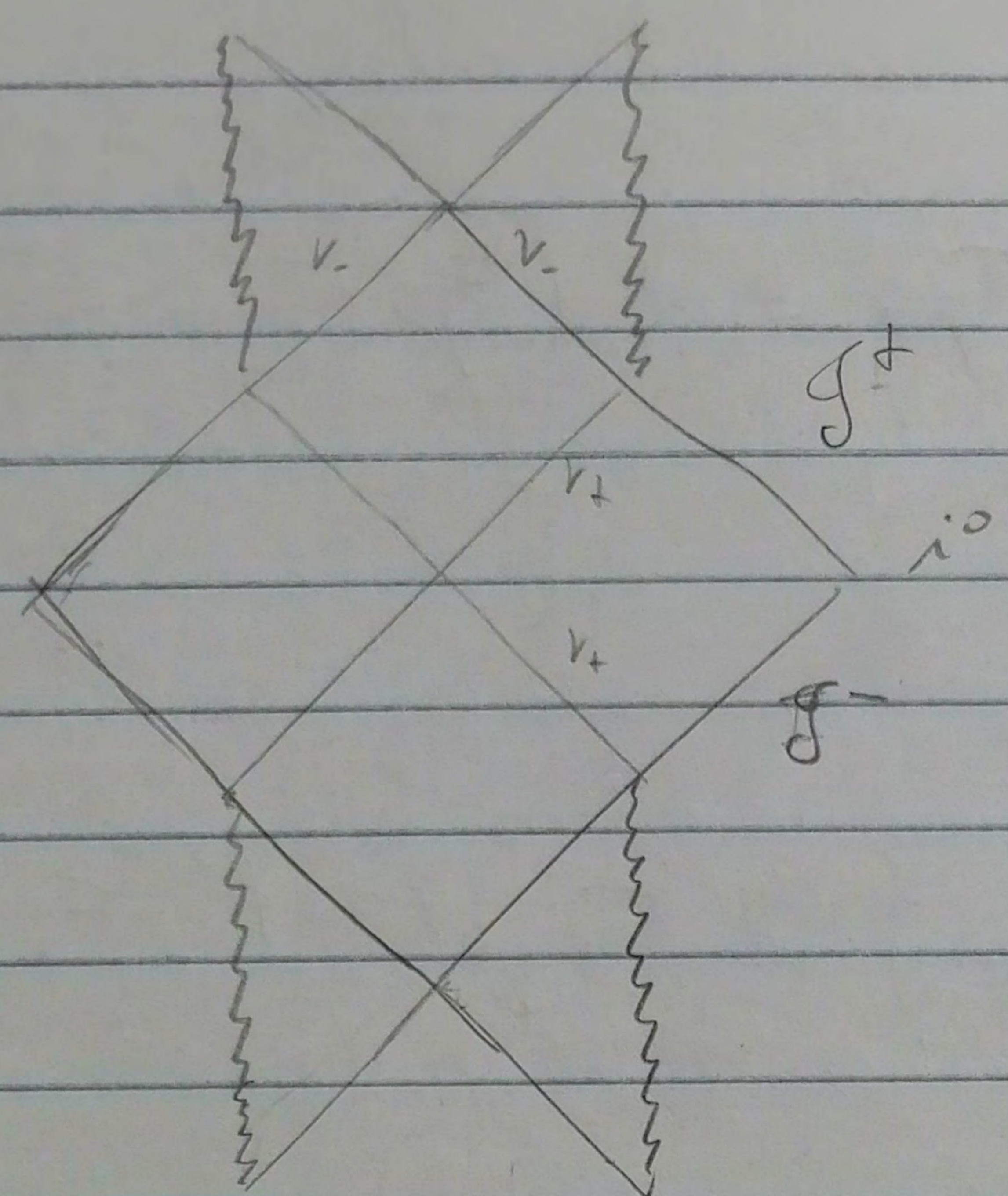
In general $Q^2 = Q_e^2 + Q_m^2$.

Horizon: $g_{rr}^{-1} = 0 \Rightarrow 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0$

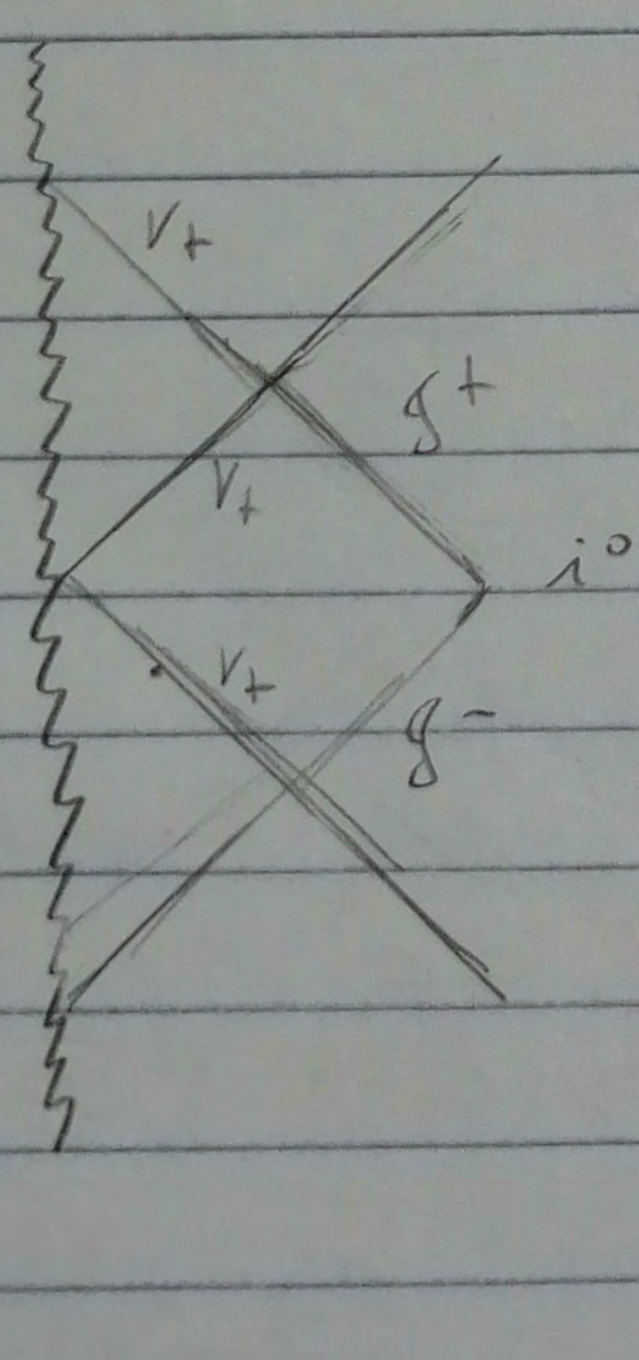
$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

- Extremal BH: $M = Q$,
- NAKED singularity: $M < |Q|$.

Penrose Diagram



Extremal BH $\quad Q = M \Rightarrow \quad r_+ = r_-$



GRAVITATIONAL WAVES

$$R_{km} = g^{il} R_{iklm}$$

$$= g^{il} \frac{1}{2} \left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l} \right) +$$

$$+ g^{il} g_{np} \left(\Gamma^n_{kl} \Gamma^p_{im} - \Gamma^n_{km} \Gamma^p_{il} \right)$$

$$g_{ik} = \underset{\substack{\shortparallel \\ g^{(0)}_{ik}}}{\eta_{ik}} + h_{ik} \qquad |h_{ik}| << 1 \quad \text{(Weak field approximation)}$$

$$\Rightarrow R_{km} = \frac{1}{2} \eta^{il} \left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l} \right) + O(h^2)$$

$$R = \eta^{km} R_{km}$$

$$\Rightarrow G_{km}(h) = T_{km} \quad (*)$$

$$\bar{h}_{ik} = h_{ik} - \frac{1}{2} h \eta_{ik} \qquad \bar{h} = h - \frac{1}{2} \cdot 4 h = -h$$

$$h_{ik} = \bar{h}_{ik} + \frac{1}{2} h \eta_{ik} = \bar{h}_{ik} - \frac{1}{2} \bar{h} \eta_{ik}$$

$$R_{km} = \frac{1}{2} \left[\frac{\partial^2 h^l{}_m}{\partial x^k \partial x^l} + \frac{\partial^2 h_k{}^i}{\partial x^i \partial x^m} - \frac{\partial^2 h}{\partial x^k \partial x^m} - \Box h_{km} \right]$$

$$= \frac{1}{2} \left[\bar{h}^l{}_{m,kl} - \cancel{\frac{1}{2} \bar{h}_{,km}} + \bar{h}^i{}_{k,im} - \cancel{\frac{1}{2} \bar{h}_{,km}} + \right.$$

$$\left. + \cancel{\bar{h}_{,km}} - \Box \bar{h}_{km} + \frac{1}{2} \Box \bar{h} \eta_{km} + O(h^2) \right]$$

R_km - 1/2 \eta_km R =0

$$= \frac{1}{2}\Big[\bar{h}^{l}{}_{m,lk} + \bar{h}^{l}{}_{k,lm} - \Box \bar{h}_{km}$$

$$+ \frac{1}{2}\Box \bar{h}\, \eta_{km} + O(h^2) \Big] =$$

$$R = \frac{1}{2}\left(2\bar{h}^{lm}{}_{,lm} - \Box \bar{h} + 2\Box \bar{h}\right) = \bar{h}^{lm}{}_{,lm} + \frac{1}{2}\Box \bar{h}. \quad (*)$$

Therefore;

$G_{km} = -\frac{1}{2}\Box \bar{h}_{km} + \cancel{\frac{1}{4}\Box \bar{h}\,\eta_{km}}$
$\quad - \cancel{\frac{1}{2}\eta_{km}\frac{1}{2}\Box \bar{h}} =$
$\quad = -\frac{1}{2}\Box \bar{h}_{km}.$

imposing the gauge $\partial_k \bar{h}^{ki} = 0$

or $\partial_k h^{ki} - \frac{1}{2}\partial^i h = 0.$

Gauge invariance of eq. (*) $\quad h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu.$

$$h' = h + 2\partial_\mu \xi^\mu$$

$$\Box h'_{km} = \Box h_{km} + \underline{\underline{\partial_k \Box \xi_m}} + \underline{\underline{\partial_m \Box \xi_k}}$$

$$\partial_k h'^{ki} - \frac{1}{2}\partial^i h' = \partial_k h^{ki} + \partial_k \partial^k \xi^i + \partial_k \partial^i \xi^k$$

$$- \frac{1}{2}\partial^i h - \frac{1}{4}\partial^i(\cancel{4}\partial_k \xi^k)$$

$$= \partial_k h^{ki} - \frac{1}{2}\partial^i h + \underline{\underline{\Box \xi^i}}$$

$\Rightarrow$ We have a residual gauge if the functions ξ^i satisfy the condition:

$$\boxed{\Box \xi_i = 0} \Rightarrow \Box h'_{ik} = 0,$$
$$\partial_i h'^{ik} - \frac{1}{2}\partial^i h' = 0.$$

Gauge invariant components: $10 - 4 - 4 = 2$. We can fix to zero 8 components of $h_{\mu\nu}$.

The linear equation for $h_{\mu\nu}$ (or $\bar{h}_{\mu\nu}$) is invariant under the gauge transformation: Gauge invariance of the field equations. Given a solution we can always generate other solutions by a coordinate transformation, ~~aka~~ gauge transformation. (Weinberg pp 253 - 256) solutions:

$\Box \bar{h}_{\alpha\beta} = 0 \Rightarrow h_{\alpha\beta} = a_{\alpha\beta} e^{ik_\lambda x^\lambda}$ with $k_\alpha k^\alpha = 0$

then the graviton has mass zero and propagate at the speed of light.

$a_{\alpha\beta} \in \mathbb{C}$.

$\partial_\alpha \bar{h}^{\alpha\beta} = 0 \Rightarrow k_\alpha a^{\alpha\beta} = 0 \Rightarrow a^{\alpha\beta}$ are 6 number and not 10.

Let us consider the gauge transformation such that

$\xi_\alpha = b_\alpha e^{ik_\lambda x^\lambda}$, $(\Box \xi_\alpha = b_\alpha \underbrace{k_\lambda k^\lambda}_{=0} e^{ik\cdot x} = 0)$ $\bar{h}_{\alpha i} = 0$ ($i = 1,2,3$), $\bar{h}^\alpha{}_\alpha = 0 \Rightarrow \bar{h}^0{}_0 + \bar{h}^i{}_i = 0$

then we can select b_α such that:

$$\bar{h}_{0\alpha} = 0, \quad \bar{h}^\alpha{}_\alpha = 0$$

$(\alpha, \beta = 0, 1, 2, 3)$

They seem: $6 + 1$ conditions, but: $\partial_\mu \bar{h}^{\mu\nu} = 0$

$\partial_0 \bar{h}^{0\nu} + \partial_i \bar{h}^{i\nu} = 0$, with $\partial_0 \bar{h}^{0\nu} = 0$

$\bar{h}^{i0} = 0 \Rightarrow \partial_i \bar{h}^{i\nu} = 0$

$\partial_i \bar{h}^{ij} = 0$ perché $\bar{h}^{i0} = 0 \Leftarrow \bar{h}_{0\alpha} = 0$

$\Rightarrow$ do $\partial_\mu \bar{h}^{\mu\nu} = 0$ we get 3 conditions instead of 4:

$\underset{\bar{h}^{0\alpha}=0}{4} + \underset{\bar{h}^\alpha{}_\alpha = 0}{1} + \underset{\partial_\mu \bar{h}^{\mu\nu} = 0}{3} = 8$

Now $\bar{h}_{\alpha\beta}$ coincides with $h_{\alpha\beta}$.

$h_{0\alpha} = 0 \Rightarrow h_{\alpha\beta}$ has only space components that satisfy the following conditions:

$\Box h_{ij} = 0$ $\qquad \frac{3 \cdot 3 - 3}{2} + 3 = 6$

$h^{ij}{}_{,j} = 0, \quad h^i{}_i = 0$ $\qquad 6 - 3 - 1 = 2$

$\Updownarrow$

$k_j h^{ij} = 0$ no polarisations in the propagation direction.

TT gauge : transversal and traceless.

If the wave propagates in the "z" direction $\Rightarrow h_{xx}, h_{xy}, h_{yx}, h_{yy}$ are non zero, but

$h_{ij} = h_{ji} \Rightarrow h_{xy} = h_{yx}$ and $h^i_{\ i} = 0$

$k_\mu = (k_0, 0, 0, k_z)$

$k_\mu h^{\mu\nu} = 0$

$k_0 h^{0\nu} + k_z h^{z\nu} = 0$

$\Rightarrow h^{0\nu} = h^{z\nu} = 0$

$h_{xx} + h_{yy} = 0.$

Finally,

$$\epsilon_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & -h_{xx} \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

$$\epsilon_\times = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us calculate the Riemann tensor in gauge TT:

$$R^i_{\ 0k0} = \frac{1}{2} h^i_{\ k,00}.$$

The tidal force gives:

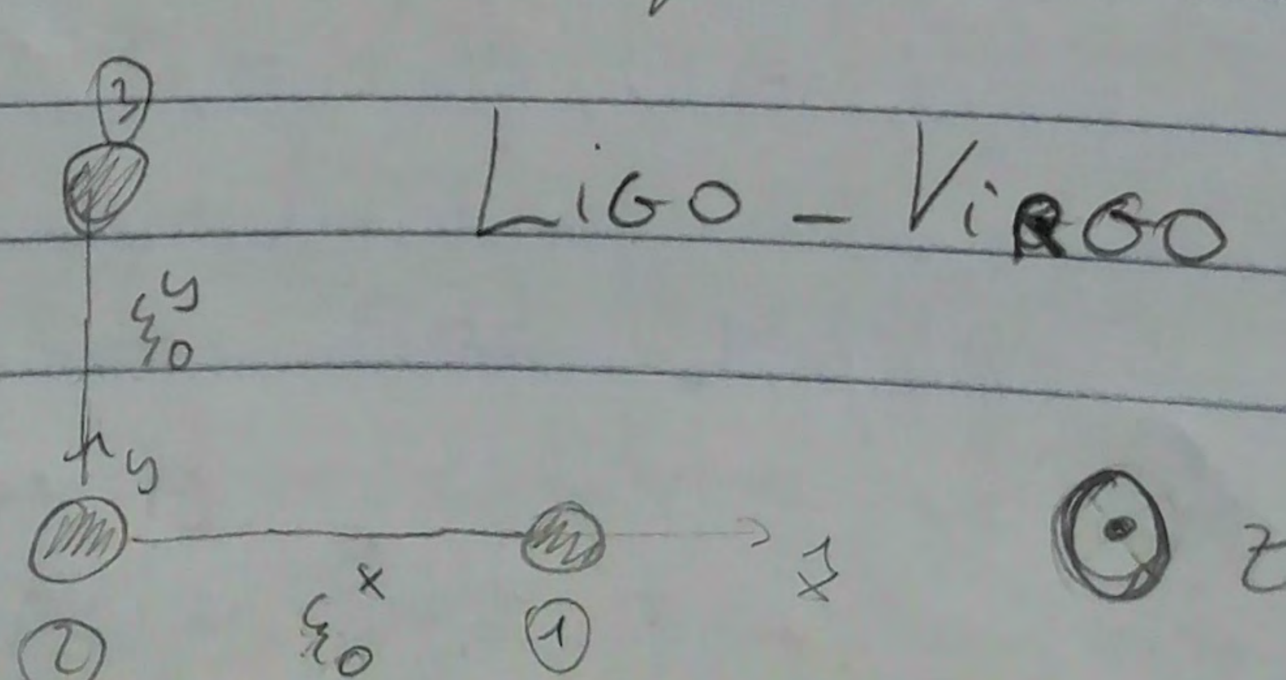
$$\frac{d^2 \xi^i}{dt^2} = -R^i_{\ 0k0}\, \xi^k = -\frac{1}{2} \ddot{h}_{ik}\, \xi^k, \quad (*)$$

for $|h_{ik}| \ll 1$ the oscillation amplitude is small respect to the average distance and we can replace $(*)$ with

$$\frac{d^2 \delta\xi^i}{dt^2} = -\frac{1}{2} \ddot{h}_{ik}\, \xi^k_0$$

$$\xi^i = \xi_0 + \delta\xi^i$$

LIGO – VIRGO



ξ_0^i : average position.

$\delta\xi^i$: deviation from ξ_0^i.

$i = x$: $\dfrac{d^2\delta\xi^x}{dt^2} = -\dfrac{1}{2}\left(\ddot{h}_{xx}\,\xi_0^x + \ddot{h}_{xy}\,\xi_0^y\right)$

$i = y$: $\dfrac{d^2\delta\xi^y}{dt^2} = -\dfrac{1}{2}\left(\ddot{h}_{yx}\,\xi_0^x + \ddot{h}_{yy}\,\xi_0^y\right)$

$h_{ik} = \epsilon_{ik}\cos(kz - \omega t)$ $\qquad k = (k_0, 0, 0, k_z)$

$i = z$: $\dfrac{d^2\delta\xi^z}{d\xi t^2} = 0$ $\qquad k_\mu k^\mu = 0 \Rightarrow k_0^2 = k_z^2$

$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = \eta_{\mu\nu}\,dx^\mu dx^\nu + h_{\mu\nu}\,dx^\mu dx^\nu \Rightarrow s = \int ds \cong (l^2 + h\,l^2)^{\frac{1}{2}}$

$s \cong \sqrt{l^2 + h\,l^2} \Rightarrow$

$\dfrac{s - l}{l} = \dfrac{\delta l}{l} = \dfrac{l\sqrt{1+h} - l}{l} \cong 1 + \frac{1}{2}h - 1 \simeq \frac{1}{2}h$

$\Rightarrow \dfrac{\Delta l}{l} \sim h$. For $h \sim 10^{-21}$:

$\Delta l = 10^{-16}\left(\dfrac{h}{10^{-21}}\right)\left(\dfrac{l}{km}\right)$ cm $\sim 10^{-16}\,10^{4}$ cm $= 10^{-12}$ cm.

$\uparrow$ $h \sim 10^{-21}$ $\quad (100\text{ Km} = 10 \cdot 10^3)$

$l \sim 10^4$

NO

With 0 source:

$$\frac{1}{2}\Box \bar{h}_{ik} = \frac{8\pi G}{c^4}\tau_{ik} \quad , \quad \bar{h}_{ik} = h_{ik} - \frac{1}{2} h\,\gamma_{ik}.$$

$$\partial_k \bar{h}^k_i = 0 \Rightarrow \partial_k \tau^k_i = 0 \quad (\text{the eq. replace } \nabla_\mu T^{\mu\nu} = 0).$$

In analogy with electromagnetism:

$$\bar{h}^k_i = -\frac{4G}{c^4}\int \frac{\tau^k_i(t-\frac{R}{c}, \vec{r}')}{R}\,dV$$

$$dV = dx'dy'dz' \quad , \quad \vec{R} = \vec{r} - \vec{r}'.$$

$$\Box \bar{h}_{ik} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial h_{ik}}{\partial R}\right) - \frac{1}{c^2}\frac{\partial^2 h_{ik}}{\partial t^2} = 0$$

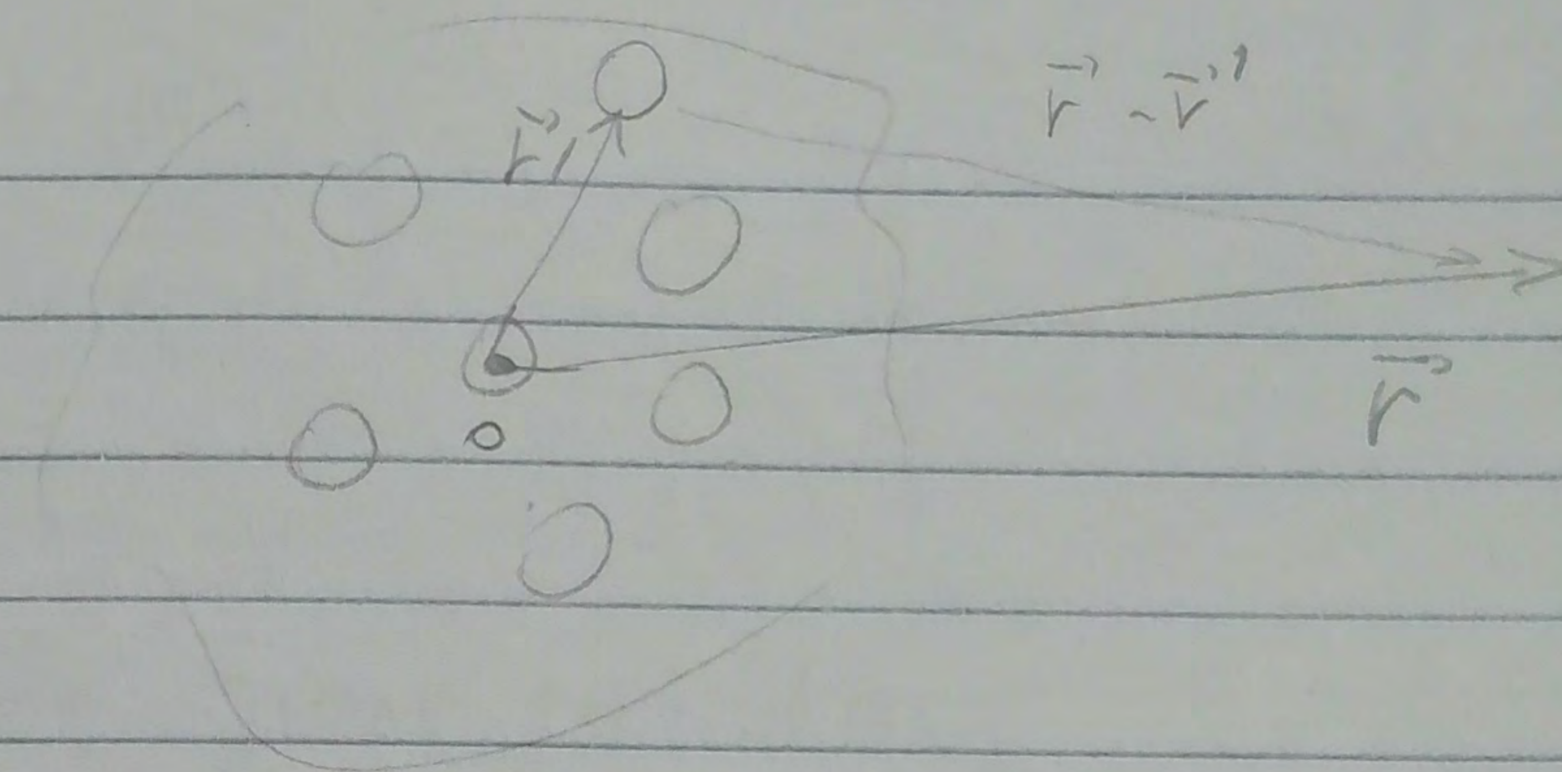
$$\text{If } \bar{h}_{ik} = \frac{\chi_{ik}(R,t)}{R} \Rightarrow \hookrightarrow \frac{\partial^2 \chi(R,t)}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2 \chi(R,t)}{\partial t^2} = 0$$

$$\Box \bar{h}_{ik} = \frac{16\pi G}{c^4}\tau_{ik}$$

$4 \to 1$ 62.5 LANDAU

$16 \to 4$

No



At large distance

$$|\vec{r}-\vec{r}'| \cong r - \frac{\vec{r}' \cdot \vec{r}}{r} + \dots$$

$$\bar{h}^{k}_{i}(\vec{r},t) \cong -\frac{4G}{c^4 r}\int \tau^{k}_{i}\left(t-\frac{r}{c},\vec{r}'\right) dV$$

$$\square \bar{h}_{ik} = 0$$

$$\square h_{ik} - \frac{1}{2}\eta_{ik}\square h = 0$$

$$h'_{ik} = h_{ik} + \partial_i \epsilon_k + \partial_k \epsilon_i$$

$$h' = h + 2\partial_i \epsilon^i$$

$$\square h'_{ik} - \frac{1}{2}\eta_{ik}\square h' = \square h_{ik} - \frac{1}{2}\eta_{ik}\square h + \partial_i \square \epsilon_k + \partial_k \square \epsilon_i - \frac{1}{2}\eta_{ik} \cdot 2\partial_i \square \epsilon^i.$$

If $\square \epsilon_i = 0 \Rightarrow \square h'_{ik} - \frac{1}{2}\eta_{ik}\square h' = \square h_{ik} - \frac{1}{2}\eta_{ik}\square h$

$G^{(1)}(h) \propto T_{\mu\nu}$

$\boxed{\Box \bar{h}_{\mu\nu}} = -16\pi G_N T_{\mu\nu}$ $\qquad (\partial_\mu \bar{h}^{\mu\nu} = 0) \quad \partial_\mu T^{\mu\nu} = 0$

To solve this eq. we first find the green function,

$$\Box_x G(x-y) = \delta^{(4)}(x-y)$$

$$\Rightarrow \bar{h}_{\mu\nu}(x) = -16\pi G_N \int G(x-y)\, T_{\mu\nu}(y)\, d^4y$$

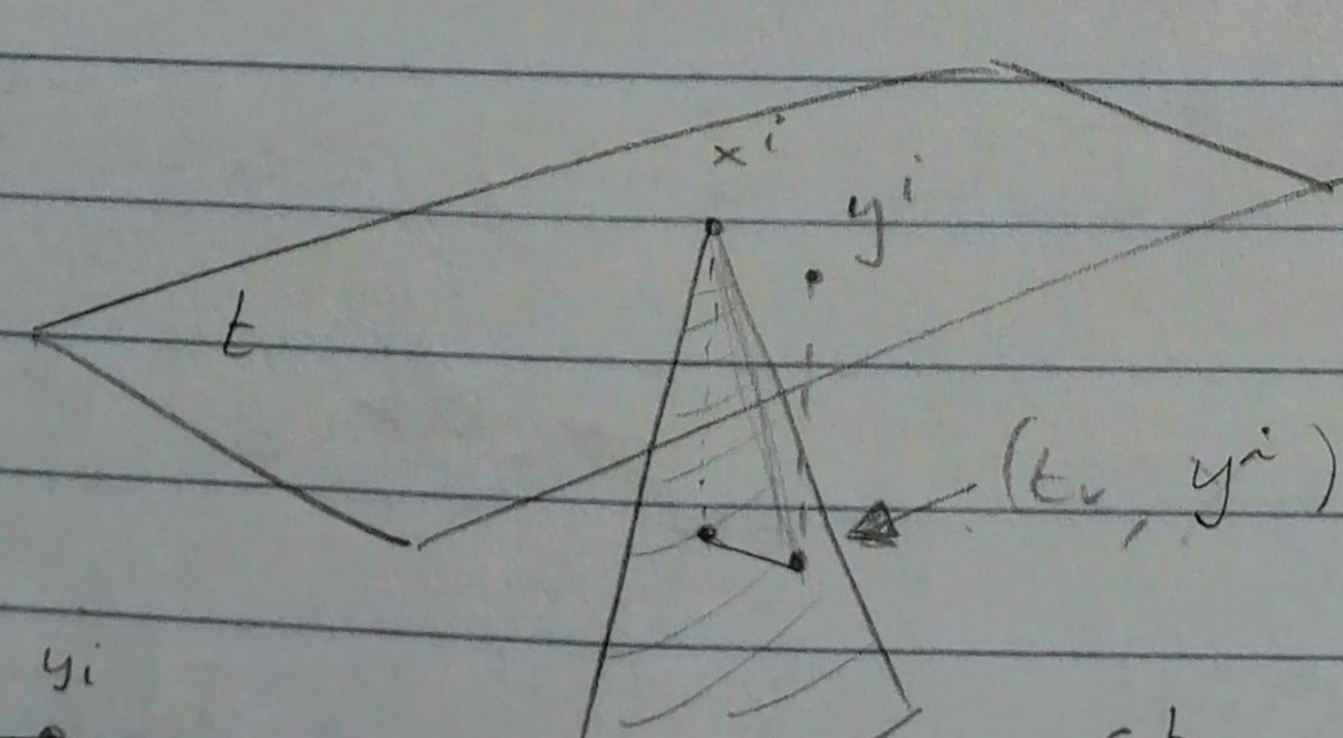
The retarded green function is:

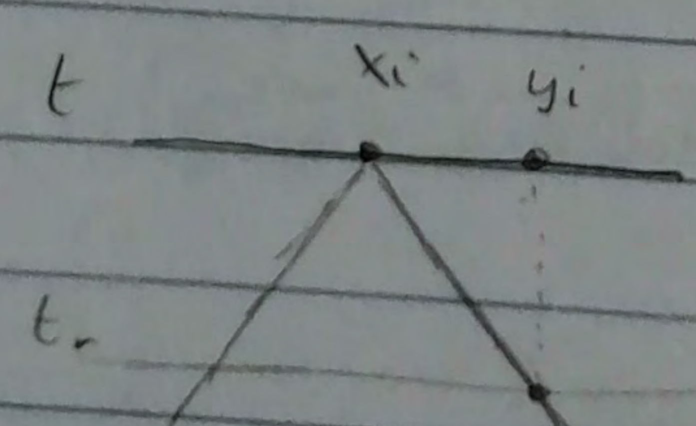
$$G(x-y) = -\frac{1}{4\pi|\vec{x}-\vec{y}|}\, \delta(|\vec{x}-\vec{y}| - (x^0 - y^0))\, \Theta(x^0 - y^0)$$

$y_0 = x^0 - |\vec{x}-\vec{y}|$ $(y_0 < x_0)$ $\qquad \Theta: \begin{cases} 1 & \text{for } x_0 > y_0 \\ 0 & \text{for } x_0 \leq y_0 \end{cases}$

Finally, $$h_{\mu\nu}(\vec{x}, \overset{x_0}{t}) = 4G \int \frac{1}{|\vec{x}-\vec{y}|}\, T_{\mu\nu}(\underbrace{\overset{x_0}{t} - |\vec{x}-\vec{y}|}_{< x_0}, \vec{y})\, d^3y$$

t_r: retarded time.





$$ct_r = ct - |\vec{x}-\vec{y}|$$

$$t_r = t - \underbrace{\frac{1}{c}|\vec{x}-\vec{y}|}_{\text{time that light needs to go from } y^i \text{ to } x^i}$$

Emitted in y^i at t_r and absorbed in x^i at time t.

Fourier transform in general:

$$\tilde{\phi}(\omega,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int dt\, e^{i\omega t}\, \phi(t,\vec{x})$$

$$\phi(t,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega t}\, \tilde{\phi}(\omega,\vec{x})$$

$$\tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int dt\, e^{i\omega t}\, \bar{h}_{\mu\nu}(t,\vec{x})$$

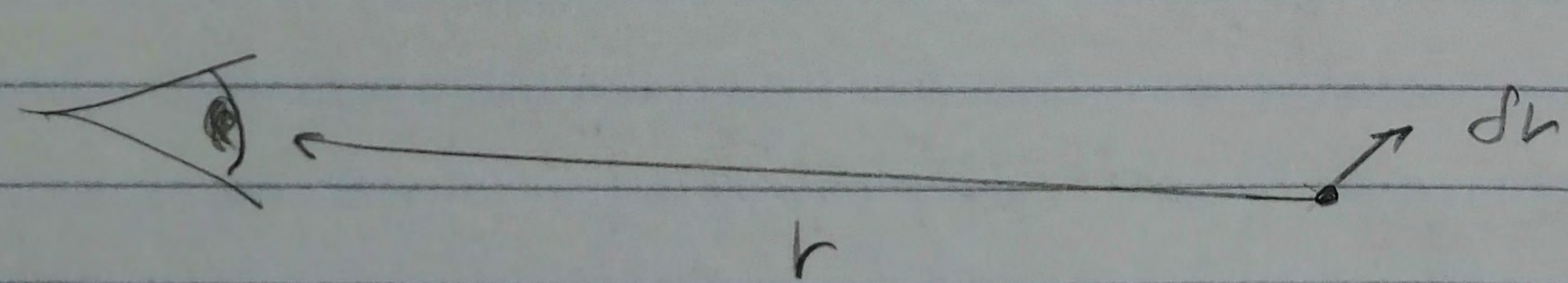
$$= \frac{4G}{\sqrt{2\pi}}\int dt\, d^3y\, e^{i\omega t}\, \frac{T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}$$

$t-|\vec{x}-\vec{y}| = t_r$

$$= \frac{4G}{\sqrt{2\pi}}\int \underbrace{dt_r\, d^3y}\ \underbrace{e^{i\omega(t_r+|\vec{x}-\vec{y}|)}}\ \frac{T_{\mu\nu}(t_r,\vec{y})}{|\vec{x}-\vec{y}|}$$

$$= 4G\int d^3y\, \frac{\tilde{T}_{\mu\nu}(\omega,\vec{y})}{|\vec{x}-\vec{y}|}\, e^{i\omega|\vec{x}-\vec{y}|} \qquad (*)$$

$|\vec{x}-\vec{y}| \to \frac{1}{r}$



$\delta r \ll r$, $\quad \delta r \ll \frac{1}{\omega} = \lambda_\omega$

Lorentz gauge

$$\partial_\mu \bar{h}^{\mu\nu}(\vec{x},t) = 0 \qquad \frac{\partial}{\partial x^\mu} \Rightarrow \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$$

$$-i\omega\, \tilde{\bar{h}}^{0\nu}(\omega,\vec{x}) + \partial_i \tilde{\bar{h}}^{i\nu}(\omega,\vec{x}) = 0$$

$$-i\omega\, \tilde{\bar{h}}^{0\nu}(\omega,\vec{x}) = -\partial_i \tilde{\bar{h}}^{i\nu}(\omega,\vec{x})$$

$$\tilde{\bar{h}}^{0\nu}(\omega,\vec{x}) = \frac{1}{i\omega}\partial_i \tilde{\bar{h}}^{i\nu}(\omega,\vec{x})$$

$$= -\frac{i}{\omega}\partial_i \tilde{\bar{h}}^{i\nu}(\omega,\vec{x})$$

$i,j: 1,2,3$

Let us integrate ⊛ by parts (only the space components),

$$\int d^3y\, \tilde{T}^{ij}(\omega,\vec{y}) = \underbrace{\int \partial_k\left(y^i \tilde{T}^{kj}(\omega,\vec{y})\right) d^3y}_{\to 0}$$

$$- \int y^i\, \partial_k \tilde{T}^{kj}(\omega,\vec{y})\, d^3y .$$

Now $\partial_\mu T^{\mu\nu} = 0 \Rightarrow -i\omega\, \tilde{T}^{0j}(\omega,\vec{y}) + \partial_i \tilde{T}^{ij}(\omega,\vec{y}) = 0$

$$\Rightarrow -\partial_k \tilde{T}^{kj} = -i\omega\, \tilde{T}^{0j}$$

$$\Rightarrow \int d^3y\, \tilde{T}^{ij}(\omega,\vec{y}) = -\int y^i\, i\omega\, \tilde{T}^{0j}(\omega,\vec{y})\, d^3y =$$

$$= -\frac{i\omega}{2}\int \left(y^i \tilde{T}^{0j} + y^j \tilde{T}^{0i}\right) d^3y =$$

Symmetrisation.

$$\underbrace{}_{=0 \text{ on } S}$$

$$= -\frac{i\omega}{2}\int\left[\partial_l\left(y^i y^j \tilde{T}^{0l}\right) - y^i y^j\left(\partial_l \tilde{T}^{0l}\right)\right] d^3y$$

$$\left(\delta^i_l y^j \tilde{T}^{0l} + y^i \delta^j_l \tilde{T}^{0l}\right) + y^i y^j \partial_l \tilde{T}^{0l}$$

$$\partial_\mu T^{\mu\nu} = 0 \overset{\nu=0}{\Rightarrow} -i\omega \tilde{T}^{00} + \partial_l \tilde{T}^{l0} = 0$$

$$\Rightarrow \partial_l \tilde{T}^{0l} = i\omega \tilde{T}^{00}$$

$$= \frac{i\omega}{2} \cdot i\omega \int d^3y \; y^i y^j \tilde{T}^{00}(\omega, \vec{y})$$

$$= -\frac{\omega^2}{2}\int d^3y \; y^i y^j \tilde{T}^{00}(\omega, \vec{y}) = -\frac{\omega^2}{2}\tilde{I}(\omega)$$

We Def. the quadrupol momentum tensor:

$$I_{ij}(\underset{t_r}{t}) = \int y^i y^j T^{00}(\underset{t_r}{t}, \vec{y})\, d^3y.$$

The solution in Fourier transform reads:

$$\tilde{\bar{h}}_{ij}(\omega, \vec{x}) = -2G\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}(\omega)$$

This is proved 5 pages later.

$$\bar{h}_{ij}(t, \vec{x}) = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}(t_r), \quad t_r = t - r$$

Gravitational radiation from a binary system.

Alonso Finn pag 248.

Circular orbit

$$(\ddot r - r\dot\theta^2) = -G\frac{M_1+M_2}{r^2} \qquad r = 2R$$

$$\mu \ddot{\vec r}_{12} = \vec F_{12}$$

$$\mu \ddot r_{12} = -G\frac{M_1M_2}{r^2}$$

$$\frac{\cancel{M_1M_2}}{M_1+M_2}\ddot{\vec r}_{12} = -G\frac{\cancel{M_1M_2}}{r^2}$$

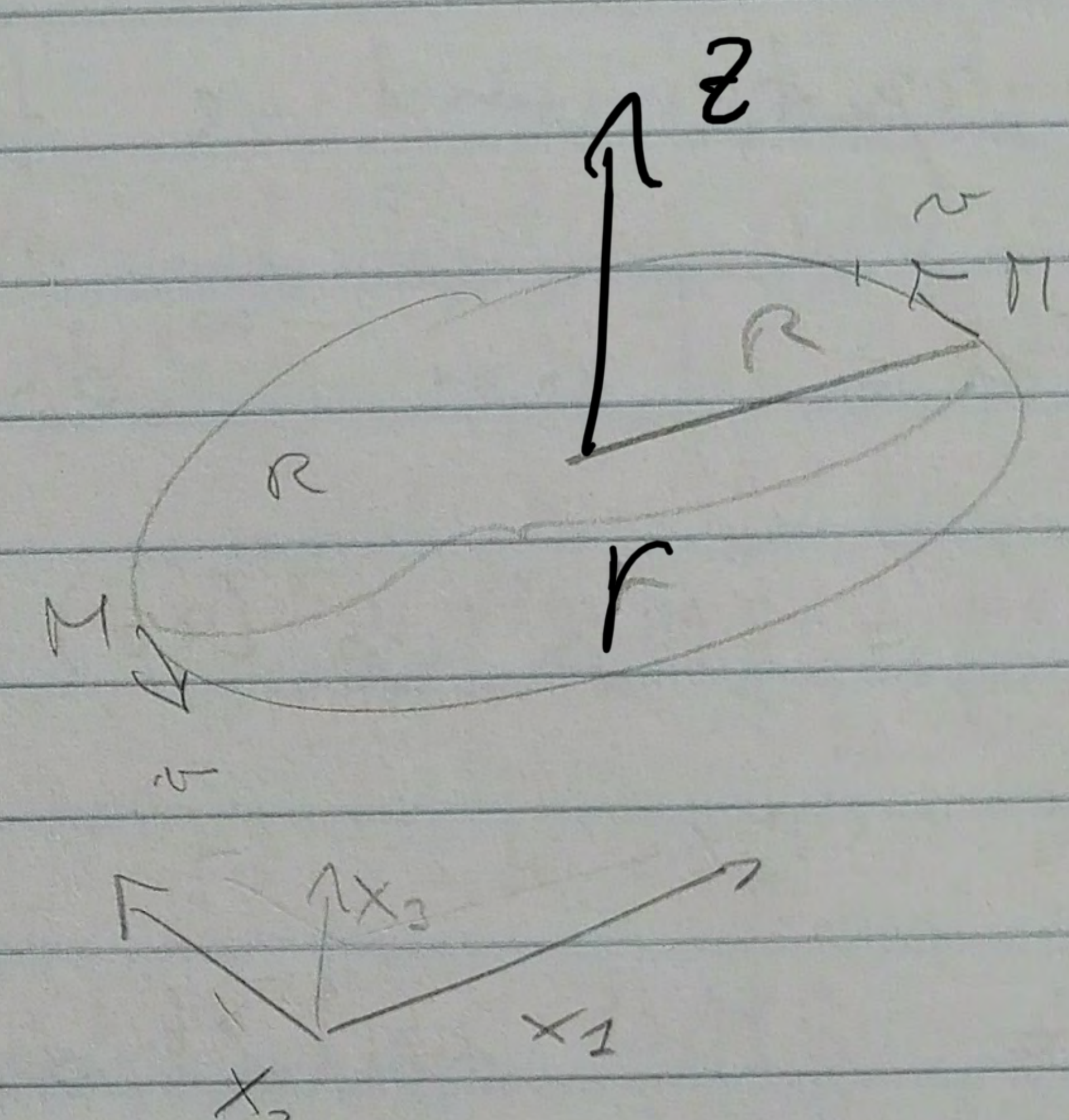
$$\ddot{\vec r}_{12} = -G\frac{(M_1+M_2)}{r^2}$$

$$\mu = \frac{M_1M_2}{M_1+M_2} \qquad M_1 = M_2 = M \Rightarrow \quad \mu = \frac{M}{2}$$

$$v = R\dot\theta \Rightarrow \dot\theta = \frac{v}{R}, \quad \dot\theta^2 = \frac{v^2}{R^2}$$

$$r\dot\theta^2 = G\frac{M_1+M_2}{r^2} \Rightarrow \cancel{2R}\frac{v^2}{R^{\cancel{2}}} = G\frac{\cancel{2}M}{(2R)^2}$$

$$\frac{v^2}{R} = \frac{GM}{(2R)^2}$$



$$\frac{v^2}{R} = \frac{GM}{(2R)^2}$$

$$v = \sqrt{\frac{GM}{4R}}$$

$$\dot\theta = \frac{v}{R}$$

$$\theta(t) = \underbrace{\frac{v}{R}}_{\Omega} t + \underbrace{\theta_0}_{=0}$$

Orbital period: $T = \frac{2\pi R}{v}$

The angular frequency is defined by: $\Omega = \frac{2\pi}{T} = \left(\frac{GM}{4R^3}\right)^{\frac{1}{2}}$

The path of the star "a" is:

(1, 2)

$$x_1^a = R\cos\Omega t, \qquad x_2^a = R\sin\Omega t$$

The frequency is:

$$f = \frac{\Omega}{2\pi} \sim \frac{c R_s^{1/2}}{10 R^{3/2}}$$

approximate frequency of the gravitational wave.

$T_{\mu\nu} = m \int d^4x \, \delta^4(x(\tau) - x) \, \dot{x}_\mu \dot{x}_\nu$ $\quad x_0 = t$

T_{00} $\quad \underbrace{dt}_{\to 1} d^3x$ $\quad \boxed{\tau = t}$ $\quad \frac{dt}{d\tau} = 1$

For the star "b":

$x_1^b = -R\cos\Omega t \quad , \quad x_2^b = -R\sin\Omega t$

The energy density is: $\quad 1 = \int \delta \, dx \to [\delta] = \frac{1}{L}$

$$T^{00}(t, \vec{x}) = M\,\delta(x_3)\Big[\delta(x_1 - R\cos\Omega t)\,\delta(x_2 - R\sin\Omega t) + \delta(x_1 + R\cos\Omega t)\,\delta(x_2 + R\sin\Omega t)\Big]$$

$\frac{M}{L^3}$

We integrate and we find $I_{ij}(t)$:

$$I_{11} = \int x_1 x_1 \, T^{00} \, dx_1 \, dx_2 \, dx_3$$

$$= 2MR^2\cos^2(\Omega t) = MR^2(1 + \cos 2\Omega t)$$

$$I_{22} = 2MR^2\sin^2\Omega t = MR^2(1 - \cos 2\Omega t)$$

$$I_{12} = I_{21} = 2MR^2\cos(\Omega t)\sin(\Omega t) = MR^2\sin(2\Omega t)$$

$$I_{i3} = 0$$

Finally,

$$\bar{h}_{ij}(\vec{x}, t) = \frac{8GM}{r}\underbrace{\Omega^2}_{= \frac{GM}{4R^3}} R^2 \begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$h \sim R_s \frac{R_s}{4R^3} \frac{R^2}{r} = \frac{R_s^2}{rR}$$

For $M_{BH} \simeq 10\,M_\odot$

$r \simeq 100\,\mathrm{Mpc}$

$r \simeq 10^{26}\,\mathrm{cm}$ $\quad$ h ~ 10^(-21) $\quad$ $R_s = 10^6\,\mathrm{cm},\ R = 10^7\,\mathrm{cm},\ r = 10^{26}\,\mathrm{cm}$

Energy emitted via gravitational radiation
(CARROL: Spacetime and geometry: an introduction to general relativity... pag 308)

Let us start with the Einstein eq. $R_{\mu\nu} = 0$ at the second order in $h_{\mu\nu}$:

$T_{\mu\nu}$

$(\Box + m^2)\phi = \lambda\phi^3$

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \epsilon^3 h^{3}_{\mu\nu} + \dots$$

solution of the linear eq. — solution of the quadratic eq.

$\phi = \phi^{(0)} + \phi^{(1)}$

$\begin{cases} (\Box + m^2)\phi^{(0)} = 0 \\ (\Box + m^2)(\phi^{(0)} + \phi^{(1)}) = \lambda(\phi^{(0)} + \phi^{(1)})^3 \end{cases}$

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$$

1º order in $h_{\mu\nu}$. $O(h)$ — 2º order in $h_{\mu\nu}$ $O(h^2)$

$\begin{cases} (\Box + m^2)\phi^{(1)} = \lambda \phi^{(0)3} \\ (\Box + m^2)\phi^{(0)} = 0 \end{cases}$

$h^{(2)}$

$h^{(1)}$

$h^{(2)} \ll h^{(1)}$

$\epsilon^0 R^{(0)}_{\mu\nu}(\eta) = 0$ in flat space.

$\epsilon^1 R^{(1)}_{\mu\nu}(h^{(1)}) = 0 \leftarrow$ first order perturbation.

The 2º order perturbation $h^{(2)}_{\mu\nu}$ will be determined by the 2º order eq.:

first order in $h^{(2)}$ — second order in $h^{(1)}$!

$$\epsilon^2\left(R^{(1)}_{\mu\nu}(h^{(2)}) + R^{(2)}_{\mu\nu}(h^{(1)})\right) = 0 \qquad \equiv R^{(2)}(h^{(2)}, h^{(1)})$$

$$R^{(2)}_{\mu\nu} = \frac{1}{2} h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} + \frac{1}{4}(\partial_\mu h_{\rho\sigma})\partial_\nu h^{\rho\sigma}) + (\partial^\sigma h^\rho{}_\nu)(\partial_\sigma h_{\rho\mu}) - h^{\rho\sigma}\partial_\rho\partial_\mu h_{\nu\sigma}$$

$\rightarrow f \sim 10^2\, s^{-1}, h \sim 10^{-21}$.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$+ \frac{1}{2} \partial_\sigma \left(h^{\rho\sigma} \partial_\rho h_{\mu\nu} \right) - \frac{1}{4} \left(\partial_\rho h_{\mu\nu} \right) \left(\partial^\rho h \right)$$

$$- \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2} \partial^\rho h \right) \partial_{(\mu} h_{\nu)\rho}$$

$G_{\mu\nu} = 0$ at the ε^2 order gives:

$$\varepsilon^2 : \underbrace{R^{(1)}_{\mu\nu}(h^{(2)}) - \frac{1}{2} \eta^{\rho\sigma} R^{(1)}_{\rho\sigma}(h^{(2)}) \eta_{\mu\nu}}_{G^{(1)}_{\mu\nu}(h^{(2)})} = 8\pi G_N t_{\mu\nu},$$

Where $t_{\mu\nu} = -\frac{1}{8\pi G_N} \left(R^{(2)}_{\mu\nu}(h^{(1)}) - \frac{1}{2} \eta^{\rho\sigma} R^{(2)}_{\rho\sigma}(h^{(1)}) \eta_{\mu\nu} \right).$

we did not included terms $h^{(1)\rho\sigma} R^{(1)}_{\rho\sigma}(h^{(1)})$ because $R^{(1)}_{\mu\nu}(h^{(1)}) = 0$

$\partial_\mu t^{\mu\nu} = 0$ in flat space. $\Leftarrow \partial_\mu G^{\mu\nu} = 0.$

$G^{(1)}_{\mu\nu}(h^{(2)}) = 8\pi G_N t_{\mu\nu} \Rightarrow -\frac{1}{2} \Box \bar{h}^{(2)}_{\mu\nu} = 8\pi t_{\mu\nu}.$

However $t^{\mu\nu}$ is not a tensor in the theory, it is not invariant under gauge transformations (infinitesimal diff.)

Weinberg pg 260 $\langle t_{\mu\nu} \rangle$ is gauge invariant.

We average over several wavelengths, $\langle \dots \rangle$. (*)

See also Chapter 7 about energy of the gravitational field

From practical standpoint, $\langle \partial_\mu (X) \rangle = 0$,

and $\langle A (\partial_\mu B) \rangle = - \langle (\partial_\mu A) B \rangle.$

(*) $\langle t_{\mu\nu} \rangle = \int_{L^4} d^4x \, t_{\mu\nu}$

$L \gg \frac{1}{|\vec{k}|} \equiv \lambda_{\vec{k}}$

In TT GAUGE :

$h^{(1)}_{\mu\nu}$

$$\boxed{\partial_\mu h^{\mu\nu}_{TT} = 0, \quad h^{TT} = 0, \quad \Box h^{TT}_{\mu\nu} = 0}.$$

$$R^{(2)\,TT}_{\mu\nu} = \frac{1}{2} h^{\rho\sigma}_{TT} \partial_\mu \partial_\nu h^{TT}_{\rho\sigma} + \frac{1}{4} (\partial_\mu h^{TT}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{TT}) +$$

$$+ \frac{1}{2} \eta^{\rho\lambda} (\partial^\sigma h^{TT}_{\rho\nu})(\partial_\sigma h^{TT}_{\lambda\mu})$$

$$- \cancel{\frac{1}{2} (\partial^\sigma h^{TT}_{\rho\nu})(\partial^\rho h^{TT}_{\sigma\mu})} - \cancel{h^{\rho\sigma}_{TT} \partial_\rho \partial_\mu h^{TT}_{\nu\sigma}}$$

$$+ \cancel{\frac{1}{2} h^{\rho\sigma}_{TT} \partial_\sigma \partial_\rho h_{\mu\nu}}.$$

integration by parts

$$\langle R^{(2)\,TT}_{\mu\nu} \rangle = -\frac{1}{4} \langle (\partial_\mu h^{TT}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{TT}) + 2 \eta^{\rho\lambda} \underbrace{(\Box h^{TT}_{\rho\nu})}_{=0} h^{TT}_{\lambda\mu} \rangle$$

$$\Rightarrow \langle R^{(2)\,TT}_{\mu\nu} \rangle = -\frac{1}{4} \langle (\partial_\mu h^{TT}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{TT}) \rangle$$

$$\langle \eta^{\mu\nu} R^{(2)\,TT}_{\mu\nu} \rangle = 0$$

$\int$ by parts and $\Box h^{TT}_{\mu\nu} = 0$.

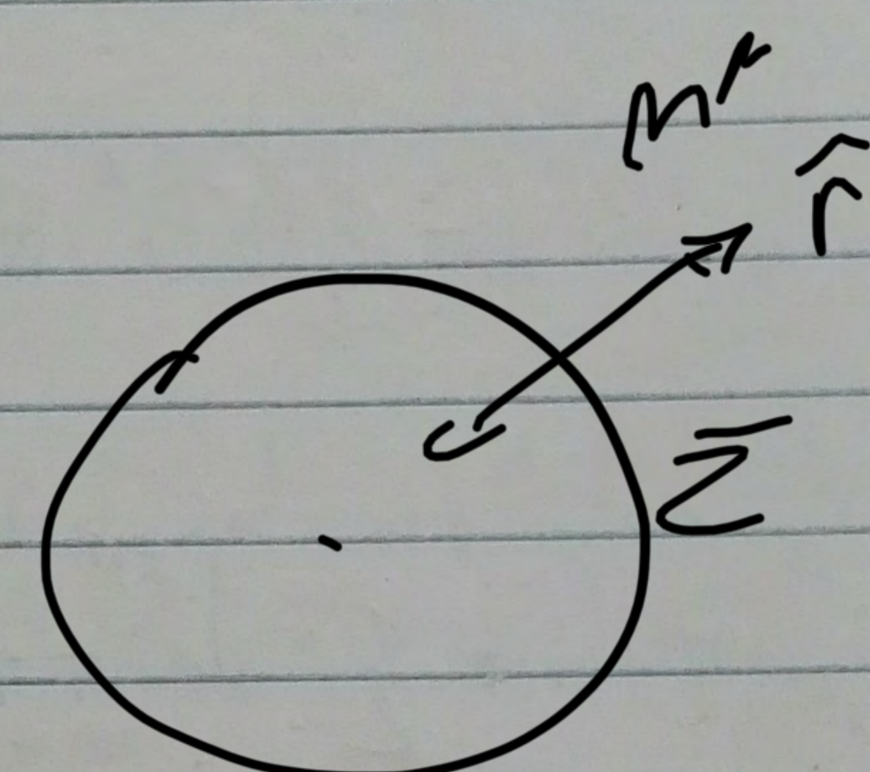
$$h_{ij}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finally,

$$t_{\mu\nu} = \frac{1}{32\pi G_N} \langle (\partial_\mu h^{TT}_{e\sigma})(\partial_\nu h^{e\sigma}_{TT}) \rangle$$

$$t_{0r} = \frac{1}{32\pi G_N} \langle (\partial_0 h^{TT}_{ij})(\partial_r h^{TT\, ij}) \rangle$$

The total energy contained in gravitational radiation on a surface Σ of constant t is:



$$E = \int_\Sigma t_{00}\, d^3x$$

The energy radiated to infinity is

$$\Delta E = \int P\, dt, \qquad \underset{\substack{\uparrow \\ \text{Power}}}{P} = \int_{S^{(2)}_\infty} t_{0\mu}\, n^\mu r^2\, d\Omega$$

In polar coordinates $n^\mu = (0, 1, 0, 0)$.

We introduce the projector

$$P_{ij} = \delta_{ij} - n_i n_j$$

n^i is in the propagating direction of the wave $\Rightarrow$

P_{ij} project on the 2-sphere at infinity.

Give a symmetric spatial tensor X_{ij} we can construct the transverse-traceless using P_{ij}, namely:

$$X_{ij}^{TT} = \left(P_i^{\ k} P_j^{\ l} - \frac{1}{2} P_{ij} P^{kl}\right) X_{kl},$$

$$\partial_i X^{ij}_{TT} = 0, \quad X^{TT\,i}{}_i = 0.$$

Now: $\bar{h}_{ij}^{TT} = h_{ij}^{TT} = \frac{2G_N}{r} \frac{d^2 I_{ij}^{TT}}{dt^2}(t-r)_{tr}$.

We introduce the "reduced quadrupol moment",

$$J_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I_{kl} \delta^{kl},$$

$$J^i{}_i = 0. \quad (J_{ij} \text{ is the traceless part of } I_{ij})$$

(J_{ij} appears has coefficient of $\frac{1}{r^5}$ in the multipole expansion of the Newton potential:

$$\phi = -\frac{GM}{r} - \frac{G}{r^3} D_i \underbrace{X^i}_{r \cdot \hat{X}^i} - \frac{3G}{2r^5} J_{ij} x^i x^j + \ldots \;)$$

Therefore, $h_{ij}^{TT} = \frac{2G_N}{r} \frac{d^2 J_{ij}^{TT}}{dt^2}(r-t)$.

$\vec{D} = \sum_a m_a \vec{r}_a$

$\left(\sum_a m_a \vec{r}_a = 0 \text{ in C.M. reference frame.}\right)$ $\quad D_i = \int d^3x\, T^{00}(x)\, x_i$

$$\sum_a q_a \vec{r}_a \neq 0$$

We have to compute $n^\mu t_{0\mu} = t_{0r}$,

$$\partial_0 h_{ij}^{TT} = \frac{2G}{r} \frac{d^3 J_{ij}^{TT}}{dt^3}(t-r)$$

$$\partial_r h_{ij}^{TT} = -\frac{2G}{r^2} \frac{d^2 J_{ij}^{TT}}{dt^2}(t-r) + \frac{2G}{r} \underbrace{\frac{\partial}{\partial r} \frac{d^2}{dt^2} J_{ij}^{TT}(t-r)}_{-\frac{d^3}{dt^3} J_{ij}^{TT}(t-r)}$$

because depends on $t-r$.

$$\underset{r\to\infty}{\simeq} -\frac{2G}{r} \frac{d^3}{dt^3} J_{ij}^{TT}(t-r).$$

$$t_{0r} = -\frac{G}{8\pi r^2} \left\langle \left(\frac{d^3}{dt^3} J_{ij}^{TT} \right) \left(\frac{d^3}{dt^3} J^{ij}_{TT} \right) \right\rangle$$

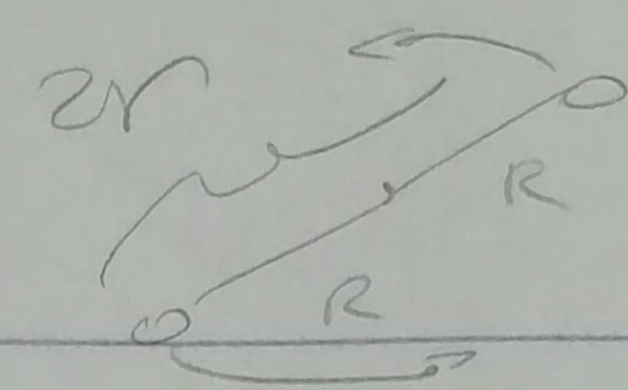
Now we come back to J_{ij} from J_{ij}^{TT}.

$$P = -\frac{G}{5} \left\langle \left(\frac{d^3}{dt^3} J_{ij} \right) \left(\frac{d^3 J^{ij}}{dt^3} \right) \right\rangle$$

For the binary system $P = -\frac{2}{5} \frac{G^4 M^5}{R^5}$

or $P = -\frac{128}{5} G M^2 R^4 \Omega^6$ for $M_1 \neq M_2$ $\frac{M^2}{G}$

$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{M}{2}$ $r = 2R$ $P_2 = -\frac{32}{5} G \mu^2 r^4 \Omega^6$



$$P = -\frac{d\mathcal{E}}{dt}$$

$$\mathcal{E} = -\frac{G M_1 M_2}{2r} \quad ?$$

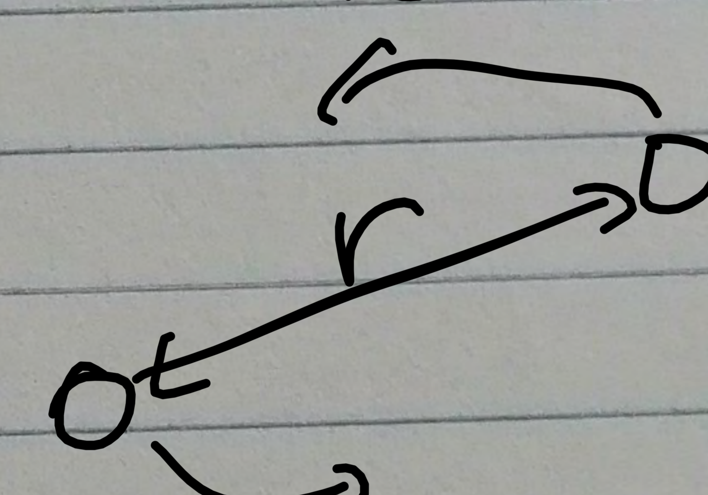
$$\left(\begin{matrix} r = 2R \\ r \equiv R \end{matrix} \right\} \Rightarrow 2r \Big)$$

$$\frac{d\mathcal{E}}{dt} = -\frac{G M_1 M_2}{2}\left(-\frac{1}{r^2}\right)\frac{dr}{dt}$$

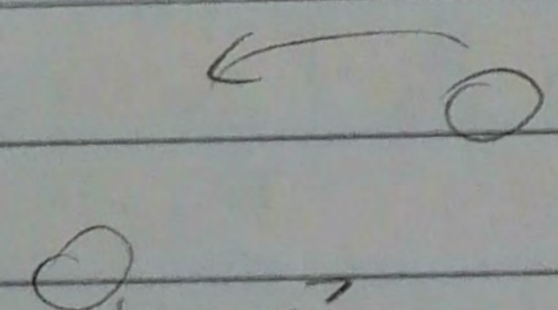
$$\Rightarrow \dot{r} = \frac{2r^2}{G M_1 M_2}\frac{d\mathcal{E}}{dt} = -\frac{2r^2}{G M_1 M_2} I$$

$$\dot{r} = -\frac{64 G^3 M_1 M_2 (M_1 + M_2)}{5 c^5 r^3}.$$

$$\frac{dr}{dt} < 0$$



Observed by Huls and Taylor.



Pulsars.

for $m_1 = m_2 \equiv M$, $\mu = \frac{M}{2}$, $r = 2R$

$$P = -\frac{32}{5} G \frac{M^2}{4} (2R)^4 \Omega^6$$

$$= -\frac{32}{5} G \frac{M^2}{\cancel{4}} \cancel{16}^{4} R^4 \Omega^6$$

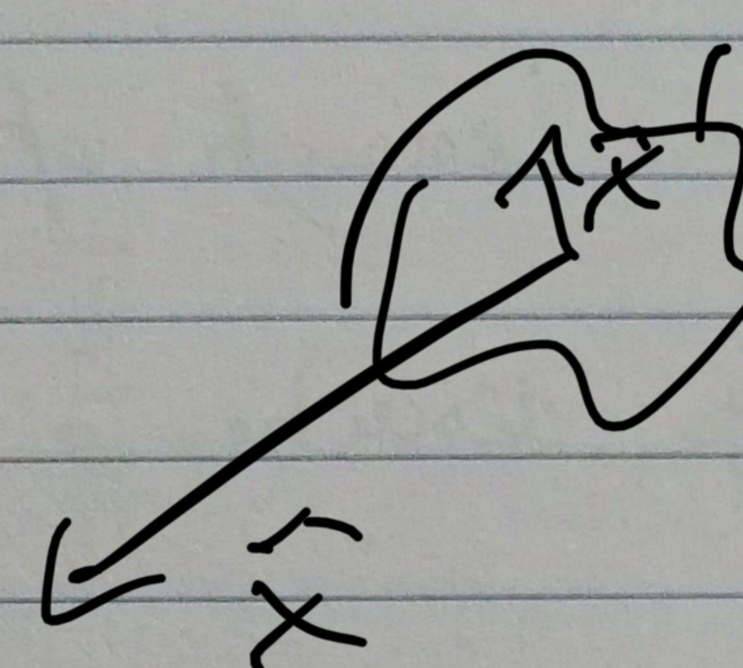
$$= -\frac{128}{5} G M^2 R^4 \Omega^6$$

$|\vec{x}-\vec{x}'| \simeq r - \vec{x}' \cdot \hat{x}$, because:

$$(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}') = (x_i - x'_i)(x_i - x'_i)$$
$$= x_i x_i - x_i x'_i - x'_i x_i - x'_i x'_i$$
$$\sim r^2 - 2 x_i x'_i$$

$$|\vec{x}-\vec{x}'| = \sqrt{(\vec{x}-\vec{x}')^2} = r\left(1 - \frac{1}{2}\frac{2\vec{x}'\cdot\vec{x}}{r^2} - \frac{1}{2}\frac{x'_i x'_i}{r} + \cdots\right)$$
$$= r - \frac{\vec{x}'\cdot\vec{x}}{r} + \cdots$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r\left(1 - 2\frac{\vec{x}'\cdot\vec{x}}{r^2} - \frac{\vec{x}'\cdot\vec{x}'}{r}\right)^{\frac{1}{2}}}$$



$$\cdots = \frac{1}{r}\left(1 + \underbrace{\frac{\vec{x}'\cdot\vec{x}}{r^2}}_{} + \cdots\right)$$

$$\simeq \frac{\vec{x}'\cdot\hat{r}\,\cancel{r}}{r^{\cancel{2}}} = \frac{\vec{x}'\cdot\hat{r}}{r} \longrightarrow \vec{D}$$

$$\frac{\boxed{\sum_a m_a \vec{x}'_a}\cdot\hat{r}}{r^2}$$

Note:

$$\tilde{I}_{ij}(\omega) = \frac{1}{\sqrt{2\pi}}\int dt_r\, e^{i\omega t_r} I_{ij}(t_r)$$

$$I_{ij}(t_r) = \frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega t_r}\tilde{I}_{ij}(\omega)$$

$$\ddot{I}_{ij}(t) = \frac{1}{\sqrt{2\pi}}\int d\omega\,(-\omega^2)\, e^{-i\omega t_r}\tilde{I}_{ij}(\omega)$$

$\ddot{I}_{ij}(t) = \frac{d^2 I_{ij}}{dt^2}$, $t_r = t - r$

On the other hand:

$$\bar{h}_{ij}(t,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega t}\tilde{\bar{h}}_{ij}(\omega,\vec{x})$$

$$\Rightarrow \bar{h}_{ij}(t,x) = \frac{2G}{r}\frac{d^2 I_{ij}}{dt^2}(t_r) \Leftrightarrow \tilde{h}_{ij}(\omega,\vec{x}) = -\frac{2G}{r}e^{i\omega r}\omega^2\tilde{I}_{ij}(\omega)$$

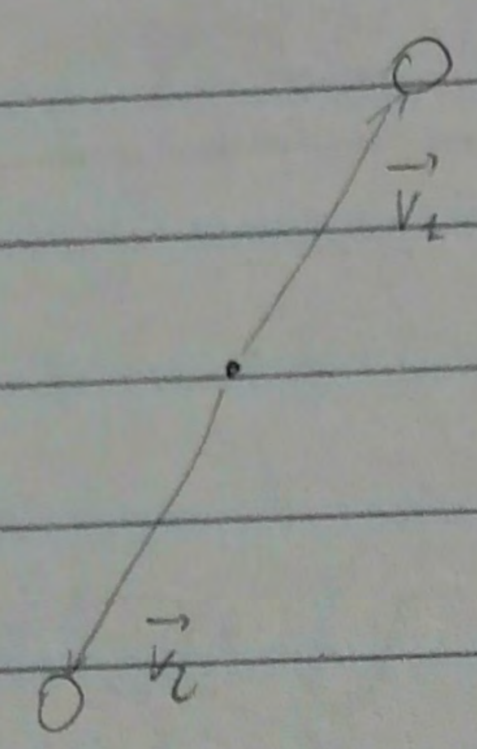
Remark

In a point we can take normal Riemann coordinates

$$g_{\mu\nu} = \eta_{\mu\nu}\,, \quad \underset{\underset{\Gamma = 0}{\Uparrow}}{\partial g} = 0 \quad \text{locally}.$$

Therefore, we can define locally the energy of the gravitational field giving $g_{\mu\nu}$ and $\partial g_{\mu\nu}$ at precisely any point.

If we average over several wave-lenght, we may hope to capture enough of the physical curvature in a small region to describe a gauge invariant measure.



$$m_1 \ddot{\vec{r}}_1 = \frac{-G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|} \hat{r}_{12}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$m_2 \ddot{\vec{r}}_2 = \frac{-G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|} \vec{r}_{21}$$

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \frac{-G(m_1 + m_2)}{|\vec{r}_1 - \vec{r}_2|} \hat{r}_{12}$$

$h_{\mu\nu} \to \Omega^2 h_{\mu\nu}$ $\qquad x \to \Omega^{-1} x$

$\int d^4x \sqrt{g}\, x^2 \left[-h_{\mu\nu} \mathcal{D}^\rho \mathcal{D}_\rho h^{\mu\nu} \right.$

$\Omega^4 \; \Omega^{-2} \left[\cancel{\Omega^2} \; \Omega^{-1} \Omega^{-1} \; \cancel{\Omega^{-2}} \right]$

$\sqrt{(\Omega^2)^4}$

Ω^2

$\int \sqrt{g}\, x^2 h_{\mu\nu} \, \mathcal{K}^{\mu\nu}{}_{\rho\sigma} \, h^{\rho\sigma}$

$\Omega^4 \; x^{-2} \; \Omega^2 \quad \Omega^2$

$\mathcal{O}_{hh\,\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} = x^{-4} \mathcal{K}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma}$

$\to \Omega^2 \, x^{-4} \, \mathcal{K}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma}$

$\mathcal{K} h \to \Omega^2 h$

$h_{\mu\nu} = SM \, h_{\mu\nu}^{sd}$

Cosmology

$(-+++)$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}$$

$R \equiv a$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G \rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3}$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^{(2)}\right],$$

$$U^\mu U_\mu = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = -1$$

$$\Downarrow$$

$$U_\mu U^\mu = -1$$

[Comoving Perfect fluid]: $U^\mu = (-1, 0, 0, 0) = \frac{dx^\mu}{d\tau}$

$$T_{\mu\nu} = (\rho + P) U_\mu U_\nu + P g_{\mu\nu}$$

$$= \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} P & \\ 0 & & & \end{pmatrix}$$

$\uparrow$

$g_{00} = -1$

$U^i = 0.$

$T_{0i} = 0 \Rightarrow$ No heat conduction,

$T_{ij} = 0 \Rightarrow$ No viscosity.
$i \neq j$

$$T^\mu{}_\mu = g^{\mu\nu} T_{\mu\nu} = -\rho + 3P$$

$g_{00} = -1$, g_{ij} s.t. $g^{ij} g_{ij} = 3$.

I compute:

$T_{\mu\nu} T^{\mu\nu} =$

$$= \left[(\rho+P) U_\mu U_\nu + P g_{\mu\nu}\right]\left[(\rho+P) U^\mu U^\nu + P g^{\mu\nu}\right]$$

$$= (\rho+P)^2(-1)^2 + 2P(\rho+P)(-1) + 4P^2$$

$$= (\rho+P)^2 - 2P(\rho+P) + 4P^2$$

For Radiation:

$$T^{\mu\nu} T_{\mu\nu} = \left(\rho + \tfrac{1}{3}\rho\right)^2 - \tfrac{2}{3}\rho\left(\rho + \tfrac{1}{3}\rho\right) + 4\left(\tfrac{1}{3}\rho\right)^2$$

$$= \left(\tfrac{4}{3}\rho\right)^2 - \tfrac{2}{3}\rho\tfrac{4}{3}\rho + \tfrac{4}{9}\rho^2$$

$$= \left(\frac{16}{9} - \frac{8}{9} + \frac{4}{9}\right)\rho^2$$

$$= \frac{12}{9}\rho^2$$

Einstein eq. o. m:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

$$\nabla^\mu G_{\mu\nu} \equiv 0 \Rightarrow \nabla^\mu T_{\mu\nu} = 0$$

$$\nabla^\mu T_{\mu\nu} = 0 \Rightarrow \nabla_\mu T^\mu{}_\nu = 0$$

for $\nu = 0$: $\nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu{}_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda{}_{\mu 0} T^\mu{}_\lambda$

$$= \partial_0 T^0{}_0 - \frac{3\dot{a}}{a}(\rho + P)$$

$$= -\partial_0 \rho - \frac{3\dot{a}}{a}(\rho + P) \quad (*)$$

We can choose an equation of state $P = P(\rho)$, for example

$P = \omega \rho$ (ω can be constant or not).

$(*)$ becomes $\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a} \overset{\omega = \text{const}}{\Rightarrow} \lg \rho = -3(1+\omega) \lg a$

If $\omega = \text{const} \Rightarrow \rho = a^{-3(1+\omega)}$

The Null dominant energy conditions $\Rightarrow |\omega| \leq 1$.

For Dust Matter

$P = 0 \Rightarrow w = 0 \Rightarrow \rho \propto a^{-3}$ or $\rho \cdot a^{3} = \text{const.}$

For RADIATION

$$T^{\mu\nu} = F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4} g^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}$$

$$T^{\mu}{}_{\mu} = F^{\mu\lambda} F_{\mu\lambda} - \frac{1}{\cancel{4}} \cdot \cancel{4}\, F^{\lambda\sigma} F_{\lambda\sigma} = 0$$

$\Rightarrow P = \frac{1}{3} \rho \Rightarrow w = \frac{1}{3} \Rightarrow \rho \propto a^{-3(1+\frac{1}{3})} \propto a^{-4}$

$\Rightarrow \rho a^{4} = \text{const.}$

Cosmological constant

$$S' = \frac{1}{16\pi G} \int \sqrt{g}(R - 2\Lambda)$$

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & P\, g_{ij} & \\ 0 & & & \end{pmatrix} = (\rho + P) U_\mu U_\nu + P g_{\mu\nu}$$

because $T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S'}{\delta g^{\mu\nu}}$

$P = -\rho_\Lambda = -\Lambda \Leftarrow$

$T_{00} = P g_{00} = -\Lambda g_{00}$, $T_{11} = P g_{11} = -\Lambda g_{11}$

$\Rightarrow w = -1$

$$\delta S' = \frac{1}{16\pi G} \int \sqrt{g}\, G_{\mu\nu} \delta g^{\mu\nu} + - \cancel{2}\Lambda \left(\frac{1}{\cancel{2}}\right) \sqrt{g}\, g_{\mu\nu} \delta g^{\mu\nu}$$

EOM $\Rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$

$\Rightarrow T^{\Lambda}_{\mu\nu} = -\Lambda g_{\mu\nu}.$

$\rho \propto a^0$

Einstein Equations without cosmological constant:

$$R_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

$(\mu,\nu) = (0,0)$: $-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p)$. (*)

$(\mu,\nu) = (i,j)$: $\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 4\pi G(\rho - p)$. (**)

There is only one distinct eq. for (i,j) because of isotropy.

(*) → (**) eliminate the second derivative and we get:

(1) $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$

and

(2) $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$

$\Big\}$ Friedmann eq.

$H = \frac{\dot{a}}{a}$ Hubble parameter, which characterise the expansion

Today: $H_0 = 100h \frac{km}{sec \times Mpc}$, $h \simeq 0.7$

Hubble distance: $d_H = H_0^{-1} c = 3.00 \times 10^3 h^{-1}$ Mpc

Hubble time: $t_H = H_0^{-1} = 9.78 \times 10^9 h^{-1}$ yer

Decelleration parameter: $q = -\frac{a\ddot{a}}{\dot{a}^2}$, which measures the rate of change of the rate of expansion.

Density parameter:

$$\Omega = \frac{8\pi G \rho}{3H^2} = \frac{\rho}{\rho_{crit}}$$

Using Friedmann eq. we can rewrite it as:

$$\Omega - 1 = \frac{K}{H^2 a^2}$$

We have: $\rho < \rho_{crit} \Leftrightarrow \Omega < 1 \Leftrightarrow K < 0 \Leftrightarrow$ OPEN

$\rho = \rho_{crit} \Leftrightarrow \Omega = 1 \Leftrightarrow K = 0 \Leftrightarrow$ flat

$\rho > \rho_{crit} \Leftrightarrow \Omega > 1 \Leftrightarrow K > 0 \Leftrightarrow$ closed

Solutions of the Einstein-Friedmann eq. for Dust matter

$P=0 \Rightarrow \rho a^3 = \text{const} \Rightarrow \rho = \frac{C}{a^3}$

① $k=0$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\frac{C}{a^3}$$

$$\frac{\dot a^2}{a^2} \propto \frac{1}{a^3}, \quad \dot a^2 \propto \frac{1}{a}, \quad a\dot a^2 = \text{const}$$

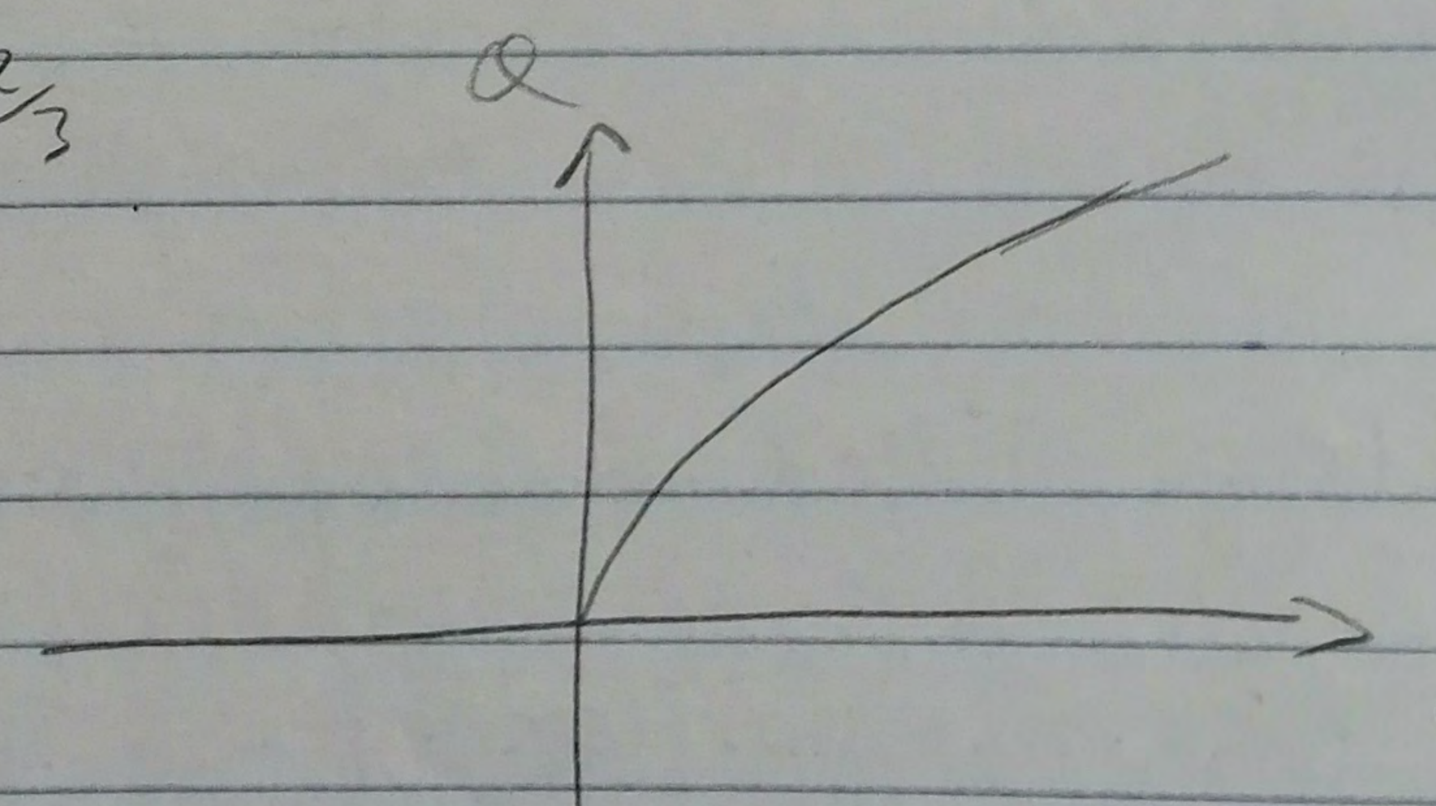
$$\sqrt{a}\,\frac{da}{dt} \propto 1 \qquad \frac{a^{\frac{1}{2}+1}}{\frac{1}{2}+1} \propto t$$

$$a^{\frac{3}{2}} \propto t$$

$$a \propto t^{\frac{2}{3}}$$

$$a = \left(\frac{3}{2}\, a_*^{\frac{1}{2}}\, t\right)^{\frac{2}{3}}$$

$\dot a \rightarrow 0$ for $t \rightarrow \infty$.



Remark: From Einstein Eq.

$$① \quad \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

I take the derivate respect to t:

$$\frac{2\dot a}{a}\left(\frac{\ddot a\, a - \dot a^2}{a^2}\right) = \frac{d}{dt}\left(\frac{8\pi G}{3}\rho\right) - k\left(\frac{-2a\dot a}{a^4}\right)$$

$$2\left(\frac{\dot{a}}{a}\right)\underbrace{\left(\frac{\ddot{a}}{a}\right)}_{(*)(*)} - \frac{2\dot{a}}{a}\overbrace{\left(\frac{\dot{a}}{a}\right)^2}^{\frac{8\pi G}{3}\rho - \frac{k}{a^2}} = \frac{8\pi G}{3}\frac{d\rho}{dt} + \frac{2k\dot{a}}{a^3}$$

$$2\left(\frac{\dot{a}}{a}\right)\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] = \underbrace{\frac{8\pi G}{3}\dot{\rho} + \frac{2k\dot{a}}{a^3}}$$

$$2\left(\frac{\dot{a}}{a}\right)\left[-2\left(\frac{\dot{a}}{a}\right)^2 - \frac{2k}{a^2} + 4\pi G(\rho - p) - \left(\frac{\dot{a}}{a}\right)^2\right] =$$

$$\frac{2\dot{a}}{a}\left[-3\underbrace{\left(\frac{\dot{a}}{a}\right)^2}_{\frac{8\pi G}{3}\rho - \frac{k}{a^2}} - \frac{2k}{a^2} + 4\pi G(\rho - p)\right] = \frac{8\pi G}{3}\dot{\rho} + \frac{2k\dot{a}}{a^3}$$

$$\frac{2\dot{a}}{a}\left(-8\pi G\rho + \frac{3k}{a^2} - \frac{2k}{a^2} + 4\pi G\rho - 4\pi G p\right) = \frac{8\pi G}{3}\dot{\rho} + \frac{2k}{a^3}\dot{a}$$

$$2\left(\frac{\dot{a}}{a}\right)\left(-4\pi G\rho + \cancel{\frac{k}{a^2}} - 4\pi G p\right) = \frac{8\pi G}{3}\dot{\rho} + \cancel{\frac{2k}{a^3}\dot{a}}$$

$$-\cancel{8\pi G}\,\frac{\dot{a}}{a}(\rho + p) = \frac{\cancel{8\pi G}}{3}\dot{\rho}$$

$$\boxed{-\frac{3\dot{a}}{a}(\rho + p) = \dot{\rho}} \iff \boxed{\frac{d}{dt}(\rho a^3) + p\frac{da^3}{dt} = 0}$$

Verify: $\dot{\rho}a^3 + 3a^2\dot{a}\rho + p\,3a^2\dot{a} = 0 \quad / a^3$

$\dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3p\frac{\dot{a}}{a} = 0$ ok

This equation is equivalent to $dE = -p\,dV$

$E = \rho a^3$, $V = a^3$
$= \rho V$
spatial volume

$dE + p\,dV = T\,dS = dQ$

Hot big-bang $\Rightarrow dS = 0 \Rightarrow S$ is max
$\Rightarrow$ paradox (no adiabatic expansion)

$\rho a^3 = \text{const} = \rho_0 a_0^3$

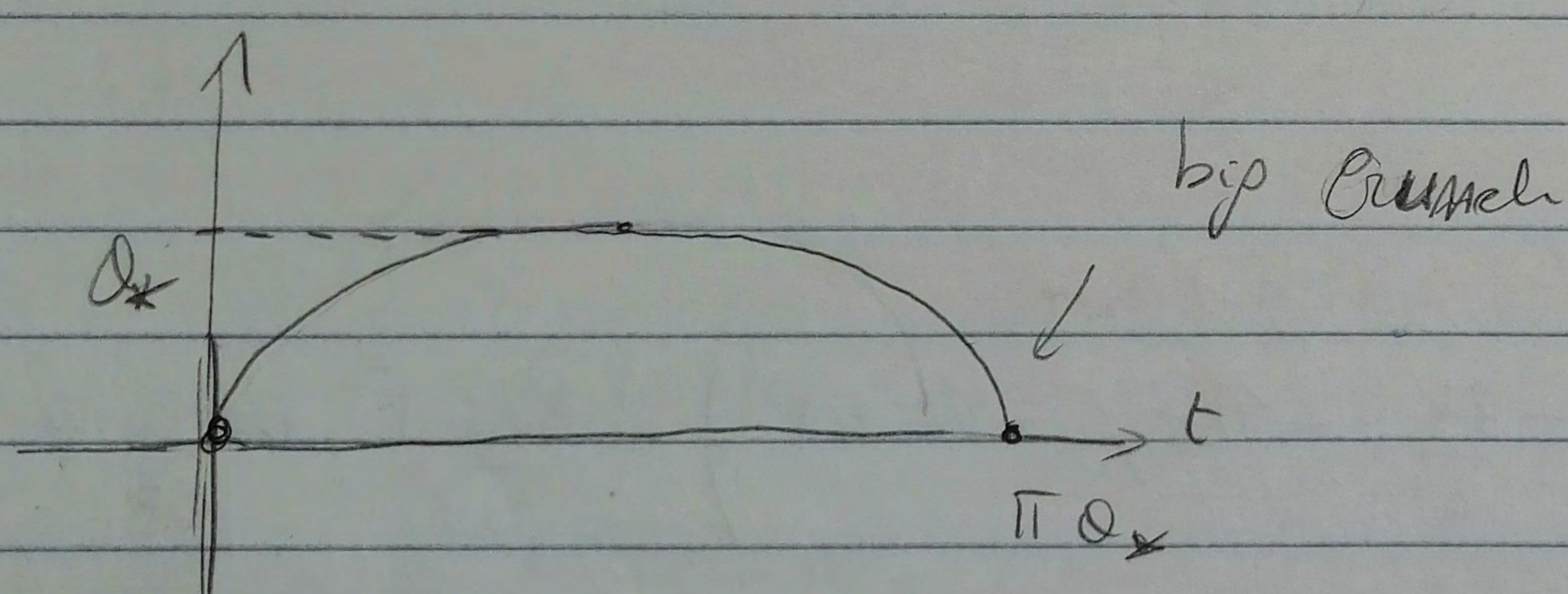
② $k = 1$

$$\left(\frac{\dot a}{a}\right)^2 + \frac{\overset{=1}{k}}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a^3}$$

$$\dot a^2 + k = \frac{8\pi G \rho_0 a_0^3}{a} \equiv \frac{a_*}{a}$$

Solution: $a = \frac{1}{2} a_* (1 - \cos\eta)$,

$$t = \frac{1}{2} a_* (\eta - \sin\eta) \quad \text{(Cycloid)}$$

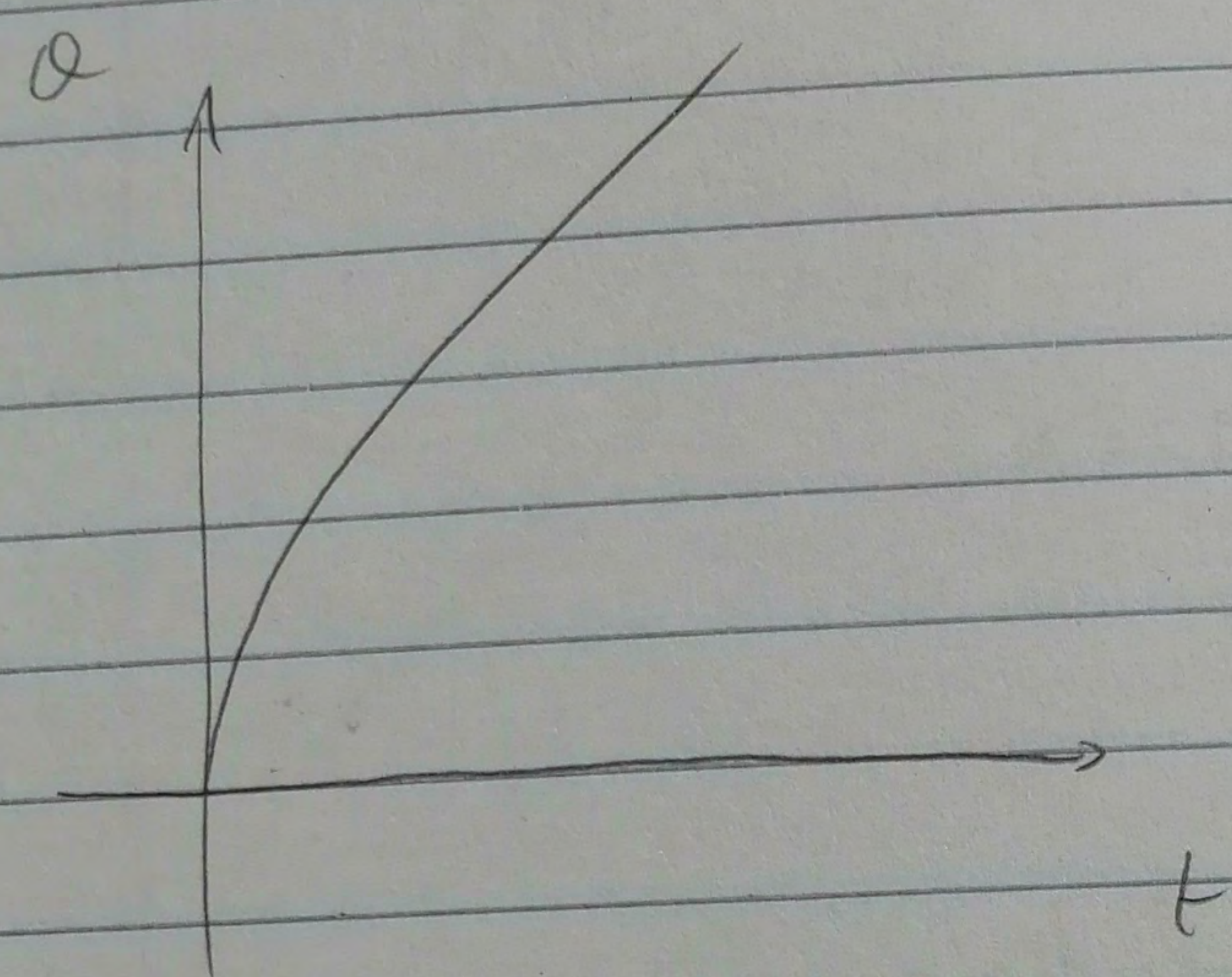


We have "2" singularities in $t = 0$ and $t = \pi a_*$

③ $k = -1$

$$a = \frac{1}{2} a_* (\cosh \eta - 1)$$

$$t = \frac{1}{2} a_* (\sinh \eta - \eta)$$



$$\dot{a} \xrightarrow[t \to \infty]{} c.$$

The Tolman's Model

$$P = \frac{1}{3}\rho \Rightarrow \frac{d}{dt}(\rho a^3) + P \frac{da^3}{dt} = 0$$

$$\frac{d}{dt}(\rho a^3) + \frac{1}{3}\rho \frac{da^3}{dt} = 0$$

$$\frac{d}{dt}(\rho a^4) = 0$$

$$\frac{d}{dt}(\rho a^3 \cdot a) = 0$$

$$\left(\frac{d}{dt}(\rho a^3)\right) \cdot a + \frac{3}{3}\rho a^3 \frac{da}{dt} = 0$$

$$\frac{1}{a} \times \left(a \left(\frac{d}{dt} \rho a^3\right) + \frac{1}{3}\rho\, 3 a^3 \frac{da}{dt} = 0 \right)$$

$$\frac{d}{dt}(\rho a^3) + \frac{1}{3}\rho \underbrace{3 a^2 \frac{da}{dt}}_{\frac{d a^3}{dt}} = 0$$

EOM:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho$$

$$\rho a^4 = \rho_0 a_0^4 = \text{const}$$

$$\frac{8\pi}{3}\rho_0 a_0^4 = a_*^2$$

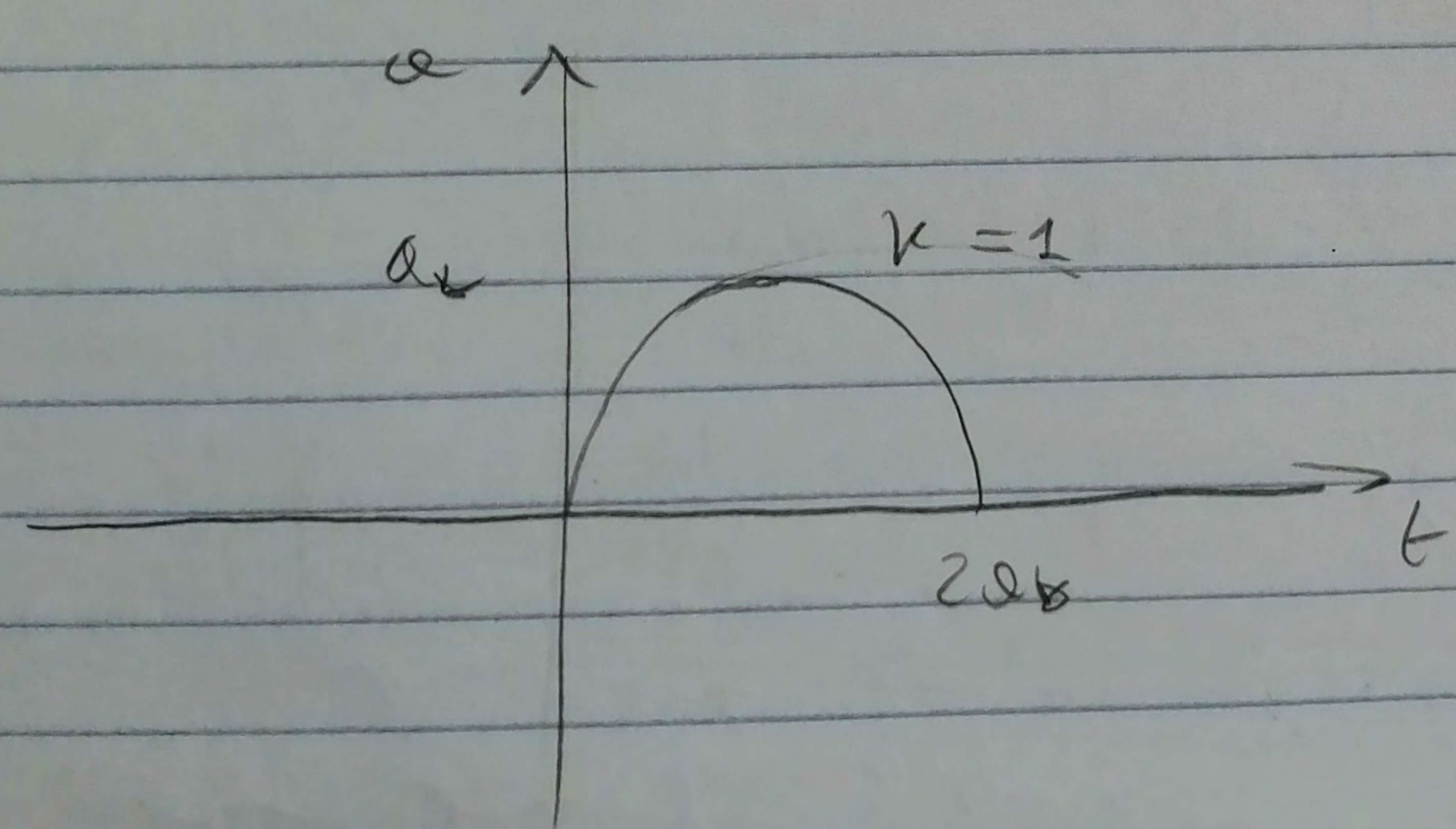
$$\Rightarrow \quad \dot{a}^2 + k = \frac{a_*^2}{a^2}$$

For $k = 1, 0, -1$, the solutions are:

a) $a = \sqrt{t(2a_* - t)}$ for $k = 1$.

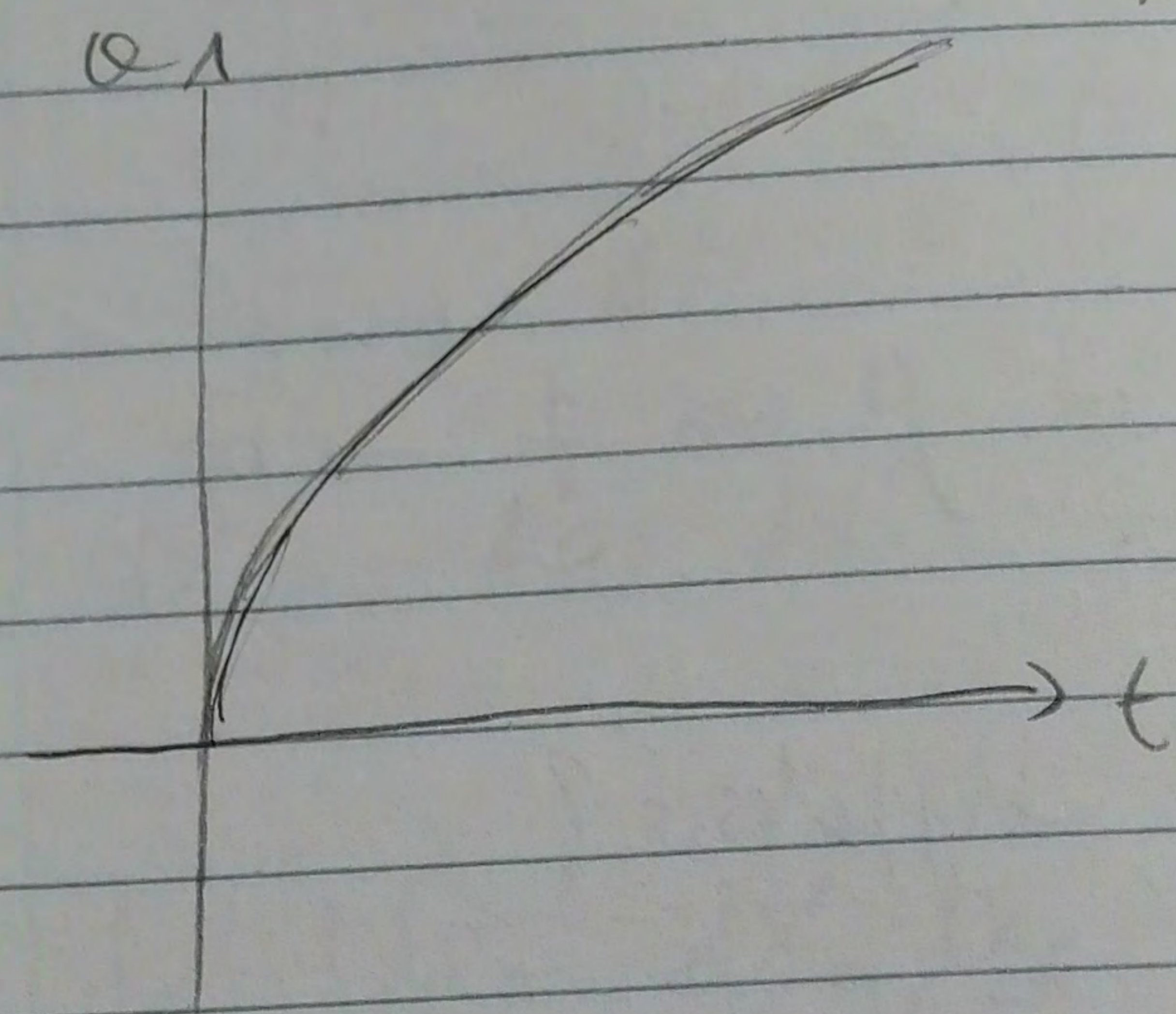
b) $a = \sqrt{2a_* t}$ for $k = 0$.

c) $a = \sqrt{t(2a_* + t)}$ for $k = -1$.

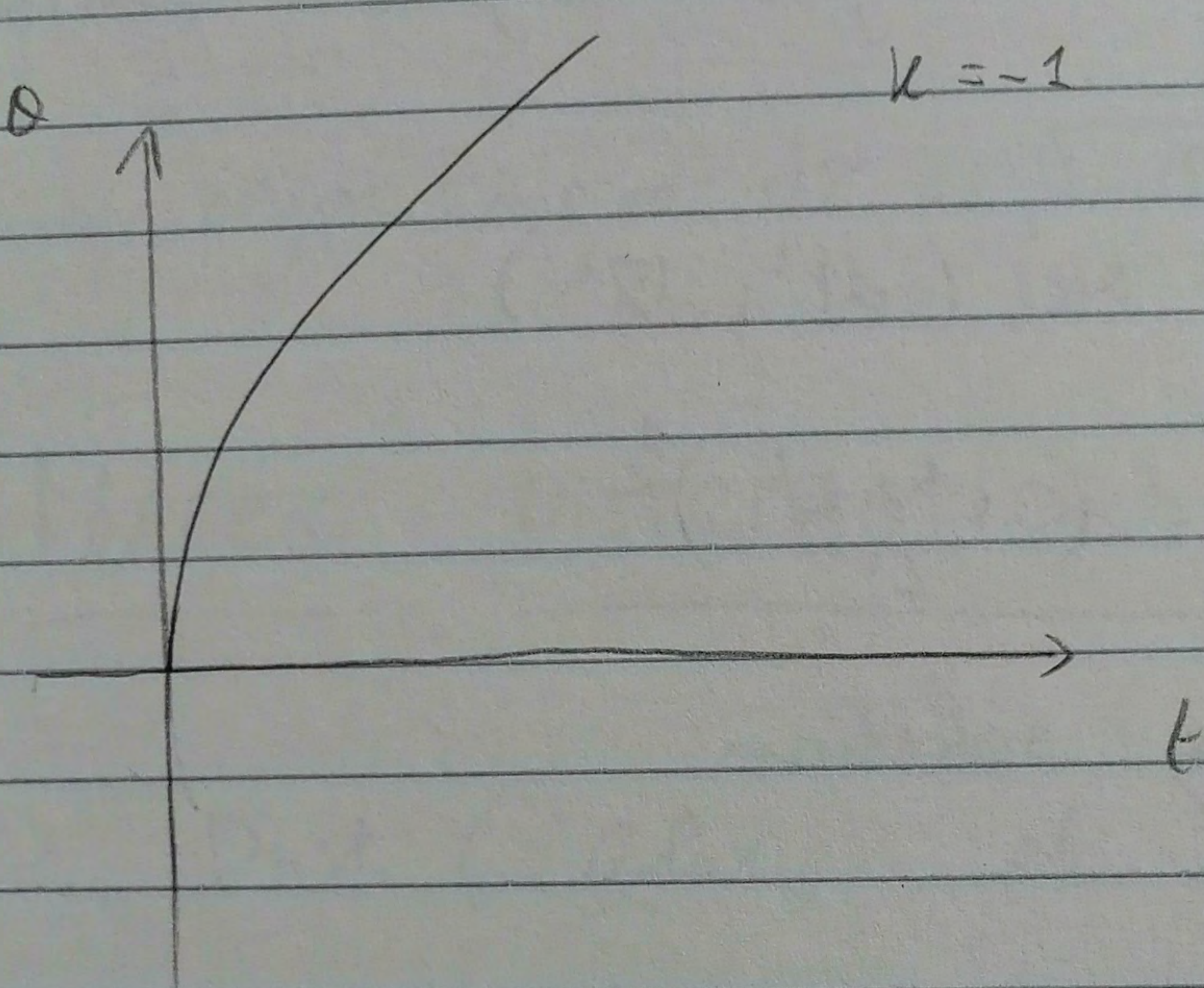


semi-cycle

$k = 0$ parabol



$k = -1$ Hyperbole



Syngularity

$\rho \sim \frac{1}{a^3}$ (Dust) or $\rho \sim \frac{1}{a^4}$ (Radiation)

for $a \to 0$ (or $a = \pi a_*$ for $k = 1$) $\rho \to \infty$.

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

$$R = 8\pi G \left(T - \frac{1}{2} 4 \cdot T \right) = -8\pi G \, 2T$$
$$= -16\pi G T$$

$T = 0$ for radiation.

$T = \rho$ for dust $\Rightarrow$ $R \propto \rho \propto \frac{1}{a^3} \xrightarrow{t \to 0} \infty$

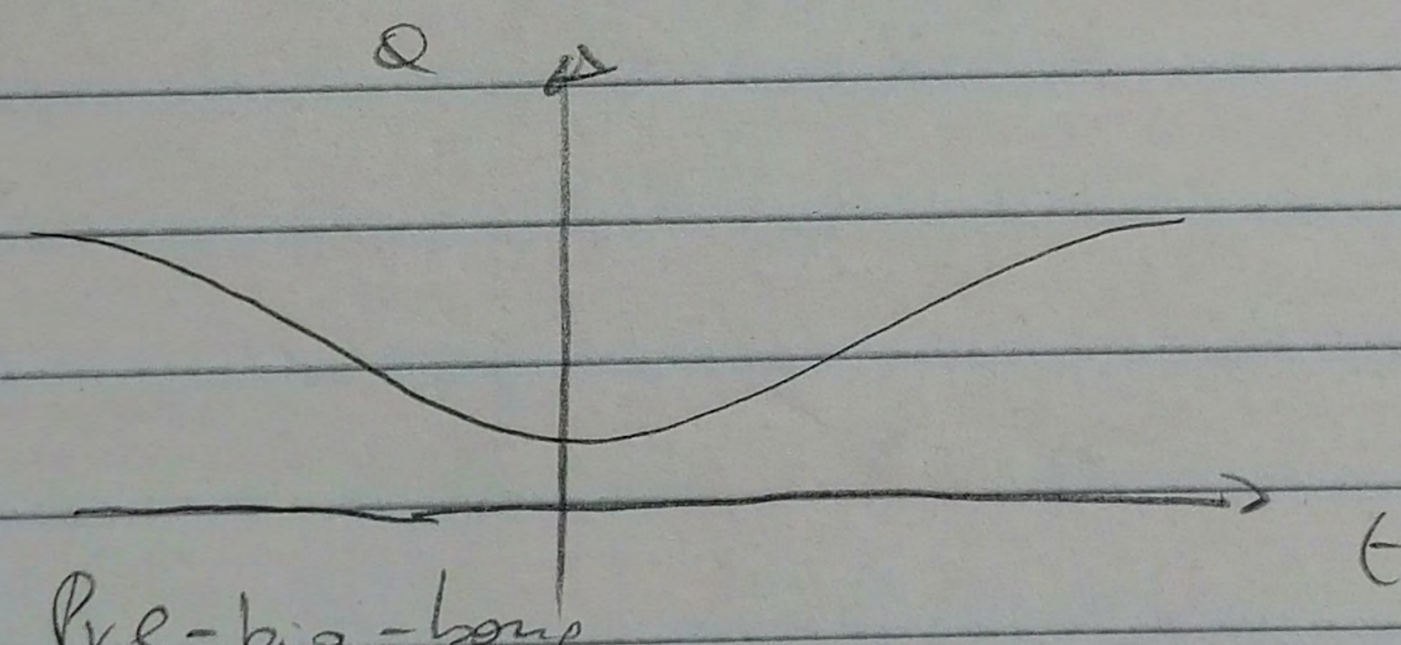
Is this due to the cosmological hypotesis?

No! This is general as proved by Hawking and Penrose.

Conformal Gravity : $ds^{*2} = a(t)^2 (-dt^2 + d\vec{x}^2)$

$$a(t) = \left(\frac{t^2 + L^2}{L^2} \right)^{\frac{1}{2}}$$

bounce solution



No singularity.

For radiation $R_{\mu\nu} R^{\mu\nu} = (8\pi G)^2 \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)\left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right)$

$$= (8\pi G)^2 \left(T_{\mu\nu}^2 - \cancel{\frac{1}{2} T^2} - \cancel{\frac{1}{2} T^2} + \cancel{\frac{1}{4} \cdot 4} T^2\right)$$

$$= (8\pi G)^2 T_{\mu\nu}^2 = (8\pi G)^2 T_{\mu\nu} T^{\mu\nu}$$

$$= (8\pi G)^2 \frac{12}{9} \rho^2 \xrightarrow[t \to 0]{} \infty .$$

Hartle : $(-1, +++)$

Remind

$$U^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{dx^i}{d\tau}\right) \qquad x^0 \equiv t$$

$$ds^2 = -dt^2 + d\vec{x}^2 = -dt^2\left(1 - \frac{d\vec{x}^2}{dt^2}\right) \Rightarrow ds^2 = -d\tau^2$$

$$U^\mu U_\mu = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{ds^2}{d\tau^2} = -1$$

proper time $\Rightarrow d\tau = \sqrt{-ds^2} \underset{\uparrow \, SR}{=} \sqrt{-(-dt^2 + dx^2)} = \sqrt{1-v^2}\, dt$

$$S = -m\int d\tau = -m\int dt\left(1 - \tfrac{1}{2}v^2 + \dots\right) = -\int m\, dt + \tfrac{1}{2}mv^2 + \dots$$

MATTER DISTRIBUTION

a) Dust (Galaxies, etc.)

b) Black body radiation

c) Massless particles (neutrinos, gravitons, etc).

$\rho_d \sim 10^{-30}\, g/cm^3 \quad (v \ll c)$

$\rho_{em} \sim \frac{8\pi^5}{15c^5}\frac{k^4}{h^3}T^4 = 4 \cdot 10^{-34}\, g/cm^3.$

CMB : $kT \sim 2 \cdot 10^{-4}\, eV$, $T \sim 2.7\, K$.

We expect $\rho_r \sim 10^{-33} \,\mathrm{gr}/\mathrm{cm}^3$ (photons, neutrinos, etc)

Total density:

$$\rho(t) = \rho_m(t) + \rho_r(t) = \rho_{m,0}\left(\frac{a_0}{a(t)}\right)^3 + \rho_{r,0}\left(\frac{a_0}{a(t)}\right)^4$$

Comparing with observations

From Einstein EOM:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \quad (*)^1$$

$$\Rightarrow \text{Today}: H_0 = \frac{8\pi G}{3}\rho_0 + \frac{\Lambda}{3} - \frac{k}{a_0^2} \quad (*)^2$$

We define the value of ρ for $\Lambda = 0$ and such that $k = 0$, namely:

$$\rho_c = \frac{3}{8\pi} H^2 \quad \Rightarrow \text{today} \quad \rho_{0c} = \frac{3}{8\pi} H_0^2$$

We define: $\Omega = \frac{\rho}{\rho_c}$,

$$\Omega_\Lambda = \frac{\Lambda}{3H^2},$$

$$\Omega_k = -\frac{k}{a^2 H^2}$$

$\Rightarrow (*)^1$ and $(*)^2$ became:

$\Omega + \Omega_\Lambda + \Omega_k = 1$ and $\Omega_0 + \Omega_{0k} + \Omega_{0\Lambda} = 1$.

Cosmological Redshift

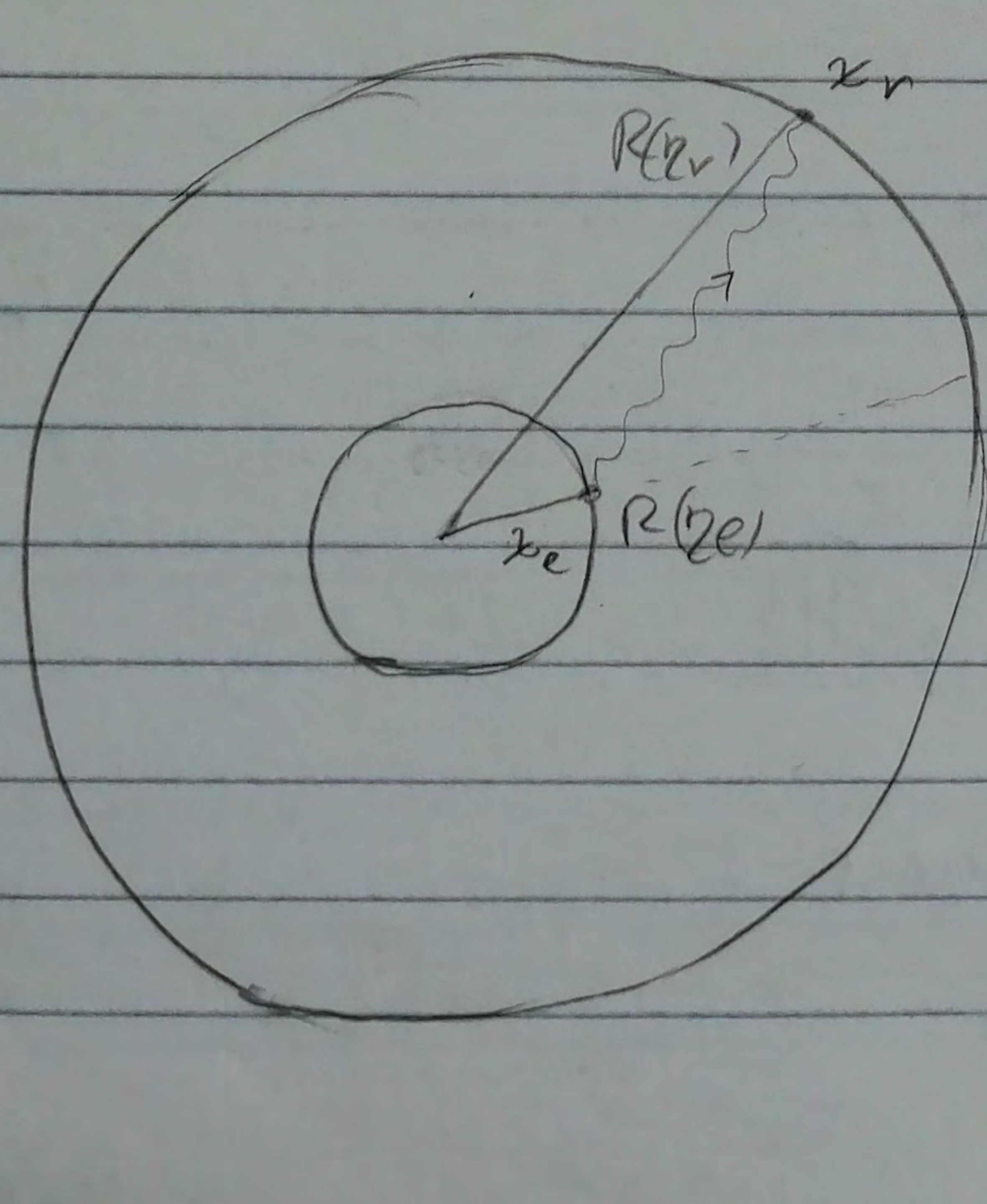
We consider the metric in coordinates $(\eta, r, \theta, \varphi)$

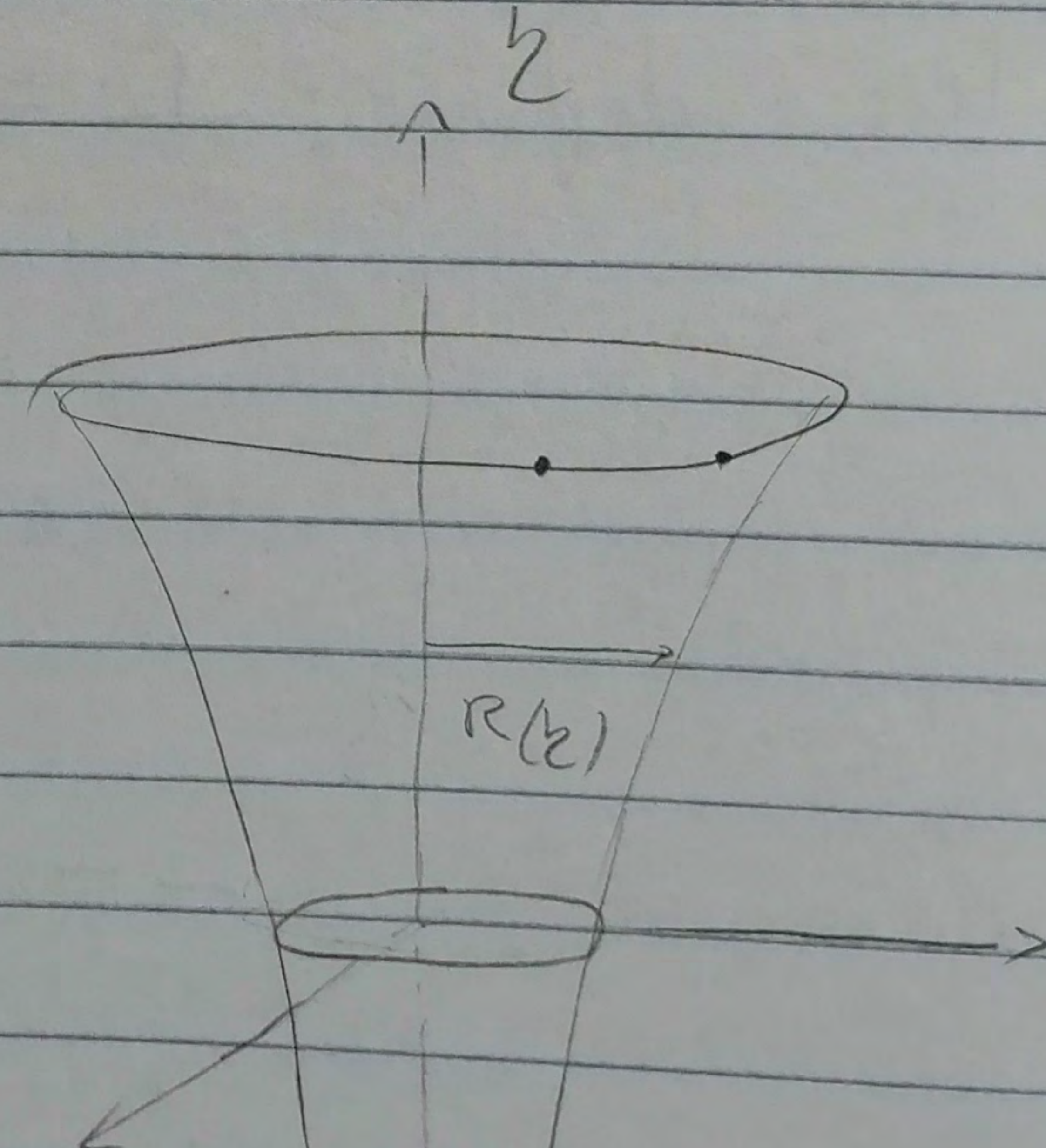
$$ds^2 = R^2(\eta)\left(d\eta^2 - d\chi^2 - \Sigma^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right)$$

$$\Sigma = \begin{cases} \chi , & k = 0 , \\ \sin\chi , & k = 1 , \\ \sinh\chi , & k = -1 . \end{cases} \qquad \begin{aligned} &\chi \in [0, \pi] \\ &\theta \in [0, \pi] \\ &\varphi \in [0, 2\pi] \end{aligned}$$

For light $ds^2 = 0 \Rightarrow d\eta = \pm d\chi$

$$\uparrow \begin{cases} \theta = \text{const.} \\ \varphi = \text{const.} \end{cases}$$





Emission : (η_e, x_e)

received : (η_r, x_r)

For $x_r > x_e$:

$$\eta_r - \eta_e = x_r - x_e$$

The sourse and receiver are at rest
$\Rightarrow d\eta_r - d\eta_e = dx_r - dx_e$, but $dx_r = dx_e = 0 \Rightarrow d\eta_r = d\eta_e$.
$\Rightarrow d\tau_r = R(\eta_r)\, d\eta_r$

$$d\tau_e = R(\eta_e)\, d\eta_e$$

$$\Rightarrow \frac{d\tau_r}{d\tau_e} = \frac{R(\eta_r)}{R(\eta_e)}.$$

time interval between two signals.

We define $z = \frac{\Delta\lambda}{\lambda}$

$$\Rightarrow 1 + z = 1 + \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{\lambda_r}{\lambda_e} = \frac{R_r}{R_e}$$

$$z = \frac{R_r - R_e}{R_e} \leftarrow$$ cosmological Redshift.

$z = \frac{R(\eta_r) - R(\eta_e)}{R(\eta_e)}$, for $x_r \sim x_e$ and so $\eta_r \sim \eta_e$, and z small :

$$z \simeq \frac{1}{R(\eta_e)} \left[\cancel{R(\eta_e)} + (\eta_r - \eta_e) \frac{dR}{d\eta}\Big|_{\eta = \eta_e} - \cancel{R(\eta_e)} \right]$$

$$\Rightarrow z \simeq \frac{1}{R}\frac{dR}{d\eta}(\eta_r - \eta_e) = \frac{1}{R}\frac{dR}{d\eta}\cdot(x_r - x_e).$$

$$= \frac{dR}{dt}(x_r - x_e)$$

The distance between source and receiver is:

$$\ell \simeq R(x_r - x_e) \quad \text{(up to second order terms)}$$

$$\Rightarrow z \simeq \left(\frac{dR}{dt}\right)\frac{1}{R}\ell = \underset{\underset{\frac{\dot{R}}{R}}{\parallel}}{H}\cdot\ell$$

HABBLE LOW: $z \simeq H\cdot\ell$

Cosmological redshit is proportional to the distance from the source.

<u>Next order</u> (necessory beouse there are sources at $z > 1$)

We define $\ell = R_r(\eta_r - \eta_e)$ ← distance at the instant of reception.

We can prove:

$$z = H_0\ell + \frac{1}{2}(1+q_0)H_0^2\ell^2$$

today

$$q_0 = -\frac{R_r\ddot{R}_r}{\dot{R}_r^2} = -\frac{\ddot{R}_r}{H_0^2 R_r}$$

today

From z we can get q_0.

Flatness and naturalness problem

$$\frac{\rho - \rho_c}{\rho_c} = \frac{3k}{8\pi G_N \rho R^2} \qquad \Omega - 1 = \frac{k}{H^2 R^2} = \frac{k}{\frac{8\pi G}{3} \rho R^2}$$

We know that $\left| \frac{\rho - \rho_c}{\rho_c} \right| = O(1)$ $(\rho \approx \rho_c)$,

but $R(t) \sim t^{2/3}$ or $R(t) \sim t^{1/2}$

$\rho \sim 1/R^3$ MATTER DOMINATED UNIVERSE

$\rho \sim 1/R^4$ RADIATION

Therefore, $\frac{\rho - \rho_c}{\rho_c} \sim t^{2/3}$ or t.

If $\frac{\rho - \rho_c}{\rho_c} = O(1)$ now for $t = 10^{28}$ cm, then

$$\frac{\left(\frac{\rho - \rho_c}{\rho_c}\right)_0}{\left(\frac{\rho - \rho_c}{\rho_c}\right)_{t_P}} = \frac{(10^{28} \text{cm})^{2/3}}{t_P^{2/3}} = \left(\frac{10^{28} \text{ cm}}{10^{-33} \text{ cm}}\right)^{2/3}$$

$$\Rightarrow \left(\frac{\rho - \rho_c}{\rho_c}\right)_{t_P} = \left(\frac{\rho - \rho_c}{\rho_c}\right)_0 \cdot 10^{-61},$$

which means $\rho \equiv \rho_c$ at the Planck scale.

This is a very fine tuning of the initial condition.

The Horizon problem

$$ds^2 = -dt^2 + R^2(t)\, d\vec{x}^2$$

For light: $ds^2 = 0 \Rightarrow X(t) = \int_0^t \frac{c\,dt'}{R(t')}$

The Proper distance is: $d_H(t) = R(t)\, X(t)$.

For $R \sim t^{2/3}$ or $R \sim t^{1/2}$

$$d_H \cong c\,t^{\frac{2}{3}} \cdot \int dt' \frac{1}{t^{2/3}} = c\,\cancel{t^{2/3}} \cdot \frac{\overbrace{\cancel{t}^{-\frac{2}{3}+1}}^{t^{\frac{1}{3}}}}{-\frac{2}{3}+1} = 3ct$$

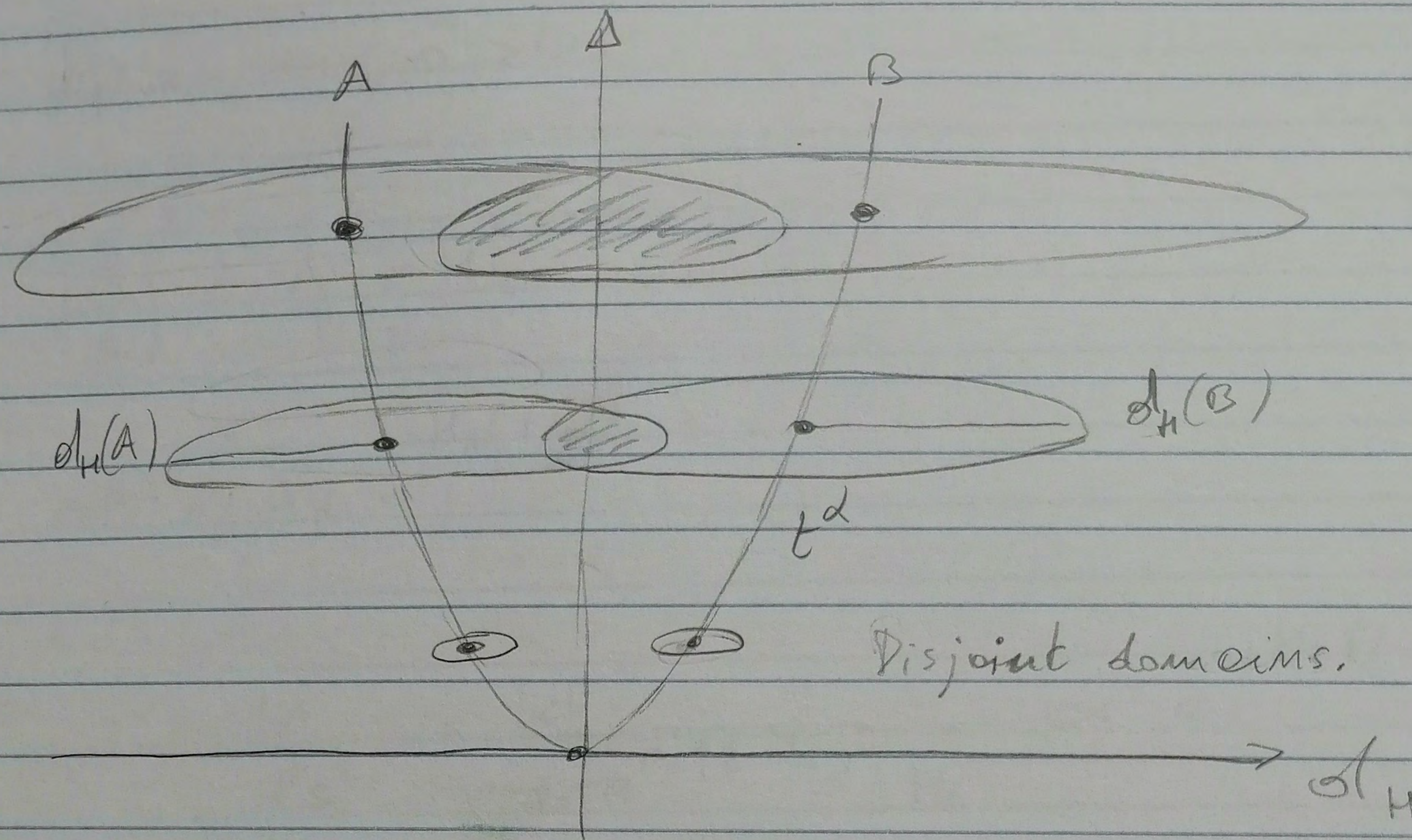
or

$$d_H \cong c\,t^{\frac{1}{2}} \cdot \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2ct$$

Now $d_H(t_0) = c H_0^{-1} \approx 10^{26}$ m.

We also know that the time when the background microwaves radiation decoupled is $t_d = 0.5 \times 10^{13}$ sec.

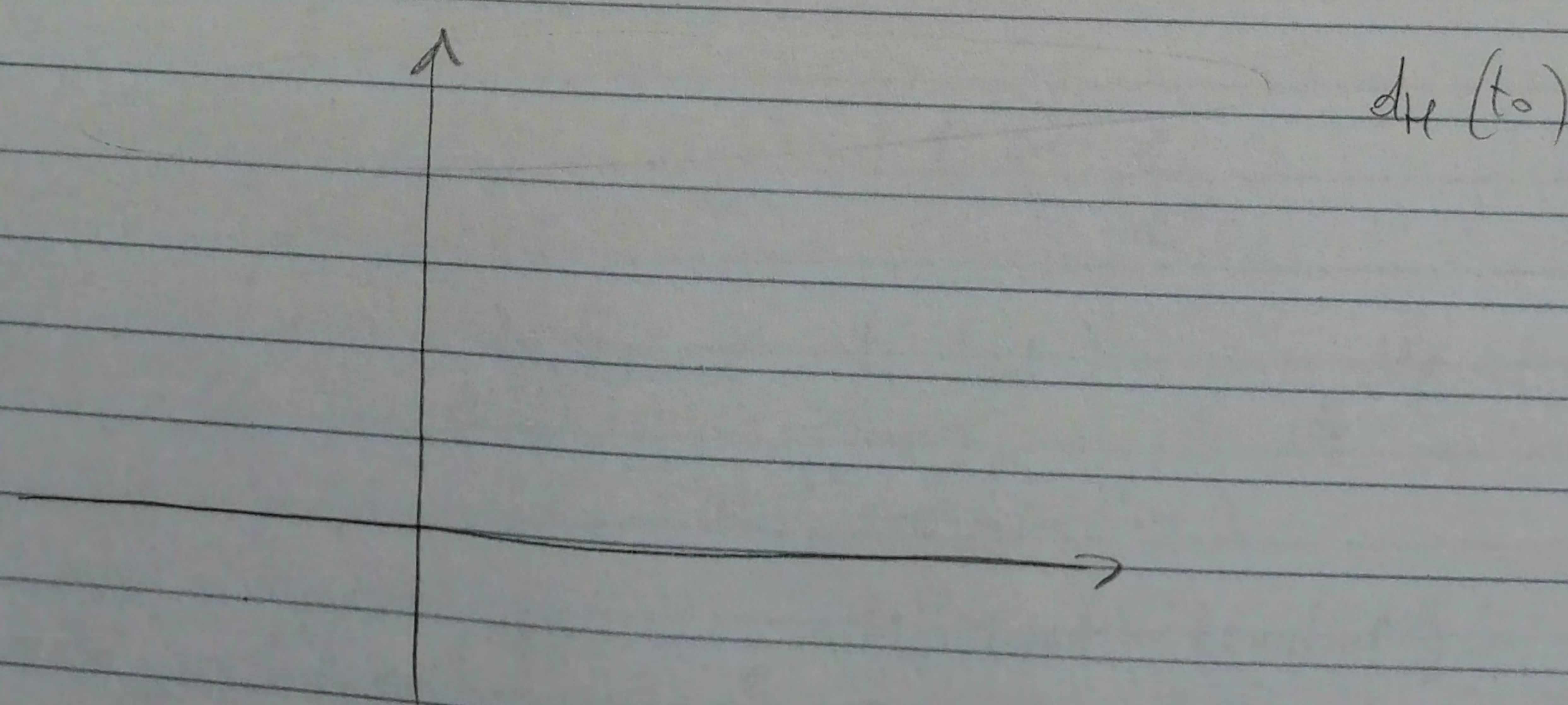
$t_0 = 3\times10^{17}$ sec.



From the theory:

$$\frac{d_H(t_0)}{d_H(t_d)} = \frac{3\cdot10^{17}}{0.5\cdot10^{13}} \sim 10^4$$ $\simeq$ 4 degrees of sky.

$$d_H(t_0) = 10^4\, d_H(t_d)$$



Inflation

Inflation $\Leftrightarrow \ddot{a} > 0$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} = H_\Lambda^2$$

$$a(t) = a_1 e^{H_\Lambda (t - t_1)}$$

① $$|\Omega_{TOT} - 1| = \frac{|k|}{H^2 a^2} = \frac{|k|}{a_1 e^{H_\Lambda (t - t_1)} H_\Lambda^2} \propto e^{-H_\Lambda (t - t_1)}$$

Present value $|\Omega_{TOT} - 1| \leq 0.1$

→ For radiation $|\Omega_{TOT} - 1| \propto t$, therefore, if

$$|\Omega_{TOT}(t_0) - 1| \leq 0.1 \Rightarrow |\Omega_{TOT}(10^{-34}\,\text{sec}) - 1| \leq 3 \cdot 10^{-53}.$$

→ During inflation: $$|\Omega_{TOT}(t) - 1| \propto \frac{1}{a^2}$$

⇒ We need: $$\frac{a_{final}}{a_{initial}} = e^{H_\Lambda \underbrace{(t_{final} - t_{initial})}_{(10^{-36}\,\text{sec} - 10^{-34}\,\text{sec})}} \cong e^{H_\Lambda \Delta t} > 10^{30}$$

so that: $$\frac{|\Omega_{TOT}(t_{final}) - 1|}{|\Omega_{TOT}(t_{initial}) - 1|} = e^{-H_\Lambda (t_{final} - t_{initial})} > 10^{-30}.$$

↑ Must be

$$e^{-H_\Lambda 10^{-34}} = 10^{-30}, \quad H_\Lambda \cdot 10^{-34} \gtrsim 30$$

$$H_\Lambda \cong 10^{36}\,\text{sec}^{-1}$$

$t_P = 10^{-44}\,sec \Rightarrow H_\Lambda = \frac{10^{36}}{sec} = \frac{10^{-8}}{t_P}$

Therefore, whatever the initial condition are the inflation will make $|\Omega_{TOT} - 1| \ll\ll 1$ at the end of the inflationary expansion.

$$e^{-(p-k)^4} \quad e^{-p^4}$$

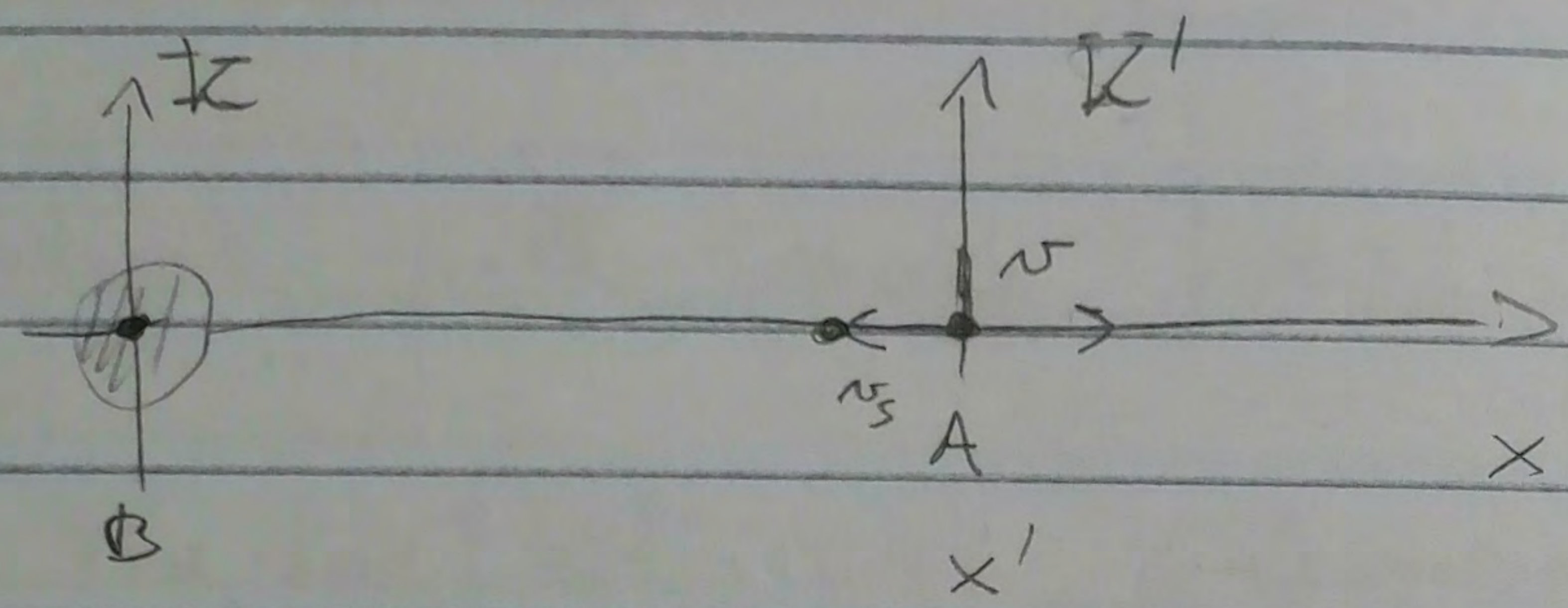
$$e^{-(p^2 - 2\underbrace{p \cdot k}_{p_0 \cdot k_0 + \vec{p} \cdot \vec{k}} + k^2)}$$

$$p_0^2 + \vec{p}^{\,2}$$

$$e^{-\left[(p_0 - k_0)^2 + (\vec{p} - \vec{k})^2\right]}$$

$$\int e^{-(x-a)^2} e^{-(y-b)^2} \, dx\, dy$$

Tachyonic communication (Two way)

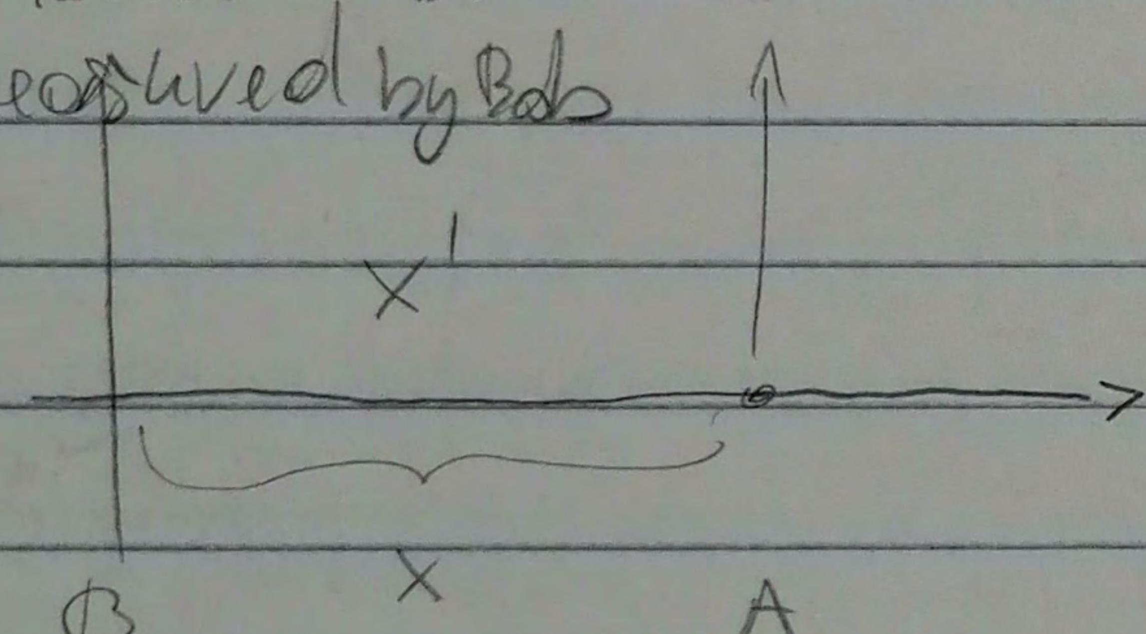


$v_s = a c$, $a > 1$; Bob got a message from Alice at the instant "t". At that instant "t" Alice position for Bob is $x = a \cdot t$.

$$(t, x) = (t, \underbrace{c \cdot a \cdot t}_{x})$$

time needed at the signal to reach Bob measured by Bob

Space travels by the signal (Tachyon) when it reaches Bob as measured by Bob.



$x' = L'$ position of Alice measured by Alice.

1) $x' = (x - vt)\gamma(v) = (cat - vt)\gamma(v) = \gamma(v)(ca - v)t$

2) $t' = \left(t - \frac{v}{c^2}x\right)\gamma(v) = \left(t - \frac{v\,c a t}{c^2}\right)\gamma(v) = \gamma(v)\left(1 - \frac{v c a}{c^2}\right)t$
$= \gamma(v)\left(1 - \frac{v a}{c}\right)t$

Alice receives the reply when she is in $x' = L'$ (distance measured by her).

Using 1 $x' = L' = \gamma(v)(c \cdot a - v)t \Rightarrow t = \dfrac{L'}{\gamma(v)(c \cdot a - v)}$

Using ② $t' = \cancel{\gamma(v)}\left(1 - \frac{va}{c}\right) \frac{L'}{(ca - v)\cancel{\gamma(v)}}$

$$= \frac{1 - \frac{va}{c}}{ca - v} L'.$$

For Alice the signal took $t'_0 = \frac{L'}{ca}$ to reach Bob, then

she got the message back after $T' = t'_0 + t' = \circledast \longrightarrow$

$$(R_{ic} - \Lambda) F (R_{ic} - \Lambda) = RFR + RF\Lambda + \Lambda FR + \Lambda^2 F$$

$$= R_{ic} F R_{ic} + 2 R_{ic} F \Lambda + \Lambda^2 F$$

$$\Lambda^2 F = S - (RFR + 2RF\Lambda)$$

$$M^4$$

$$\downarrow$$

$$\nabla_{g'}(R(g') - 2\Lambda_c) = \nabla_g\left(R(g) - 2\Lambda_{cc} + R_{ic} F R_{ic} + 2 R_{ic} F \Lambda_{cc} + \Lambda_{cc}^2 F\right)$$

? ?

$$\frac{M^4 M^6}{M^8} = M$$

$$= \nabla_g\left(R(g) - 2\Lambda_{cc} + \mathcal{O}\right)$$

ES.

$$F = \alpha \frac{Riem^3}{\Lambda^8}$$

$$\mathcal{O} = \Lambda_{cc}^2 F$$

$$\alpha \Lambda_{cc}^2 \frac{Riem^3}{\Lambda^8} + \alpha \frac{Ric\, Riem\, Ric^3}{\Lambda^8}$$

$$F = \frac{\tilde{F}}{\Lambda^2}$$

$$\frac{Riem^3}{M_P^4} + \frac{Ric\, Riem\, Ric^3}{M^8}$$

$$= R - 2\Lambda + R_{ic} \frac{\tilde{F}}{\Lambda^2} R_{ic} + 2\frac{\tilde{F}}{\Lambda} R_{ic} + \tilde{F}$$

$$\boxed{\Lambda_{cc}^2 = \Lambda^2}$$

$$R_{ic} \frac{\tilde{F}}{\Lambda} \cancel{R_{ic}} = -2 \tilde{F} R_{ic}$$

$$\alpha \Lambda_{cc}^2 \frac{E^6}{\Lambda^8} + \beta \frac{E^{10}}{\Lambda^8}$$

$$\tilde{F} R_{ic} = -2\Lambda \tilde{F}$$

$$\Lambda_{cc}^2 < E^4 \qquad E < \sqrt{\Lambda_{cc}}$$

$$T' = t_0' + t' = \frac{L'}{a \cdot c} + t' = \left(\frac{1}{ac} + \frac{1 - \frac{va}{c}}{ca - v} \right) L'$$

$$T' = 0 \quad \text{for} \quad v = \frac{2ac}{1+a^2},$$

$$T' < 0 \quad \text{if} \quad v > \frac{2ac}{1+a^2},$$

for $a = 1$, $v = c$,

for $a < 1$, $\underset{\underset{a=\frac{1}{2}}{\uparrow}}{v =} \frac{c}{1+\frac{1}{4}} = \frac{4c}{5} < c$,

for $a > 1$, $\underset{\underset{a=2}{\uparrow}}{v =} \frac{4c}{1+4} = \frac{4}{5} c$

Therefore, $v < c$ but for $a > 1$ $T' < 0 \Rightarrow$

Alice receives the signal back before to send it.

v_s is the same for the two B and A.

v_s changes in agreement with special relativity.

$$v_s' = \frac{v - v_s}{1 - \frac{v v_s}{c^2}} = \frac{v - ac}{1 - \frac{a\cancel{c} v}{c^{\cancel{2}}}} = \frac{v - ac}{1 - \frac{av}{c}}$$

$$a > 1 \Rightarrow v_s' = \frac{-\left(a - \frac{v}{c}\right)}{1 - a\frac{v}{c}}$$

$$= \frac{\frac{c}{2} - 3c}{1 - \frac{\cancel{c}}{2} 3 \cancel{c} \frac{1}{\cancel{c}^2}} = \frac{c\left(\frac{1}{2} - 3\right)}{1 - \frac{3}{2}} =$$

$v_s = 3c, a = 3$

$v = \frac{c}{2}$

$$= \frac{c\left(\frac{1}{2} - \frac{6}{2}\right)}{-\frac{1}{2}}$$

$$= -\cancel{2}c\left(-\frac{5}{\cancel{2}}\right)$$

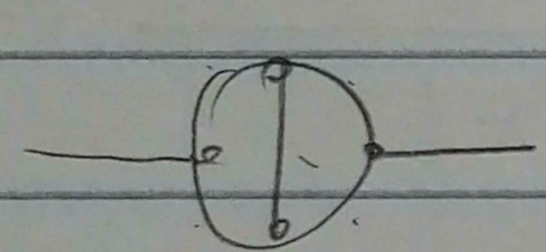
$$= 5c$$

Se $v_s = 3c \Rightarrow v_s' = 5c$

$$\Rightarrow T' = \frac{L'}{v_s'} + \left(\frac{1 - \frac{va}{c}}{ca - v}\right) L'$$

$$= \frac{L'}{\left(\frac{v - ac}{1 - \frac{av}{c}}\right)} + \text{"} = L'\left[\frac{1 - \frac{av}{c}}{v - ac} + \frac{1 - \frac{va}{c}}{ca - v}\right]$$

$$= 0$$



$$\int \frac{(d^4k)^2}{M_p^{10}} \left(\frac{1}{k^2}\right)^5 M_p^2 k^2 (\Lambda)^3 \quad \delta(\ldots)$$

$$\frac{\Lambda^3}{M_p^8} \int d^8K \ \frac{1}{k^{10}} \ k^2 \qquad \delta(\ldots) \quad \Rightarrow \left(\frac{\Lambda^3}{M_p^8}\right) \int d^4x$$

$$\frac{M^{12}}{M^8} \quad M^4$$

$$S_{EH-\Lambda}(g') = S_{EH-\Lambda}(g) + E_{EH-\Lambda} \cdot F \cdot E_{E-\Lambda} = S'(g)$$

$$= S^{(L)}_{EH-\Lambda}(g) + \mathcal{O}^{(L+1)}(g) + \mathcal{O}^{(L+2)}(g)$$

operator $\mathcal{O}(\mathcal{O}^{(L+1)})$

$$\left(\sqrt{g}\,(R(g)-\Lambda) + \overset{M^2}{\Lambda} \tfrac{1}{\epsilon} F \overset{M^2}{\Lambda} \right) \left(Ric \tfrac{1}{\epsilon} F \,\Lambda + Ric \tfrac{1}{\epsilon} F \,Ric \right)$$

$$F = \sum_{i=1}^{L} \alpha_i R^{i+1} \cancel{/ (M_p^{2(L-1)} \cdot \Lambda^2)} \,, \quad [\alpha_i] = M^0 .$$

$$S_{ct} = \left(\frac{1}{M_p^2}\right)^{L-1} \int d^4x \sqrt{-g}\, R^{L+1} \longrightarrow (R + \mathrm{Const})^{L+1}$$

$[\mathrm{Const}] = M^2$

$$M^{-2\cancel{L}+2} \quad M^{-4} \quad M^{2(\cancel{L}+1)}$$

$$\int d^4x\, \delta^4(x) = 1$$

$$M^4 \quad \frac{1}{M^4}$$

$L = 1 : R^2$

$L = 2 : R^3$

$L = 3 : R^4$

$\vdots$

$L = L : R^{L+1}$

$$S = \int \sqrt{-g}\, d^4x \left(g + \frac{\delta S}{\delta g} \delta g + \dots \right)$$

$$S + G_{\mu\nu} \delta g^{\mu\nu}$$

$$= \int \sqrt{-g}\, d^4x \cdot S(g + \delta g)$$ with Λ.

$$(G_{\mu\nu} + \Lambda g_{\mu\nu})\, \delta g^{\mu\nu}$$

$$\delta g^{\mu\nu} = \alpha g^{\mu\nu} (\dots)$$

$$\alpha G(\;) + \alpha \Lambda (\dots)$$