i-color

N

$M(r)$

$\|$

$F(r)$

$$\dot{R}^2 = \frac{\ell\, G(R)\, \mu r^3}{R}$$

$$G(R) = \frac{R^3\, G_0}{R^3 + (G_0 M)^3}$$

$(G_0 M)$ offre L^3

$$\frac{\dot{R}^2}{R^2} = \frac{\ell\, G(R)}{R^3}\, \mu r^3$$

$$\frac{\dot{R}^2}{R^2} = \frac{2\, G_0}{R^3 + \bar{X}}\ (\mu r^3)$$

For $\bar{X} = (G_0 M)\ell_p^2 = (G_0 \mu r^3)\ell_p^2$

$R = \alpha r$

$$\frac{\dot{\alpha}^2 r^2}{\alpha^2 r^2} = \frac{2 G_0\ (\mu r^3)}{\alpha^3 r^3 + G_0 \mu r^3 \ell_p^2}$$

($G_0 \mu r^3$: L)

$$\frac{\dot{\alpha}}{\alpha^2} = \frac{\ell G_0 \mu}{\alpha^3 + (G_0 \mu \ell_p^2)}$$

$1 \quad \frac{M}{M^2} \quad \frac{L^3}{L^3}$

$\frac{1}{M \cdot L}$

$$\frac{\dot{\alpha}^2}{\alpha^2} = \frac{2 G_0 \mu}{\alpha^3 + C}$$

$[\alpha] = H^0$

$[r] = L$

$[R] = L$

$\left[\frac{G_0 \mu r^3}{r^3}\right] = \frac{1}{L^2}$

$C = G_0 \mu \ell_p \qquad [C] = H$

$\boxed{\mu = \frac{4\pi}{3}\rho}$

$$C = 0 \Rightarrow \left(\frac{\dot{\alpha}}{\alpha}\right)^2 = \frac{2 G_0 \mu}{\alpha^3} \Rightarrow \frac{\dot{\alpha}}{\alpha} = -\sqrt{2 G_0 \mu}\ \frac{1}{\alpha^{3/2}}$$

$$\dot{\alpha} = -\sqrt{2 G_0 \mu}\ \frac{1}{\alpha^{1/2}}$$

$$\alpha^{\frac{1}{2}}\, d\alpha = -dt\, \sqrt{2 G_0 \mu}$$

$$\alpha^{3/2} = -t\, \sqrt{2 G_0 \mu}$$

$$a^{3/2} = -\frac{3}{2}\sqrt{2G_0\mu}\cdot t + \text{const}$$

$$a = -\left(\frac{3}{2}\sqrt{2G_0\mu}\right)^{2/3} t^{2/3} + \text{const} \qquad \text{Dust}$$

$$a(t=\underbrace{\frac{1}{\frac{3}{2}\sqrt{2G_0\mu}}}) = 0 \quad \Rightarrow \quad a^{3/2} - \cancel{\frac{3}{2}\sqrt{2G_0\mu}} = -t\cdot\frac{3}{2}\sqrt{2G_0\mu}$$

$$a^{3/2} = \left(1 - \frac{3}{2}\sqrt{2G_0\mu}\, t\right)$$

$$\Rightarrow \frac{3}{2}\sqrt{2G_0\mu}\, t = 1 - a^{3/2} \quad \Rightarrow \quad \sqrt{2G_0\mu}\cdot t = \frac{2}{3}(1-a^{3/2})$$

$$\frac{3}{\sqrt{2}}\sqrt{G_0\mu}\, t = 1 - a^{3/2}$$

Or à moi :

$$\frac{\dot{a}}{a} = -\sqrt{\frac{2G_0\mu}{a^3+c}}$$

$$C = G_0\mu\ell_p^2$$

$$\dot{a} = -\sqrt{\frac{a^2}{a^3+c}}\cdot\sqrt{2G_0\mu} \qquad G_0\mu = 1$$

$$\ell_p = 1$$

$$\sqrt{\frac{a^3+c}{a^2}}\, da = -\sqrt{2G_0\mu}\, t + \text{const}$$

Finite QG in cut-off regularization scheme

(Spoiled finite QG)

$$-2\kappa_4^{-2}\left[R + R\,\gamma_0'(\Box)\,R + R_{ic}\,\gamma_2'(\Box)\,R_{ic} + Riem\,\gamma_4'(\Box)\,Riem\right]$$

or

$$\gamma_{0,2,4} = \gamma_i\left(\frac{e^{H_i}-1}{\Box}\right)$$

GB_{NL} Non dovrebbe dare contributo alla Λ_{cc} beta function.

$\boxed{D=4}$ $\quad \gamma_0 = -\dfrac{(D-2)\left(e^{H_0}-1\right) + D\left(e^{H_2}-1\right)}{4(D-1)\,\Box} + \gamma_4'$

$$= -\frac{2\left(e^{H_0}-1\right) + 4\left(e^{H_2}-1\right)}{12\,\Box} + \gamma_4'$$

$$= -\frac{\left(e^{H_0}-1\right) + 2\left(e^{H_2}-1\right)}{6\,\Box} + \gamma_4' = \qquad H_0 = H_2 \equiv H \qquad -\frac{1}{2}\,\frac{e^{H}-1}{\Box}$$

$$\gamma_2 = \frac{e^{H_2}-1}{\Box} - 4\gamma_4'$$

$$H_2 = \frac{1}{2}\left(\gamma_E + \Gamma\left(0, P_{\gamma+1}^{(2)}\right) + \ln\left(P_{\gamma+1}^{(2)}\right)^2\right) \qquad P_{\gamma+1}^{(2)} \to \infty$$

$$\left(e^{H_2}-1\right)\frac{1}{\Box} \qquad (\Box^n + a^2 D^{n+1})\frac{1}{\Box} = \Box^{n-1} + a^2 D^{n+2}$$

$$e^{H_2} \xrightarrow{P_{\gamma+1}^{(2)} \to \infty} e^{\frac{1}{2}\gamma_E \sqrt{\left(P_{\gamma+1}^{(2)}\right)^2}} = e^{\gamma_E/2}\left(\omega_2\,\Box^{\gamma+1} + Q_2 R_{\mu\nu}^2\,\Box^{\gamma-1}\right)$$

$$e^{H_0} \to e^{\frac{1}{2}\gamma_E \sqrt{P_{\gamma+1}^{(2)2}}} = e^{\gamma_E/2}\left(\omega_0\,\Box^{\gamma+1} + Q_0 R^2\,\Box^{\gamma-1} + G_0 R\,\Box^{\gamma}\right)$$

$$\to (\Box^4 + Q_0 R\,\Box^3)(\Box^4 + Q_0 R\,\Box^3)$$

$\gamma = 3 \Leftrightarrow \sqrt{(\Box^4)^2}$ NONLOCAL KILLERS

$$e^{H_4} \to e^{\frac{1}{2}\gamma_E}\left(\omega_4\,\Box^{\gamma+1}\right) \qquad \gamma+1 = 4m$$

CS. $\gamma=7$, $\gamma+1=8$, $\gamma-1=6$, $\gamma=$

$\Box$? has positive spectrum in any background !!!

4)

Koshdev dS

$$f_2 = 0 \qquad c_{1,0} = 0$$

$$\text{Spin 2}: \quad 1 + \frac{2}{M_p^2}\left(2\left(\Box - \frac{\bar{R}}{3}\right) f_3\left(\Box + \frac{\bar{R}}{3}\right)\right) = e^{H_3}$$

$$f_3\left(\Box + \frac{\bar{R}}{3}\right) = + \frac{M_p^2}{2}\left(\frac{e^{H_3\left(\Box - \frac{\bar{R}}{3}\right)} - 1}{2\left(\Box - \frac{\bar{R}}{3}\right)}\right) \Rightarrow f_3(\Box) = \frac{M_p^2}{2}\,\frac{e^{H_3\left(\Box - \frac{2}{3}\bar{R}\right)} - 1}{2\left(\Box - \frac{2}{3}\bar{R}\right)}$$

$$\text{Spin 0}: \quad 1 - \frac{2}{M_p^2}\left(6\Box + 2\bar{R}\right) f_1(\Box) = e^{H_1\left(\Box + \frac{\bar{R}}{3}\right)}$$

$$\updownarrow$$

$$e^{H_1\left(\Box + \frac{\bar{R}}{3}\right)}\left(1 - \frac{\Box}{m_s^2}\right)$$

$$f_1(\Box) = -\frac{M_p^2}{2}\,\frac{1}{6}\,\frac{e^{H_1\left(\Box + \frac{\bar{R}}{3}\right)} - 1}{\left(\Box + \frac{\bar{R}}{3}\right)}$$

$$\bar{R} = 0 \Rightarrow f_3 = \frac{M_p^2}{2}\,\frac{e^{H_3(\Box)} - 1}{2\Box}, \qquad f_1(0) = -\frac{M_p^2}{2}\,\frac{1}{6}\,\frac{e^{H_1(\Box)} - 1}{\Box}$$

$$\mathcal{L} = R - \frac{1}{6}R\left(\frac{e^{H_1} - 1}{\Box}\right)R + C\,\frac{e^{H_3} - 1}{2\Box}\,C$$

$$\frac{M_p^2}{2} = 2u_4^{-2} = \frac{2}{M_4^2} = \frac{u_4^2}{2}$$

$$C_{1,0} = 0 \Rightarrow H_1 = \frac{1}{c}\left(\gamma_E + P(0, P_{\gamma+1}^{(1)})^2 + \beta^{UR}_{\gamma+1}\right)$$

$$\gamma = 3 \Rightarrow$$

$$P_{\gamma+1}^{(1)} = \left(\Box + \frac{R}{3}\right)^2(\Box)^2 \qquad \Box^8 \qquad \left(P_{\gamma+1}^{(1)}\right)^2 = \left(\Box + \frac{R}{3}\right)^4(\Box)^4$$

$$f_1(0) = -\frac{M_p^2}{2}\,\frac{1}{6}\,\frac{H_1\left(\left(\Box + \frac{R}{3}\right)^2\Box^2\right)}{\Box + \frac{R}{3}}\Bigg|_{\Box = 0} = 0$$

$$P^2 \; P^{2\cdot 8} = P^{18}$$

$\gamma+1 = 8 \qquad (\gamma = 7)$

posso prendere $H_1 = H_3 = \frac{1}{2}\left(Y_6 + \int (0, P_{\gamma+1}^2) + g\, P_{\gamma+1}^2 \right)$

$$P_{\gamma+1}^2 = \left(\Box + \frac{R}{3}\right)^4 \Box^4 \left(\Box - \frac{2}{3}R\right)^8$$

$$P_{\gamma+1}^{P(0)} = 0 \qquad P^4 \; P^4 \cdot P^8 = P^{16}$$

In questo modo il termine di Fermi è overall

posso anche prendere $\gamma+1 = 12$

$$P_{\gamma+1}^2 = \left(\Box + \frac{R}{3}\right)^4 \left(\Box - \frac{2}{3}R\right)^8 \Box^{12}$$

$$\downarrow$$

$$\left(\omega_8 \Box^6 + \omega_{\gamma-1} \Box^5 + \omega_{\gamma-2} \Box^4\right)^2$$

$\gamma+1 = 2n+2$

$n=1 \qquad 2$

$n=2 \qquad 4$

$n=3 \qquad 8$

$n=4$

$$-\frac{1}{6} + \frac{2}{3}$$

$$\frac{-1+4}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{V^{-1}-1}{\Box} = \frac{V^{-1}-e^{\gamma\varepsilon/2}\sqrt{\beta_{\gamma+1}^2}-1}{\Box} + \frac{e^{\gamma\varepsilon/2}\sqrt{\beta_{\gamma+1}^2}}{\Box}$$

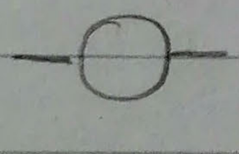
$h(\Box)$ $\quad e^{H}$ $\quad$ smooth convergent integrals $\quad e^{\gamma\varepsilon/2}\frac{\beta_{\gamma+1}}{\Box}$, $\frac{\beta_{\gamma+1}>0}{\neq\Box}$.

$\frac{1}{k^2}$

kin: $h\left(e^{H}\Box\right)h = \tilde{h}\,\Box\,\tilde{h} \Rightarrow \tilde{h} = e^{H/2}h \qquad \left(e^{H/2}\tilde{h}\right)$

Interactions: $h^2\Box e^{H}h = \left(e^{-H/2}\tilde{h}\right)\left(e^{-H/2}\tilde{h}\right)\Box e^{H}\left(e^{-H/2}\tilde{h}\right)$

$$h^2\Box^2 f h = \left(e^{-H/2}\tilde{h}\right)\left(e^{-H/2}\tilde{h}\right)\Box^2 f(\Box)\left(e^{-H/2}\tilde{h}\right)$$



$$\beta^2\beta^2\int d^4k\ \frac{e^{H(k)}}{k^2}\ \frac{e^{-H(k+p)}}{(k+p)^2} = R^2\int d^4k\ \frac{1}{k^2}\ \frac{1}{(k+p)^2}\ \frac{e^{H(k)}}{e^{H(k+p)}}$$

$$e^{H} = e^{H} - e^{\gamma\varepsilon/2}\sqrt{\beta^2} + e^{\gamma\varepsilon/2}\sqrt{\beta^2}$$

$$\cong e^{-z^2} + \sqrt{\beta^2} = \sqrt{\beta^2}\left(1 + \frac{e^{-z^2}}{\sqrt{\beta^2}}\right)$$

$$= \int \frac{h\,\partial^2 h\ \ e^{H}-1\ \partial^2 h}{\Box}$$

$$\frac{1}{e^{H}} \sim \frac{1}{\sqrt{\beta^2}}\left(1 - \frac{e^{-z^2}}{\sqrt{\beta^2}} + \dots\right)\int d^4k\ \frac{1}{k^{2\gamma+4}}\ \frac{k^{2\gamma+2}}{(k+p)^{2\gamma+2}}\ \frac{k^{2\gamma+2}}{(k+p)^{2\gamma+4}}$$

$\left(e^{H(k)-H(k+p)} - 1\right) + 1$ — log. divergent

ζ_0 at "∞". smooth function.

$\frac{1}{z}$ as $H = \frac{1}{z}h_\varepsilon + \Gamma(0,\varepsilon^4) + \oint\varepsilon$

Anselmi-Wimbauer-file

$$\int_{\mu\nu} = \phi^{\alpha'}\hat{\beta}^{\alpha}_{\mu'\nu'} = \phi^2\hat{\beta}_{\mu\nu}$$

$\zeta_{\mu\nu}$ is exact solution $\Rightarrow \hat{\beta}_{\mu\nu} = \sigma^2(\zeta_{\mu\nu})$ $\quad + \beta z^4(-dz^2 + dx^2)$

$\phi = \sigma^{-1}$ $\quad$ exact solution.

$\hat{\beta}^{\alpha}_{\mu'\nu'} = S\,\hat{\beta}_{\mu'\nu'} = \frac{1}{\phi^2}\phi^2\zeta_{\mu\nu} = \zeta_{\mu\nu}$

$\phi^{\alpha} = S^{-1/2}\phi = \phi\frac{1}{\phi} = 1$ $\quad \Rightarrow \phi^{\alpha'}\hat{\beta}^{\alpha}_{\mu'\nu'} = \zeta_{\mu\nu}$

$$h(z) = \frac{e^{H(z)} - 1}{z}$$

$$= \frac{V^{-1} - e^{\gamma_{E/2}}\sqrt{P(z)}}{z} + \frac{e^{\gamma_{E/2}}\sqrt{P(z)}}{z}$$

$$\frac{h(z_1)}{z_1 - z_2} + \frac{h(z_2)}{z_2 - z_1}$$

$$= \frac{e^{H(z_1)} - 1}{z_1(z_1 - z_2)} + \frac{e^{H(z_2)} - 1}{z_2(z_2 - z_1)}$$

$$= \frac{e^{H(z_1)}}{z_1(z_1 - z_2)} + \frac{e^{H(z_2)}}{z_2(z_1 - z_1)} - \left(\frac{1}{z_1(z_1 - z_2)} + \frac{1}{z_2(z_2 - z_1)}\right)$$

$$- z_2(z_1 - z_1)$$

$$- \frac{1}{z_1 - z_2}\left(\frac{z_2 - z_1}{z_1 z_2}\right) = \frac{1}{z_1 z_2}$$

$$= \frac{e^{H(z_1)}}{z_1(z_1 - z_1)} + \frac{e^{H(z_2)}}{z_2(z_2 - z_1)} + \frac{1}{z_1 z_2}$$

$$= \frac{1}{z_1 - z_2}\left[\frac{e^{H(z_1)}}{z_1} - \frac{e^{H(z_2)}}{z_2}\right] + \frac{1}{z_1 z_2}$$

$$= \frac{1}{z_1 - z_2}\left[\frac{V^{-1}(z_1) - e^{\gamma_{E/2}}\sqrt{B_{N+1}^2(z_1)} + e^{\gamma_{E/2}}\sqrt{B_{N+1}^2(z_1)}}{z_1} - \frac{V^{-1}(z_2) - e^{\gamma_{E/2}}\sqrt{B_{N+1}^2(z_2)} + e^{\gamma_{E/2}}\sqrt{B_{N+1}^2(z_2)}}{z_2}\right.$$

$$\left. + \frac{z_1 - z_2}{z_1 z_2}\right]$$

$$\int d^4p \; \frac{1}{p^{2(\alpha+\gamma)}} \; \left(\frac{1}{22}\right) p^2$$

$$2\alpha + 4 - 2 > 4$$

$$2\alpha > 2$$

$$\alpha > 1$$

2-Loops

$$(\epsilon_{\alpha\beta} e^{i\ell_1 x})(e^{i\ell_2 x} \epsilon^{\alpha\beta}) \qquad \ell = 3$$

scalar

$$\delta^2\Box = 2 h_{\alpha\beta} h^{\gamma\alpha} \nabla^\beta \nabla_\gamma + 2 h_{\alpha\beta}(\nabla^\alpha h^{\beta\gamma})\nabla_\gamma + 2 h_{\alpha\beta}(\nabla_\gamma h^{\gamma\alpha})\nabla^\beta - h_{\alpha\beta}(\nabla^\alpha h)\nabla^\beta$$

$$- h_{\alpha\beta}(\nabla^\gamma h^{\alpha\beta})\nabla_\gamma \quad (\text{SCALAR})$$

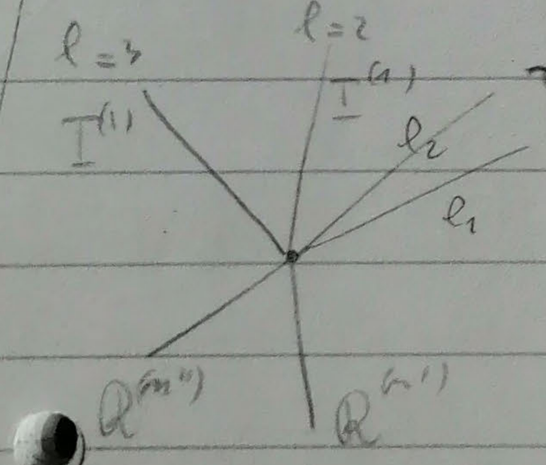
$$2 h_{\alpha\beta} \ell_2^\alpha h^{\beta\gamma}\nabla_\gamma + 2 h_{\alpha\beta} \ell_{2\gamma} h^{\gamma\alpha}\nabla^\beta - h_{\alpha\beta} \ell^\alpha h \nabla^\beta \ldots$$

In short: $I^{(2)}$

$$\left[\underset{\ell_1\ell_2}{hh}\partial^2 + \underset{\ell_2\ \ell_1}{h(\partial h)}\partial + \underset{\ell_2\ \ell_1}{h(\partial^2 h)} \right]$$

$$Q_{i+1}^2 + \ell_1 Q_{i+1} + \ell_1^2 \qquad Q_1 \equiv \overbrace{\ell_1 + \ell_2 + q_2 + q_3 + q_4}^{-q_5}$$

$$(Q_2^2 + \ell_1 Q_2 + \ell_1^2)(Q_3^2 + q_2 Q_3 + q_2^2)(q_4^2 + q_3 q_4 + q_3^2)$$



$$Q_1 = \ell_1 + \ell_2 + q_2 + q_3 + q_4 = \ell_1 + \ell_2 + Q_2 \qquad R^{(n+1)}_{q_4}$$
$$Q_2 = q_2 + q_3 + q_4 = q_2 + Q_3$$
$$Q_3 = q_3 + q_4 = q_3 + Q_4$$
$$Q_4 = q_4$$

$$\Rightarrow q_2 = Q_2 - Q_3, \quad \ell_1 + \ell_2 = Q_1 - Q_2,$$
$$q_3 = Q_3 - Q_4,$$
$$q_4 = Q_4.$$

$$I^{(m_i)} \subset \ell_{i_1}, \ldots, \ell_{m_i}, \partial, \partial$$

$$Q_{i+1}\ell_{i'} + \ell_{i'}^2 + Q_{i+1}^2$$

$$(Q_2^2 + \ell_1 Q_2 + \ell_1^2)\ (\alpha Q_3^2 + \gamma Q_3(Q_2 - Q_3) + \varepsilon(Q_2 - Q_3)^2) \cdot$$
$$\cdot (\alpha Q_4^2 + \beta(Q_3 - Q_4)Q_4 + \gamma(Q_3 - Q_4)^2) =$$

$$= \qquad = \qquad (a_1 Q_3^2 + a_2 Q_2 Q_3 + a_3 Q_2^2) \cdot$$
$$\cdot (b_1 Q_4^2 + b_2 Q_3 Q_4 + b_3 Q_3^2)$$

$l=3$, optimus 2-legs

$$(h_2 q_2 + c(q_2)q + c_2 q_4 q)\,(h_2 q_2 + c(q_2)q + c_2 q_4 q)\,(h_2 q_2 + c(q_2)q + c_2 q_4 q)$$

$q_2^a q_1^b \qquad q_1^a q_1^b \qquad q_2^a \qquad q_1^b \qquad q_1^a q_2^b \qquad q_4$

$$\left[a_1\left(q_2^a + q_2^b + q_3^a + q_3^b + q_4\right)^2 + a_2 q_2^b\left(q_2^a + q_2^b + q_3^a + q_3^b + q_4\right) + a_3 q_2^{b\,2}\right]$$

$$\left[b_1\left(q_3^a + q_3^b + q_4\right)^2 + b_2\, q_2^b\left(q_3^a + q_3^b + q_4\right) + b_3\, q_2^{b\,2}\right]$$

$$\left[c_1\, q_4^2 + c_2 q_3^b\, q_4 + c_3\, q_3^{b\,2}\right] \quad (\star)$$

↖ depinde de 6 momente: $p_2^{a,b}, q_3^{a,b}, q_4$ și masa q_1^a

$$Q_1 = q_1^a + q_1^b + q_2^a + q_2^b + q_3^a + q_3^b + q_4 = q_1^a + q_1^b + Q_2$$

$$Q_2 = q_2^a + q_2^b + q_3^a + q_3^b + q_4 = q_2^a + q_2^b + Q_3$$

$$Q_3 = q_3^a + q_3^b + q_4 = q_3^a + q_3^b + Q_4$$

$$Q_4 = q_4$$

$$Q_2 = Q_1 - (q_1^a + q_1^b) \quad (-q_5)$$

$$\Rightarrow q_1^a + q_1^b = Q_1 - Q_2 \qquad \Rightarrow (\star) = \left[a_1 Q_2^2 + a_2 q_1^b Q_2 + a_3 q_1^{b\,2}\right]$$

$$q_2^a + q_2^b = Q_2 - Q_3 \qquad \cdot\left[b_1 Q_3^2 + b_2 q_2^b Q_3 + b_3 q_2^{b\,2}\right]$$

$$q_3^a + q_3^b = Q_3 - Q_4 \qquad \left[c_1 Q_4^2 + c_2 q_3^b Q_4 + c_3 q_3^{b\,2}\right]$$

$q_4 = Q_4$

conform: ↑ depinde de 6 momente + 1

7 variabile: $\frac{h(Q_2)}{Q_2^2 - Q_1^2}\,\frac{1}{(Q_2^2 - Q_3^2)(Q_3^2 - Q_4^2)}\,q_{1,1}^b\, Q_2, q_2^b, Q_3, Q_4, q_3^b + Q_1$

$$\int dq_1^b\, f(Q_1, Q_2, Q_3, Q_4, q_1^b, q_2^b, q_3^b)\left(\frac{1}{q_1^{a\,2} - i\epsilon}\right)^{\gamma+2}\left(\frac{1}{(Q_1 - Q_2 - q_1^b)^2 - i\epsilon}\right)^{\gamma+2}$$

$Q_1 = Q_2 + q_2^a + q_1^b$

↖ în total 7 variabile: pierdem pol denominator cu Q_1: $\frac{1}{(Q_1^2 - Q_2^2)(\ldots)}$

Se integro in $dq_1^e \Rightarrow q_1^b, q_2^{e,b}, q_3^{e,b}, q_4$ sono costanti, ma

$$q_1^e = Q_1 - q_1^b - q_2^e - q_2^b - q_3^e - q_3^b - q_4 \quad \Rightarrow \quad dq_1^e = dQ_1$$

const.

$$ds^2 = s(r)\left[-\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^{(2)}\right]$$

$$s(r)\left[-d\tau^2 + \frac{r_0}{r}dl^2 + r^2 d\Omega^{(2)}\right]$$

$$r = \left[\frac{3}{2}(\rho-\tau)\right]^{2/3} r_0^{1/3} \qquad r^{3/2} = \frac{3}{2}(\rho-\tau)\, r_0^{\frac{1}{2}}$$

$$\rho - \tau = \frac{2}{3}\frac{1}{\sqrt{r_0}}\, r^{3/2}$$

$$dV^{(3)} = \int \sqrt{\frac{r_0}{r}}\sqrt{s(r)}\cdot dp \quad 4\pi r^2 s(r)$$

$$\rho = \tau + \frac{2}{3}\frac{1}{r_0^{1/2}} r^{3/2}$$

$$(\rho-\tau)^2 = \frac{4}{9}\frac{1}{r_0} r^3$$

$$= 4\pi\sqrt{\frac{r_0}{r}\, s(r)}\, r^2 s(r)\, dp = 4\pi\sqrt{r_0 r^3\left(1+\frac{L^4}{r^4}\right)}\, s(r)\, dp$$

$$^{(3)}V = 4\pi\int \sqrt{\frac{r_0}{r}}\, r^2 dp = 4\pi\int \sqrt{r_0}\, r^{3/2}\, dp$$

$$L=0 \Rightarrow V^{(3)} = \underbrace{4\pi r^3}_{L=0} = 4\sqrt{r_0}\int \frac{3}{2}(\rho-\tau) r_0^{1/2}\, dp$$

$$dr^2 + r^2 d\Omega^2 \qquad \sqrt{h/g} = \sqrt{r^4\sin^2\theta} = \sin\theta\, r^2$$

$$\int dr\, r^2\, d\varphi \sin\theta\, d\theta = \frac{r^3}{3} 4\pi$$

$$\sqrt{^{(3)}g} = \sqrt{s(r)\frac{r_0}{r}\, r^4} = \sqrt{r_0 r^3 s^2(r)} = \sqrt{r_0}\left(r\, s(r)\right)^{3/2}$$

$$^{(3)}V = 4\pi\int dp\, \sqrt{r_0}\left(r + \frac{L^4}{r^3}\right)^{\frac{3}{2}} \underset{L=0}{\rightarrow} 4\pi\int dp\, \sqrt{r_0}\, r^{3/2} = 4\pi\sqrt{r_0}\int \frac{3}{2}(\rho-\tau) r_0^{1/2}\, dp + C$$

$$= 4\pi\frac{3}{2} r_0\left(\frac{\rho^2}{2} - \tau\rho\right) + C$$

$$= 3\pi r_0 (\rho^2 - 2\tau\rho) + C$$

$$V(\rho=\tau) = 0 \Rightarrow 3\pi r_0(\tau^2 - 2\tau^2) + C = 0$$

$$C = 3\pi r_0 \tau^2 \Rightarrow V = 3\pi r_0 (\rho-\tau)^2 = \frac{4}{3}\pi r^3$$

$$V^{(3)} = 4\pi \int \sqrt{r_f}\, dp \left(r + \frac{L^4}{r^3} \right)^{3/2}$$

$$= 4\pi \sqrt{r_f} \int dp \left[\left(\frac{3}{2}(p-z)\right)^{2/3} r_f^{1/3} + \frac{L^4}{\left(\frac{3}{2}(p-z)\right)^2 r_f} \right]^{3/2}$$

$$= 4\pi \sqrt{r_f} \int_z^d dp \frac{L^6}{\left(\frac{3}{2}(p-z)\right)^2 r_f^{3/2}}$$

$$I = \int_z^d \left[\left(\frac{3}{2}(p-z)\right)^{-1/3}\right]^2 dp$$

$$= \frac{1}{\left[\frac{3}{2}(p-z)\right]^{1/3}} \frac{2}{3} r_f^{1/3} \int \frac{dp}{r_f^{1/2}}$$

$$= \frac{r_f^{1/2}\, dp}{r^{1/2}}$$

$$dp = \frac{r^{1/2}}{r_f^{1/2}} dr$$

$$4\pi \sqrt{r_f} \int_0^p \frac{L^6}{r^{5/2}} \frac{r^{1/2}}{\sqrt{r_f}} dr$$

$$= 4\pi L^4 \int_0^p \frac{1}{r^4} dr = 4\pi L^4 \frac{r^{-4+1}}{-4+1}$$

$$= -\frac{4\pi L^4}{3} \frac{1}{r^3}$$

$$R + R_{ij}(1+\Delta^{2\gamma})R^{ij} + R_{ij}^{2}(\Delta^{2\gamma-4})R^{ij2} \qquad \int(d\omega\, d^{D}k)^{L}$$

$$S_{\cdot} = (z+D)L - I(2\gamma+4) + V(2\gamma+4) \qquad zz = 2\gamma+4 \qquad z=\gamma+2$$

$$(z+D)L - \underbrace{(2\gamma+4)}_{zz}(L-1) = (z+D)L - zz(L-1) = (z+D-zz)L + zz$$

$$= (D-z)L + zz \qquad D=3 \qquad 0 \le z \le D$$

$L=1 \Rightarrow \delta = z+3$, $z=1 \Rightarrow \delta=4 \Rightarrow k^2, K_{ij}^2, {}^{(3)}R, \lambda, R_{ij}^2, R^2$

$\to L=2 \Rightarrow \delta = 2(D-z)+zz$

$= 2D - zz + zz = 2D$

$\to L=3 \Rightarrow (D-z)3 + zz = D - z < 0 \quad z > D$

$$S = \frac{1}{k^2}\int_R dt \int_\sigma d^3x \sqrt{h}\Big(K_{ij}^2 - K^2 + {}^{(3)}R + \alpha\, {}^{(3)}R_{ij}(1+x\Delta^{2\gamma}){}^{(3)}R^{ij} + \beta\, {}^{(3)}R^2(1+y\Delta^{2\delta}){}^{(3)}R^2$$

$$\alpha = -\frac{3}{8}\beta, \qquad + s^{(2)} R_{ij}^2(\Delta^{2\gamma-4})R^{ij2} + s^{(0)} R^2(\Delta^{2\gamma-4})R^2 +$$

$x = y$,

$\underline{\beta < 0}$. $\qquad + r^{(2)} K_{ij}^2(\Delta^{2\gamma-4})K_{ij}^2 + r^{(0)} K^2(\Delta^{2\gamma-4})K^2\Big)$

$$\int d\omega\, d^3k \left(\frac{1}{\omega^2 + K^{2\gamma+4}}\right)^{I} (\omega^2)^{V}, \; K^{2V} \qquad I - V = L - 1$$

$$I = V, \; L = 1$$

$$\Lambda_\omega \; \Lambda_k^{3-(2\gamma+4)(I-V)} \qquad (1+3) - (2\gamma+4)(L-1)$$

$$\int d^4x \left(\alpha\, {}^{(3)}R^2 + \beta\, {}^{(3)}R_{ij}^2\right)\sqrt{g} = \beta \int d^4x \sqrt{g}\left(-\frac{3}{8}\, {}^{(3)}R^2 + {}^{(3)}R^{ij2}\right)$$

$$= \Gamma \int d^4x\, \Gamma_g \frac{\delta W[q]}{\delta q_{ij}} G_{ij,ke} \frac{\delta W[q]}{\delta q_{ke}}$$

$$\to \beta \int d^4x \sqrt{g}\left[-\frac{3}{8}\, {}^{(3)}R(1+\Delta^{2\gamma}){}^{(3)}R + {}^{(3)}R_{ij}(1+\Delta^{2\gamma}){}^{(3)}R_{ij}\right] =$$

$$\text{“} \quad \text{“} \qquad \left[\frac{\delta W[q]}{\delta q_{ij}} \underbrace{(1+\Delta^{2\gamma}) G_{ij,ke}}_{\tilde{G}_{ij,ke}} \frac{\delta W[q]}{\delta q_{ke}}\right].$$

$$\frac{1}{-k^2+\beta\vec{k}^4} \rightarrow \frac{1}{-k^2\gamma(\vec{k}^2)+\beta(\vec{k})\vec{k}^4}$$

$$\frac{1}{\omega^2\gamma(\vec{k}^2)-\vec{k}^2\gamma(\vec{k}^2)+\beta(\vec{k}^2)\vec{k}^4}$$

$$\gamma(\vec{k}^2)=1+\gamma\vec{k}^{2z} \qquad \beta(\vec{k}^2)=1+\beta\vec{k}^{2z-2}$$

$$\omega^2+\gamma\omega^2\vec{k}^{2z}-\vec{k}^2-\vec{k}^2\gamma\vec{k}^{2z}+\vec{k}^4+\beta\vec{k}^{2z+2}$$

$$\omega^2=\frac{\vec{k}^2\gamma(\vec{k}^{2z})-\beta(\vec{k}^2)\vec{k}^4}{\gamma(\vec{k}^2)}$$

$$\int(d\omega\, d^3\vec{k})^L\left(\frac{1}{\omega^2\vec{k}^{2z}+\vec{k}^{2z+2}}\right)^E\Big/\Big(\omega^2\vec{k}^{2z}+\vec{k}^{2z+2}\Big)^V=\vec{k}^2\left(1-\frac{\beta(\vec{k}^2)}{\gamma(\vec{k}^2)}\vec{k}^2\right)$$

$$=\vec{k}^2\left(1-\frac{1+\beta\vec{k}^{2z-2}}{1+\gamma\vec{k}^{2z}}\frac{\vec{k}^2}{\Lambda^2}\right)$$

$L=1$, $E=V$

$$\int d\omega\, d^3\vec{k}\,\frac{1}{\vec{k}^{2z}(\omega^2+\vec{k}^2)}\,\omega^2\vec{k}^{2z}$$

$\Lambda\omega$

$$=\vec{k}^2\left(1-\frac{\beta}{\gamma\Lambda^2}\right)$$

$$\int d^4p\,\frac{1}{p^2}\frac{1}{(p+k)^2}\sim\int dx\int d^4p\,\frac{1}{(p^2+c)^2}$$

$$\int dx\;|p^3|\,dp\,\frac{1}{(p^2+c)^2}\;\int d\Omega_p$$

$$\vec{d} = \hat{d} + \frac{\bar{d}}{m} \qquad \vec{D} = \hat{D} + \frac{\bar{D}}{m}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad R = -G$$

$$R_{\mu\nu} = G_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} G$$

$$\mathcal{L} = \sqrt{g}\left[R + 2\left(G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} G\right) F \varphi^{\mu\nu} + \left(\varphi^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \varphi\right) F \varphi_{\mu\nu}\right]$$

$$= \sqrt{g}\left(R + 2 G_{\mu\nu} F \varphi^{\mu\nu} - G F \varphi + \varphi^{\mu\nu} F \varphi_{\mu\nu} - \frac{1}{2} \varphi F \varphi\right)$$

$$\rightarrow \mathcal{L} = \sqrt{g}\left(R + 2 G_{\mu\nu} F \varphi^{\mu\nu} + \varphi^{\mu\nu} F \varphi_{\mu\nu} - G F \chi - \frac{1}{2} \chi F \chi\right)$$

$\varphi^{\mu\nu}$ F fixo Φ^{eo}

$$\frac{\delta \mathcal{L}(S)}{\delta \chi} = 0 \Rightarrow -FG - F\chi = 0$$

$$\chi = -G$$

$$\frac{\delta \mathcal{L}}{\delta \varphi^{\mu\nu}} = 0 \Rightarrow 2FG_{\mu\nu} + 2F\varphi_{\mu\nu} = 0$$

$$\boxed{\varphi_{\mu\nu} = -G_{\mu\nu}} \Rightarrow \nabla^{\mu} \varphi_{\mu\nu} = 0 \quad \text{6 condições.}$$

Teorias $\Rightarrow \varphi = -G \Rightarrow$ on shell $\varphi^{\mu}_{\mu} = \chi$

$$\mathcal{L} = \sqrt{g}\left(R + 2 G_{\mu\nu} F (-G^{\mu\nu}) + G_{\mu\nu} F G^{\mu\nu} + GFG - \frac{1}{2} G F G\right)$$

$$\left(R - 2 G_{\mu\nu} F G^{\mu\nu} + G_{\mu\nu} F G^{\mu\nu} + \frac{1}{2} GFG\right)$$

$$\left(R - G_{\mu\nu} F G^{\mu\nu} + \frac{1}{2} GFG\right)$$

$$R - G_{\mu\nu} F \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) + \frac{1}{2} GFG$$

$$R - G_{\mu\nu} F R^{\mu\nu} + \cancel{\frac{1}{2} GF\underbrace{(R)}_{-G}} + \cancel{\frac{1}{2} GFG}$$

$$R - G_{\mu\nu} F R^{\mu\nu}. \quad \not{\alpha}$$

on shell

$2 + (10 - 4) + 1 - 1$ $\Rightarrow$ perde $\varphi^{\mu}_{\mu} = \chi$

$2 + 6 = 8$

Se impomos $\nabla^{\mu} \varphi_{\mu\nu} = 0$ off-shell $\Rightarrow$ invariance $\delta(\varphi_{\mu\nu}) = F^{-1}(\xi_{\mu} \xi_{\nu} + \Omega_{\nu} \xi_{\mu})$

$D = 4$ $\qquad$ $\varphi = g^{\mu\nu}\varphi_{\mu\nu}$

$$\mathcal{L} = \sqrt{g}\left[R + 2R^{\mu\nu}F(\Box)\varphi_{\mu\nu} + \left(\varphi^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}\varphi\right)G(\Box)\varphi_{\mu\nu}\right]$$

$$\left[R + \left(2R^{\mu\nu}F(\Box) + \varphi^{\mu\nu}G(\Box)\right)\varphi_{\mu\nu} - \tfrac{1}{2}\varphi G(\Box)\varphi\right]$$

$$\frac{\delta S}{\delta \varphi_{\mu\nu}} = 0 \Rightarrow 2F\cdot R^{\mu\nu} + G(\Box)\left(\varphi^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}\varphi\right) + \left(G(\Box)\varphi^{\mu\nu} - \tfrac{1}{2}g^{\rho\sigma}g^{\mu\nu}G(\Box)\varphi_{\rho\sigma}\right)$$

$$2F\cdot R^{\mu\nu} + 2G(\Box)\varphi^{\mu\nu} - \tfrac{1}{2}G(\Box)g^{\mu\nu}\varphi - \tfrac{1}{2}G(\Box)g^{\mu\nu}\varphi = 0$$

$$2FR^{\mu\nu} + 2G\varphi^{\mu\nu} - Gg^{\mu\nu}\varphi = 0$$

$$\boxed{G\cdot\varphi_{\mu\nu} = -F\cdot G_{\mu\nu}} \qquad G\cdot\varphi = -F\left(R - \tfrac{4}{2}R\right), \quad G\varphi = F\cdot R$$

$$2F\cdot R^{\mu\nu} + 2\cdot(-F\cdot G^{\mu\nu}) - g^{\mu\nu}F\cdot R =$$

$$= \cancel{2}F\left(\cancel{R^{\mu\nu}} - \cancel{R^{\mu\nu}} + \tfrac{1}{2}g^{\mu\nu}R\right) - g^{\mu\nu}FR \equiv 0, \quad [F, G] = 0$$

$$\mathcal{L} = \sqrt{g}\left[R - 2R^{\mu\nu}\left(F\cdot G^{-1}F\right)G_{\mu\nu} + \left(-\left(G^{-1}\cdot F\,G^{\mu\nu}\right) - \tfrac{1}{2}g^{\mu\nu}\left(G^{-1}FR\right)\right)\cdot\right.$$

$$\left.\left(-FG_{\mu\nu}\right)\right]$$

$$= \sqrt{g}\left[R - 2G^{\mu\nu}\left(G^{-1}F^2\right)R_{\mu\nu} + G_{\mu\nu}\left(FG^{-1}F\right)G^{\mu\nu} + \right.$$

$$\left. + \tfrac{1}{2}(-R)\left(G^{-1}F^2\right)R\right] \qquad -R \qquad \left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R\right)$$

$$= \sqrt{g}\left[R - G^{\mu\nu}\left(G^{-1}F^2\right)R_{\mu\nu} - G_{\mu\nu}\left(G^{-1}F^2\right)\tfrac{1}{2}g^{\mu\nu}R\right.$$

$$\left. - \tfrac{1}{2}R\left(G^{-1}F^2\right)R\right] = \sqrt{g}\left[R - G^{\mu\nu}\left(G^{-1}F^2\right)R_{\mu\nu}\right]$$

$G^{-1} F^2 = \gamma^1 \qquad G = F \Rightarrow F = \gamma^1$

$$\mathcal{L} = \sqrt{g}\left[\left(g^{\mu\nu} + 2\left(\gamma^{(1)} \varphi^{\mu\nu}\right)\right) R_{\mu\nu} + \left(\varphi^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu}\varphi\right) \gamma^{(1)} \varphi_{\mu\nu}\right]$$

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \left(\gamma^{(1)} \varphi^{\mu\nu}\right)$$

$$\gamma^1 = \frac{e^H - 1}{\Box} \qquad \varphi$$

$$= \sqrt{g}\left\{\left[g^{\mu\nu} + 2\left(\frac{e^H - 1}{\Box}\right)\varphi^{\mu\nu}\right] R_{\mu\nu} + \left(\varphi^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu}\varphi\right)\left(\frac{e^H - 1}{\Box}\right)\varphi_{\mu\nu}\right\}$$

$$= \sqrt{g}\left\{\left(g^{\mu\nu} + 2(e^H - 1)\chi^{\mu\nu}\right) R_{\mu\nu} + \left(\chi^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu}\chi\right)(e^H - 1)\,\Box\chi_{\mu\nu}\right.$$

$$\left. + \tfrac{1}{2}\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi_{\mu\nu}\,\Box\chi^{\mu\nu}\right\}$$

$$\frac{\delta}{\delta\varphi^{\mu\nu}} = 0 \Rightarrow \qquad \varphi_{\mu\nu} - \Box\chi_{\mu\nu} = 0 \Rightarrow \chi_{\mu\nu} = \frac{1}{\Box}\varphi_{\mu\nu}$$

$$\downarrow$$

$$\left(+\tfrac{1}{2}\chi_{\mu\nu}\,\Box^2\chi^{\mu\nu} - \tfrac{1}{2}\chi_{\mu\nu}\,\Box^2\chi^{\mu\nu}\right) = 0$$

$$\tfrac{1}{2}\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi_{\mu\nu}\,\Box\chi^{\mu\nu} + \tfrac{1}{2}\chi_{\mu\nu}\,\Box^2\chi^{\mu\nu}$$

$$= \tfrac{1}{2}(\varphi_{\mu\nu} - \Box\chi_{\mu\nu})(\varphi^{\mu\nu} - \Box\chi^{\mu\nu})$$

$$= \tfrac{1}{2}\left(\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi_{\mu\nu}\,\Box\chi^{\mu\nu} - (\Box\chi_{\mu\nu})\varphi^{\mu\nu} + (\Box\chi_{\mu\nu})(\Box\chi^{\mu\nu})\right)$$

Quindi

$$\mathcal{L} = \sqrt{g}\left\{ \left(g^{\mu\nu} + 2\left(e^{H}-1\right)x^{\mu\nu}\right)R_{\mu\nu} + \left(x^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}x\right)\left(e^{H}-1\right)\Box x_{\mu\nu} \right.$$

$$\left. + \frac{\alpha}{2}\left(\varphi_{\mu\nu} - \Box x_{\mu\nu}\right)\left(\varphi^{\mu\nu} - \Box x^{\mu\nu}\right)\right\}$$

$$\frac{\delta \mathcal{L}}{\delta \varphi^{\mu\nu}} = 0 \Rightarrow \varphi_{\mu\nu} = \Box x_{\mu\nu} \Rightarrow x_{\mu\nu} = \frac{1}{\Box}\varphi_{\mu\nu}$$

$$\varphi_{\mu\nu}^{2} - 2\varphi_{\mu\nu}\Box x^{\mu\nu} + x_{\mu\nu}\Box^{2}x^{\mu\nu}$$

$$\sqrt{g}\Big[\left(g^{\mu\nu} + 2e^{H}x^{\mu\nu} - 2x^{\mu\nu}\right)R_{\mu\nu} + \left(x^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}x\right)e^{H}\Box x_{\mu\nu} +$$

$$- \left(x^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}x\right)\Box x_{\mu\nu} + \alpha\left(\varphi_{\mu\nu} - \Box x_{\mu\nu}\right)\left(\varphi^{\mu\nu} - \Box x^{\mu\nu}\right)\Big]$$

$$\alpha\left(\varphi_{\mu\nu}\varphi^{\mu\nu} - 2\varphi_{\mu\nu}\Box x^{\mu\nu} + x_{\mu\nu}\Box^{2}x^{\mu\nu}\right)$$

$$-\frac{1}{2}\alpha\,\tilde{\varphi}_{\mu\nu}\tilde{\varphi}^{\mu\nu} + 2\tilde{\varphi}_{\mu\nu}\left(\varphi^{\mu\nu} - \Box x^{\mu\nu}\right)$$

$$\frac{\delta \mathcal{L}}{\delta \tilde{\varphi}^{\mu\nu}} = -\alpha\,\tilde{\varphi}_{\mu\nu} + \alpha\left(\varphi_{\mu\nu} - \Box x_{\mu\nu}\right) = 0 \Rightarrow \tilde{\varphi}_{\mu\nu} = \varphi_{\mu\nu} - \Box x_{\mu\nu}$$

$$\Rightarrow -\frac{1}{2}\alpha\left(\varphi_{\mu\nu} - \Box x_{\mu\nu}\right)^{2} + \alpha\left(\varphi_{\mu\nu} - \Box x_{\mu\nu}\right)^{2}$$

$$= \frac{\alpha}{2}\left(\varphi_{\mu\nu} - \Box x_{\mu\nu}\right)^{2}$$

$$\sqrt{g}\Big[\Big(g^{\mu\nu}+2e^{H}x^{\mu\nu}-2x^{\mu\nu}\Big)R_{\mu\nu}+\Big(x^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}x\Big)e^{H}\Box x_{\mu\nu}+$$

$$-\Big(x^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}x\Big)\Box x_{\mu\nu}-\tfrac{1}{2}\alpha\tilde{\varphi}_{\mu\nu}\tilde{\varphi}^{\mu\nu}+\alpha\tilde{\varphi}_{\mu\nu}(\varphi^{\mu\nu}-\Box x^{\mu\nu})\Big]$$

$$\left.\begin{aligned}\frac{\delta\xi}{\delta\tilde{\varphi}^{\mu\nu}}=0 \quad & -\tilde{\varphi}_{\mu\nu}+\varphi_{\mu\nu}-\Box x_{\mu\nu}=0\\ \frac{\delta\xi}{\delta\varphi^{\mu\nu}}=0 \quad & \tilde{\varphi}_{\mu\nu}=0\end{aligned}\right\}\quad \tilde{\varphi}_{\mu\nu}=0$$

$$\varphi_{\mu\nu}=\Box x_{\mu\nu}$$

$$\Downarrow$$

$$x_{\mu\nu}=\frac{1}{\Box}\varphi_{\mu\nu}$$

$$\frac{\delta\xi'}{\delta x^{\mu\nu}}=0 \Rightarrow 2(e^{H}-1)R_{\mu\nu}+\underbrace{2e^{H}\Box x_{\mu\nu}}-\tfrac{1}{2}g^{\sigma\rho}\Big(g^{\alpha\beta}g_{\alpha\mu}g_{\beta\nu}\Box e^{H}x_{\sigma\rho}$$

$$+g^{\alpha\beta}x_{\alpha\beta}\overleftarrow{e^{H}}\Box g_{\mu\sigma}g_{\nu\rho}\Big)-\Big(\underbrace{2\Box x_{\mu\nu}}-\tfrac{1}{2}g^{\sigma\rho}\Big(g^{\alpha\beta}g_{\mu\nu}g_{\nu\rho}\Box x_{\sigma\rho}$$

$$x\overleftarrow{e^{H}}\Box g_{\mu\sigma}g_{\nu\rho}\Big)\Big)=0$$

$$2(e^{H}-1)R_{\mu\nu}+2(e^{H}-1)\Box x_{\mu\nu}-\tfrac{1}{2}\Big(g_{\mu\nu}\Box(e^{H}-1)x+g_{\mu\nu}(e^{H}-1)\Box x\Big)=0$$

$$(e^{H}-1)\Big\{2R_{\mu\nu}+2\Box x_{\mu\nu}-g_{\mu\nu}\Box x\Big\}=0 \qquad -2\Box\varphi_{\mu\nu}$$

$$\Downarrow$$

$$0$$

$$(e^{H}-1)\Big\{2R_{\mu\nu}+2\varphi_{\mu\nu}-g_{\mu\nu}\varphi\Big\}=0$$

$$\varphi_{\mu\nu}=-G_{\mu\nu}=-\Big(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R\Big) \qquad \varphi=-\Big(R-\tfrac{1}{2}4R\Big)=+R$$

$$R_{\mu\nu}-R_{\mu\nu}+\tfrac{1}{2}g_{\mu\nu}R-\tfrac{1}{2}g_{\mu\nu}R$$

$$\frac{1}{\Box}=-G_{\mu\nu}+\sum_{m}\Big[c_m\delta(\Box+m_m)+c_m^{*}\delta(\Box+m_m^{*})\Big] \qquad m_0=0,\quad e^{H}-1=0\Rightarrow m_e\neq m_e^{*}$$

$$R_{\mu\nu}=\sum_{k}\tilde{c}_k\Big(\delta(\Box+m_k)+c_k^{*}\delta(\Box+m_k^{*})\Big),\ m_0=0 \qquad \text{← man is solution}$$

Toy Model

$$\mathcal{L} = \frac{1}{2}\phi\, e^{H}\Box\,\phi - \frac{\lambda}{4}\phi^4 \qquad \text{EOM: } e^{H}\Box\phi - \lambda\phi^3 = 0$$

$$\phi = \phi_0 + \varphi \qquad \phi_0 \text{ s.t. } e^{\Box}\phi_0 - \lambda\phi_0^3 = 0$$

$$\mathcal{L} = \frac{1}{2}(\phi_0+\varphi)\, e^{H}\Box\,(\phi_0+\varphi) - \frac{\lambda}{4}(\phi_0+\varphi)(\phi_0+\varphi)(\phi_0+\varphi)(\phi_0+\varphi)$$

$$= \frac{1}{2}\Big\{\phi_0\, e^{H}\Box\,\phi_0 + \phi_0\, e^{H}\Box\,\varphi + \varphi\, e^{H}\Box\,\phi_0 + \varphi\, e^{H}\Box\,\varphi\Big\} - \frac{\lambda}{4}\left(\phi_0^2 + 2\varphi\phi_0 + \varphi^2\right)\left(\phi_0^2 + 2\varphi\phi_0 + \varphi^2\right)$$

$$= \frac{1}{2}\Big\{\phi_0\, e^{H}\Box\,\phi_0 + 2\phi_0\, e^{H}\Box\,\varphi + \varphi\, e^{H}\Box\,\varphi\Big\} +$$

$$-\frac{\lambda}{4}\Big(\phi_0^4 + \underline{2\varphi\phi_0^3} + \phi_0^2\varphi^2 + \underline{2\varphi\phi_0^3} + 4\varphi^2\phi_0^2 + 2\varphi^3\phi_0$$

$$+\varphi^2\phi_0^2 + 2\varphi^3\phi_0 + \varphi^4\Big)$$

$$= S(\phi_0) + \underbrace{\frac{1}{2}\Big[2\varphi\, e^{H}\Box\,\phi_0\Big]}_{\frac{\delta S}{\delta\phi}\big|_{\phi_0}\cdot\varphi} + \frac{\lambda}{4}\left(4\varphi\phi_0^3\right) +$$

$$+\frac{1}{2}\varphi\, e^{H}\Box\,\varphi - \frac{\lambda}{2}\phi_0^2\varphi^2 - \lambda\varphi^2\phi_0^2 - \frac{\lambda}{4}4\phi_0\varphi^3$$

$$-\frac{\lambda}{4}\varphi^4$$

$$\mathcal{L}_{\text{kinetic}} = \frac{1}{2}\varphi\, e^{H}\Box\,\varphi - \lambda\frac{3}{2}\phi_0^2\varphi^2$$

$$= \frac{1}{2}\varphi\left(\Box\, e^{H} - 3\lambda\phi_0^2\right)\varphi$$

⇒ Infinite number of complex conjugate poles.

Ho un polo reale ad infiniti poli complessi.

Il campo non è più massless ma massivo con massa quadra positiva se:

$$\Box - \lambda \frac{3}{2} \phi_0^2 = 0$$

$$-\kappa_0^2 + \vec{\kappa}^2 - \frac{\lambda}{2} 3 \phi_0^2 = 0$$

$$\kappa_0^2 = \vec{\kappa}^2 - \frac{3}{2} \lambda \phi_0^2 \qquad \lambda < 0 \quad \Rightarrow \underline{m^2 > 0}$$

Altro Toy model:

$$\frac{1}{2} \phi \, e^{H(\Box + m^2)} \phi \qquad - \frac{\lambda}{4} \phi^2 \, \Box \, e^{H} \phi^2$$

$$\frac{\delta S}{\delta \phi} = 0 \Rightarrow \delta\phi \cdot e^{H}(\Box + m^2)\phi - \delta\phi \frac{\lambda}{4} 2\phi (\Box e^{H} \phi^2) - \frac{\lambda}{4} \phi^2 \Box e^{H(2\phi \delta\phi)}.$$

$$e^{H}(\Box + m^2)\phi - \lambda \phi \Box e^{H} \phi^2 = 0$$

for $m = 0$ $\phi = \phi_0 =$ const exact solution, $\phi = \phi_0 + \varphi$

$$\mathcal{L} = \frac{1}{2} (\phi_0 + \varphi) e^{H} \Box (\phi_0 + \varphi) - \frac{\lambda}{4} (\phi_0^2 + \varphi^2 + 2\phi_0\varphi) \Box e^{H(\phi_0^2 + \varphi^2 + 2\phi_0\varphi)}$$

$$= S(\phi_0) + \frac{\delta S}{\delta \phi}\Big|_{\phi=\phi_0} \cdot \varphi + \underline{\frac{1}{2} \varphi e^{H} \Box \varphi} - \frac{\lambda}{4} \Big[\underbrace{\phi_0^2 \Box e^{H}(\varphi^2)}_{\leftarrow 0} + \varphi^2 \Box e^{H(\phi_0^2)} \Big]_{0}$$

$$+ 2\varphi^2 \Box e^{H(\phi_0\varphi)} + 2\phi_0\varphi \Box e^{H(\varphi^2)} + \underline{2\phi_0\varphi \Box e^{H} 2\phi_0\varphi}$$

$$+ \varphi^2 \Box e^{H} \varphi^2 \Big] \Rightarrow \mathcal{L}_{kin} = \frac{1}{2} \varphi e^{H} \Box \varphi - \lambda \phi_0^2 \varphi \Box e^{H} \varphi$$

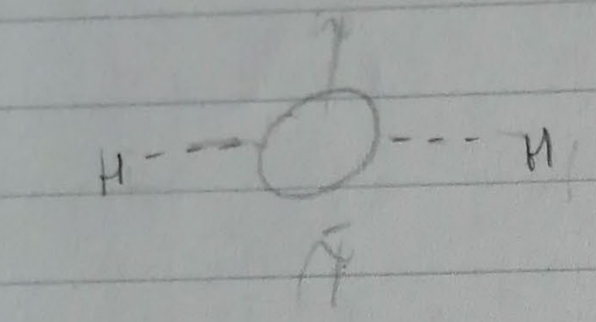
$$= \frac{1}{2} \varphi e^{H} (\Box - \lambda \phi_0^2) \varphi \quad \underline{\text{no}} \quad \underline{\text{complex}}$$

ϕ_0 exact solution but non constant:

$$\mathcal{L}_{kin} = \frac{1}{2}\varphi \,\Box e^{H}\varphi \;-\frac{\lambda}{4}\Big[\phi_0^2 \,\Box e^{H}(\varphi^2) + \varphi^2 \,\Box e^{H}(\phi_0^2) + 4\phi_0\varphi \,\Box e^{H}(\phi_0\varphi)\Big]$$

$$= \frac{1}{2}\varphi \,\Box e^{H}\varphi - \frac{\lambda}{2}\varphi^2\left[\Box e^{H}(\phi_0^2)\right] - \lambda\,\varphi\,\phi_0 \,\Box e^{H}(\phi_0\varphi)$$

$$= \frac{1}{2}\varphi \,\Box e^{H}\varphi - \frac{\lambda}{2}\varphi\left[\Box e^{H}(\phi_0^2)\right]\varphi - \lambda\,\varphi\,\phi_0 \,\Box e^{H}(\phi_0\varphi)$$



$$H^2 \int d^6p \; \frac{1}{p_1 \; p_1 + k_1}$$

$$\int d^6p \; \frac{(p+k_1)(p-k_1-k_2)}{(p^2+m^2)((p+k)^2+m^2)}$$

$$= H^2 m^2 \left(\frac{1}{\epsilon} + \log \frac{k^2+m^2}{} \right)$$

$$= H^2 m^2 \left(\log \frac{m^2}{} \right)$$

$$\mathcal{L} = H \Box H + H \left(m_H^2 \pm \frac{1}{10} m^2 \log \frac{m^2}{} \right) H$$

top

$$\underbrace{\qquad\qquad}_{H x^2}$$

$$\mathcal{L} = \phi (\Box + R) \phi + N^2$$

$$\sqrt{\int_{M^4} dx^4 \sqrt{g}}$$

Field redefinition with Λ_{cc}

$$\mathcal{L} = \sqrt{g}\left(R - 2\Lambda_{cc} + (G_{\mu\nu} + \Lambda_{cc} g_{\mu\nu})\,\gamma^{\mu}(\Box)\left(R^{\mu\nu} - g^{\mu\nu}\Lambda_{cc}\right)\right)$$

$\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} = 0 \Rightarrow G_{\mu\nu} = -\Lambda g_{\mu\nu} \Longleftrightarrow R_{\mu\nu} = \Lambda g_{\mu\nu}$.

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\Lambda_{cc} g_{\mu\nu}$$

$$R - 2R = -4\Lambda_{cc}$$

$$R = +4\Lambda_{cc} \qquad R_{\mu\nu} = 2 g_{\mu\nu}\Lambda_{cc} - \Lambda_{cc} g_{\mu\nu} = \Lambda_{cc} g_{\mu\nu}$$

$$\int \sqrt{g}\,\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda_{cc} g_{\mu\nu}\right)\gamma(\Box)\left(R^{\mu\nu} - g^{\mu\nu}\Lambda_{cc}\right) =$$

$$= \int \Big[R_{\mu\nu}\,\gamma\, R^{\mu\nu} - R\,\gamma(0)\,\Lambda_{cc} - \tfrac{1}{2} R\,\gamma\, R + 2R\,\gamma(0)\,\Lambda_{cc}$$

$$+ \Lambda_{cc}\,\gamma(0)\,R - 4\Lambda_{cc}\,\gamma(0)\,\Lambda_{cc}\Big]$$

$$= \int R_{\mu\nu}\,\gamma\, R^{\mu\nu} - \tfrac{1}{2} R\,\gamma\, R \quad + \text{TOT derivative}$$

- $\mathcal{L} = \sqrt{g}\left[R - 2\Lambda_{cc} + E_{\mu\nu} F(\Box)^{\mu\nu\epsilon\sigma} E_{\epsilon\sigma}\right]$, $E_{\mu\nu} \equiv G_{\mu\nu} + \Lambda_{cc} g_{\mu\nu}$.

$$S_{EH-\Lambda}(g') = S_{EH-\Lambda}(g) + E_{\mu\nu}(g) F^{\mu\nu\epsilon\sigma}(g)\, E_{\epsilon\sigma}(g) \xrightarrow{+\text{TOT-DER.}} S_{NL-\Lambda}(g)$$

$S(g')$ $\qquad\qquad$ $S'(g)$

$$\underbrace{\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}\right)}_{E_{\mu\nu}} F^{\mu\nu\rho\sigma} \underbrace{\left(R_{\rho\sigma} - \tfrac{1}{2} g_{\rho\sigma} R + \Lambda_{cc} g_{\rho\sigma}\right)}_{E_{\rho\sigma}} + \text{TOT-DER}$$

$$F^{\mu\nu\rho\sigma} = \gamma(\Box)\cdot\left(g^{\mu\rho} g^{\nu\sigma} - \tfrac{1}{2} g^{\mu\nu} g^{\rho\sigma}\right) \qquad \text{Vedi dopo}$$

$$-\tfrac{1}{2}\delta_{\mu\nu}R + \tfrac{1}{2}\delta_{\mu\nu}R$$

$$(G_{\mu\nu} + \Lambda_{cc}\delta_{\mu\nu})\,\gamma^{\mu}(R^{\mu\nu} - \Lambda_{cc}\delta^{\mu\nu})$$

$$= (R_{\mu\nu} - \tfrac{1}{2}R\,\delta_{\mu\nu} + \Lambda_{cc}\delta_{\mu\nu})\,\gamma^{\mu}(R^{\mu\nu} - \Lambda_{cc}\delta^{\mu\nu})$$

$$= \underbrace{R_{\mu\nu}\gamma^{\mu}R^{\mu\nu}} - R\gamma^{\mu}\Lambda_{cc} - \tfrac{1}{2}R\gamma^{\mu}R + 2R\gamma^{\mu}\Lambda_{cc}$$

$$+ \Lambda_{cc}\gamma^{\mu}R - \Lambda_{cc}\gamma^{\mu}\Lambda_{cc}$$

$$(G_{\mu\nu} + \Lambda_{cc}\delta_{\mu\nu})\,F^{\mu\nu\rho\sigma}\,(G_{\rho\sigma} + \Lambda_{cc}\delta_{\rho\sigma})$$

$$R_{\rho\sigma} - \tfrac{1}{2}\delta_{\rho\sigma}R$$

$$= (R_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}R + \Lambda_{cc}\delta_{\mu\nu})\,F^{\mu\nu\rho\sigma}\,(R_{\rho\sigma} - \tfrac{1}{2}\delta_{\rho\sigma}R + \Lambda_{cc}\delta_{\rho\sigma})$$

$$\tfrac{1}{4}(\alpha\delta^{\mu\rho}\delta^{\nu\sigma} + \beta\delta^{\mu\nu}\delta^{\rho\sigma})\gamma^{\mu}$$

$$= R_{\mu\nu}F^{\mu\nu\rho\sigma}R_{\rho\sigma} - \tfrac{1}{2}R_{\mu\nu}F^{\mu\nu\rho\sigma}\delta_{\rho\sigma}R + R_{\mu\nu}F^{\mu\nu\rho\sigma}\Lambda_{cc}\delta_{\rho\sigma}$$

$$-\tfrac{1}{2}\delta_{\mu\nu}R\,F^{\mu\nu\rho\sigma}R_{\rho\sigma} + \tfrac{1}{4}\delta_{\mu\nu}R\,F^{\mu\nu\rho\sigma}\delta_{\rho\sigma}R - \tfrac{1}{2}\Lambda_{cc}\delta_{\mu\nu}RF^{\mu\nu\rho\sigma}\delta_{\rho\sigma}$$

$$+\Lambda_{cc}\delta_{\mu\nu}F^{\mu\nu\rho\sigma}R_{\rho\sigma} - \tfrac{1}{2}\Lambda_{cc}\delta_{\mu\nu}F^{\mu\nu\rho\sigma}\delta_{\rho\sigma}R + \Lambda_{cc}\delta_{\mu\nu}F^{\mu\nu\rho\sigma}\Lambda_{cc}\delta_{\rho\sigma}$$

$$= R_{\mu\nu}\gamma^{\mu}R^{\mu\nu} + \beta R\gamma^{\mu}R - \tfrac{1}{2}R_{\mu\nu}\delta^{\mu\rho}\delta^{\nu\sigma}\gamma^{\mu}_{\rho\sigma}R - \tfrac{1}{2}R_{\mu\nu}\beta\delta^{\mu\nu}\delta^{\rho\sigma}\delta_{\rho\sigma}R$$

$$\int^{\mu\nu} \nabla_\mu \nabla_\nu \phi = S^{-1} \hat{\int}^{\mu\nu} \nabla_\mu^{(1)} \nabla_\nu^{(1)} \phi$$

— — — .

$$(G_{\mu\nu} + \Lambda_{cc} g_{\mu\nu}) \gamma^\mu \underbrace{\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)}_{G^{\mu\nu}} + \frac{1}{2} g^{\mu\nu} R - \Lambda_{cc} g^{\mu\nu}$$

$$(G_{\mu\nu} + \Lambda_{cc} g_{\mu\nu}) \gamma^\mu \left(G^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R - \Lambda_{cc} g^{\mu\nu}\right) =$$

$$= E_{\mu\nu} \gamma^\mu \left(E^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R - \Lambda_{cc} g^{\mu\nu}\right)$$

$$E_{\mu\nu} = G_{\mu\nu} + \Lambda_{cc} g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_{cc} g_{\mu\nu}$$

$$E = R - \frac{1}{2} \cdot 4 R + 4 \Lambda_{cc} = -R + 4\Lambda_{cc}, \quad E - 4\Lambda_{cc} = -R$$
$$R = 4\Lambda_{cc} - E$$

$$R = E + 4\Lambda_{cc}$$

$$\Rightarrow E_{\mu\nu} \gamma^{(\mu} \left(E^{\mu)\nu} + \frac{1}{2} g^{\mu)\nu} (-E + 4\Lambda_{cc}) - 4\Lambda_{cc} g^{\mu)\nu}\right)$$

$$= E_{\mu\nu} \gamma^{(\mu} \left(E^{\mu)\nu} - \frac{1}{2} g^{\mu)\nu} E^\alpha{}_\alpha + 2\Lambda_{cc} g^{\mu)\nu} - 4\Lambda_{cc} g^{\mu)\nu}\right)$$

$$\| \qquad E^{\mu\nu} - \frac{1}{2} g^{\mu\nu} E^\alpha{}_\alpha - 2\Lambda_{cc} g^{\mu\nu}$$

$$= E_{\mu\nu} F^{\mu\nu\rho\sigma} E_{\rho\sigma} - 2\Lambda_{cc} E_{\mu\nu} \gamma^{(\mu} g^{\mu)\nu}$$

$$F^{\mu\nu\rho\sigma} = \gamma^{(\mu} \cdot \left(g^{\mu)\rho} g^{\nu\sigma} - \frac{1}{2} g^{\mu)\nu} g^{\rho\sigma}\right)$$

$$E_{\mu\nu} = G_{\mu\nu} + \Lambda_{cc} g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_{cc} g_{\mu\nu}$$

$$E^{\alpha}{}_{\alpha} = R^{\alpha}{}_{\alpha} - \frac{1}{2} \cdot 4 R^{\alpha}{}_{\alpha} + 4 \Lambda_{cc}$$

$$E = - R + 4 \Lambda_{cc} \quad \Rightarrow \boxed{R = -E + 4\Lambda_{cc}}$$

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (-E + 4\Lambda_{cc}) + \Lambda_{cc} g_{\mu\nu}$$

$$= R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} E - 2\Lambda_{cc} g_{\mu\nu} + \Lambda_{cc} g_{\mu\nu}$$

$$= R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} E - \Lambda_{cc} g_{\mu\nu}$$

$$\boxed{R_{\mu\nu} = E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E + \Lambda_{cc} g_{\mu\nu}} \qquad -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$$

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E + \cancel{\Lambda_{cc} g_{\mu\nu}} - \frac{1}{4} g_{\mu\nu} (-E + \cancel{4\Lambda_{cc}})$$

$$= E_{\mu\nu} - \frac{1}{4} g_{\mu\nu} E$$

$$\mathcal{L} = -2\kappa_4^{-2} \sqrt{g} \Big[R - 2\Lambda_{cc} + \Big(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \Big) \gamma_{S2}^{1}(\Box, R) \Big(R^{\mu\nu} - \frac{1}{4} g^{\mu\nu} R \Big)$$

$$+ R \gamma_S R + V(\Box, R_{ic}, R) \Big]$$

$$= -2\kappa_4^{-2} \sqrt{g} \Big[R - 2\Lambda_{cc} + \Big(E_{\mu\nu} \mp \frac{1}{4} g_{\mu\nu} E \Big) \gamma_{S2}(\Box, R) \Big(E^{\mu\nu} \mp \frac{1}{4} g^{\mu\nu} E \Big) +$$

$$+ (E - 4\Lambda_{cc}) \gamma_S(\Box, R) (E - 4\Lambda_{cc}) + V(\Box, E) \Big]$$

$$= -2\kappa_4^{-2} \sqrt{g} \Big[R - 2\Lambda_{cc} + E_{\mu\nu} \gamma_{S2} E^{\mu\nu} \mp \frac{1}{4} E \gamma_{S2} E \cdot 2 + \frac{1}{16} \cdot 4 E \gamma_{S2} E$$

$$+ E \gamma_S E + \text{TOT-DER} + V(\Box, E_{\mu\nu}, E) \qquad -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$$

$$= -2\kappa_4^{-2} \sqrt{g} \Big[R - 2\Lambda_{cc} + E_{\mu\nu} \Big[\gamma_{S2}(\Box, R) \Big(g^{\mu\rho} g^{\nu\sigma} - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} \Big) + g^{\mu\nu} g^{\rho\sigma} \gamma_S \Big] E_{\rho\sigma}$$

$$+ \text{TOT-DER} + V(\Box, E_{\mu\nu}, E) \Big]$$

$\Lambda_{\alpha\beta} = 0$

$\gamma_0 = \frac{1}{\zeta} \frac{e^{H_0} - 1}{\Box}$ $\quad \gamma_2 = \frac{e^{H_2} - 1}{\Box}$

$E_{\mu\nu} = 0 \qquad E_{\mu\nu} = R_{\mu\rho} \, \mathcal{O}^{\rho}{}_{\nu}$

$g_{\mu\nu} R = g_{\mu\nu} g^{\alpha\beta} R_{\alpha\beta}$

$$\left(R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R\right) \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} - \frac{1}{2} g_{\mu\nu} R \, \gamma_0 \, g^{\alpha\beta} \underbrace{R_{\alpha\beta}}$$

$$-\frac{1}{2} g_{\mu\nu} R^{\alpha\beta} \gamma_2 \underbrace{R_{\alpha\beta}} + 2 \frac{\delta R}{\delta g^{\mu\nu}} \gamma_0 \, g^{\alpha\beta} \underbrace{R_{\alpha\beta}}$$

$$+ \frac{\delta R^{\alpha\beta}}{\delta g^{\rho\sigma}} \gamma_2 \underbrace{R_{\alpha\beta}} + \frac{\delta \Box^{\sigma}}{\delta g^{\rho\nu}} \frac{\gamma_2(\Box^{\ell}) - \gamma_0(\Box^{\sigma})}{\Box^{\sigma} - \Box^{\ell}} R \, g^{\alpha\beta} \underbrace{R_{\alpha\beta}}$$

$$+ \frac{\delta \Box^{\sigma}}{\delta g^{\rho\nu}} \frac{\gamma_2(\Box^{\ell}) - \gamma_2(\Box^{\sigma})}{\Box^{\sigma} - \Box^{\ell}} R^{\alpha\beta} \underbrace{R_{\alpha\beta}}$$

$$= \Big\{ \delta^{\alpha}_{\rho} \delta^{\beta}_{\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} - \frac{1}{2} g_{\mu\nu} R \, \gamma_0(\Box) \, g^{\alpha\beta}$$

$$-\frac{1}{2} g_{\mu\nu} R^{\alpha\beta} \gamma_2(\Box) + 2 \frac{\delta R}{\delta g^{\mu\nu}} \gamma_0(\Box) \, g^{\alpha\beta} +$$

$$+ \frac{\delta R^{\alpha\beta}}{\delta g^{\mu\nu}} \gamma_2(\Box) + \frac{\delta \Box^{\sigma}}{\delta g^{\mu\nu}} \frac{\gamma_0(\Box^{\ell}) - \gamma_0(\Box^{\sigma})}{\Box^{\sigma} - \Box^{\ell}} R \, g^{\alpha\beta} +$$

$$+ \frac{\delta \Box^{\sigma}}{\delta g^{\mu\nu}} \frac{\gamma_2(\Box^{\ell}) - \gamma_2(\Box^{\sigma})}{\Box^{\sigma} - \Box^{\ell}} R^{\alpha\beta} \Big\} R_{\alpha\beta} = 0$$

$\Rightarrow E_{\mu\nu} = \mathcal{O}_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta}$ $\qquad E_{\mu\nu} = 0 \Leftarrow R_{\alpha\beta} = 0$

$E_{\mu\nu} = 0 \Rightarrow R_{\alpha\beta} = 0$?

($\mathcal{O}_{\mu\nu}{}^{\alpha\beta}$ simile a $H^{\alpha\beta}_{\gamma\delta} \delta^{\gamma}_{\mu} \delta^{\delta}_{\nu}$)

Voglio trovare γ_0 e γ_2 in modo che l'unica soluzione sia $R_{\alpha\beta} = 0$ cioè $E_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$. Questo è un vincolo non perturbativo

Dato che $R_{vp} = 0$ è soluzione anche $\Rightarrow$ lo zero vanziane

sicuramente il problema anche a livello non perturbativo

ORDERING ISSUE

$-\frac{1}{6}+\frac{2}{3}$

$\frac{-1+4}{6} = \frac{3}{6} = \frac{1}{2}$

$\gamma_c = 0$

spin 0

$$1 - \frac{2}{\pi\rho^2}\left[\left(6\Box + 2\bar{R}\right) f_1(\Box) + \frac{1}{2}\Box f_2\left(\Box + \frac{2}{3}\bar{R}\right)\right]$$

$R = comb = 4\Lambda \Rightarrow \left[\left(\alpha\Box + \beta\bar{R}\right), \left(\alpha'\Box + \beta'\bar{R}\right)\right] = 0$ perché $\Box\bar{R} = 0$

$$f_1(\Box) = -\frac{M_p^2}{2}\frac{1}{G\left(\Box+\frac{R}{3}\right)}\left(e^{H_1} - 1\right) - \frac{M_p^2\Box}{2}\frac{1}{2G\left(\Box+\frac{R}{3}\right)}\frac{1}{\Box+\frac{R}{2}}\left(e^{H_2\left(\left(\Box+\frac{R}{2}\right)\left(\Box+\frac{R}{3}\right)\right)} - 1\right)$$

$$\frac{1}{\left(\Box+\frac{R}{3}\right)\left(\Box+\frac{R}{2}\right)}\left(e^{H_2\left[\left(\Box+\frac{R}{3}\right)\left(\Box+\frac{R}{2}\right)\right]} - 1\right)$$

2

$$f_2(\Box) = \frac{1}{\Box - \frac{R}{6}}\left(e^{H_2\left(\left(\Box-\frac{R}{6}\right)\left(\Box-\frac{R}{3}\right)\right)} - 1\right)$$

$P^{(2)}_{\gamma+1} = P^{(2)}_4 = \left(\Box - \frac{R}{6}\right)^2\left(\Box - \frac{R}{3}\right)^2$

- **Notice**

$$(\Box+\alpha R)(\Box+\beta R) = \Box^2 + \beta\Box R + \alpha R\Box + \alpha\beta R^2 \neq (\Box+\beta R)(\Box+\alpha R) = \Box^2 + \alpha\Box R + \beta R\Box + \alpha\beta R^2$$

$\alpha \neq \beta$

se $\alpha = \beta$.

$$\frac{1}{\Box}\Box\left(\Box+\alpha R\right) f \overset{?}{=} \left(\Box+\alpha R\right) f$$

$$\frac{1}{(x\Box+y)}(x\Box+y)^2\left(\Box+\alpha R\right) f \overset{?}{=} (x\Box+y)\left(\Box+\alpha R\right) f \quad ?$$

$\gamma = 3$

$$P_{\gamma+1} = \left(\Box+\frac{R}{3}\right)^2\Box^2 \Rightarrow 2 = \left(\Box+\frac{R}{3}\right)$$

$$P^2_{\gamma+1} = \left(\Box+\frac{R}{3}\right)^4\Box^4$$

P_{3+1}

$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu}$ $\quad G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$

$= -\left(\frac{\Lambda D}{D-2} - \Lambda\right) g_{\mu\nu}$ $\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} \Rightarrow R - \frac{1}{2} D R = -\Lambda D$

$= \frac{\Lambda(D-4+2)}{D-2} g_{\mu\nu}$ $\quad R\left(\frac{2-D}{2}\right) = -\Lambda D$

$$\left[(\Box + \alpha R)(\Box + \beta R)\right] f = \Box^2 f + \beta \Box R f + \alpha R \Box f + \alpha\beta R^2 f \Big| R = \frac{2\Lambda D}{D-2}$$

$$\left[(\Box + \beta R)(\Box + \alpha R)\right] f = \Box^2 f + \alpha \Box R f + \beta R \Box f + \beta\alpha R^2 f$$

come uguali $\Rightarrow \alpha = \beta$

$$\left[(\Box + \alpha R)(\Box + \beta R)(\Box + \gamma R)\right] f =$$

$$= \left(\Box^2 + \beta \Box R + \gamma R \Box + \alpha\beta R^2\right)\left(\Box f + \gamma R f\right)$$

$$= \Box^3 f + \gamma \Box^2 (R f) + \beta \Box (R \Box f) + \beta\gamma \Box (R R f)$$

$$+ \alpha R \Box (\Box f) + \alpha\gamma R \Box (R f) + \alpha\beta R^2 \Box f + \alpha\beta\gamma R^3 f$$

Ora calcolo:

$$(\Box + \alpha R)\left(\Box^2 f + \gamma \Box (R f) + \beta R (\Box f) + \beta\gamma R^2 f\right)$$

$$= \Box(\Box^2 f) + \gamma \Box(\Box(R f)) + \beta \Box(R(\Box f)) + \beta\gamma \Box(R^2 f)$$

$$+ \alpha R \Box^2 f + \alpha\gamma R \Box (R f) + \alpha\beta R R (\Box f) + \alpha\beta\gamma R^3 f$$

E' associativo !! $\leftarrow$ sono uguali anche questi: $\Box^2(Rf) \equiv \Box(\Box(Rf))$

$$(\Box + \alpha R)(\Box + \beta R) f = \Box^2 f + \beta \Box (R f) + \alpha R (\Box f) + \alpha\beta R^2 f$$

$$(\Box + \beta R)(\Box + \alpha R) f = \Box^2 f + \alpha \Box (R f) + \beta R (\Box f) + \alpha\beta R^2 f$$

sono uguali per $\alpha = \beta$

Auxiliary fields

$2\kappa_4^{-2} = \frac{M_p^2}{2}$ (Hpar Stuckelberg)

$$\mathcal{L} = 2\kappa_4^{-2}\left[R + C\,\gamma_c\,C + R\,\gamma_s\,R\right]$$

$$\gamma_c = \frac{1}{2}\,\frac{e^{H_2}-1}{\Box} \qquad \gamma_s = -\frac{1}{6}\,\frac{e^{H_0}-1}{\Box}$$

$$\frac{1}{6m^2} = -\frac{1}{6}\,\frac{e^{H_0}-1}{\Box}$$

$$\frac{1}{2M^2} = \frac{1}{2}\,\frac{e^{H_2}-1}{\Box}$$

$$\left[\left(\frac{1}{3m^2}\right)\phi\right]$$

$$\mathcal{L} = \frac{M_p^2}{2}\left[\left(1 + \phi\,\frac{1}{3m^2}\right)R - \frac{1}{6}\,\phi\,\frac{1}{m^2}\,\phi + C\,\frac{1}{2M^2}\,C\right]$$

$$\frac{\delta\Sigma'}{\delta\phi} = 0 \Rightarrow \frac{1}{3m^2}\cdot R - \frac{1}{3}\,\frac{1}{m^2}\,\phi \Rightarrow \frac{e^{H_0}-1}{\Box}(R-\phi) = 0 \Rightarrow \phi = R$$

$$\phi = R + \varphi \quad \text{s.t.} \quad \left(\frac{e^{H_0}-1}{\Box}\right)\varphi = 0$$

$$\phi = R + \sum_{m=0}^{\infty} c_m\,\delta(\Box - \lambda_m)$$

$$\delta(k^2+m^2)\,\tilde{\varphi}(k)$$

$$\delta\rho \Rightarrow \frac{1}{\left(\frac{1}{3m^2}\phi\right)+1}\,\tilde{\delta\rho}$$

$$= \frac{1}{e^{\gamma}-1+1}\,\tilde{\delta\rho} = e^{-\gamma}\,\tilde{\delta\rho}$$

$$= \delta\rho$$

$$\phi = 3m^2\left(e^{\gamma}-1\right)$$

$$m = m(\Box)$$

$$\frac{1}{3m(\Box)}\,\phi = e^{\gamma}-1$$

$$\int d^4x \int d^4k_1 \int d^4k_2\, e^{i(k_1+k_2)x}\,\tilde{\phi}(k_1)\,\frac{1}{m^2(\Box)}\,\tilde{\phi}(k_2)\,e^{\dots}$$

$$= +\int d^4k_1\,\tilde{\phi}(-k_1)\,\frac{e^{H_0(k_1)}-1}{k_1^2}\,\tilde{\phi}(k_1)$$

$$= \int d^4k\,\tilde{\phi}(-k)\,\frac{e^{H_0(k)}-1}{k^2}\sum_{m=0}^{\infty} c_m\,\delta(k^2+m^2)$$

$$\equiv 0$$

$$\frac{\rho^{(2)}}{\kappa^2\left(\frac{1}{2}\right)} = \frac{\rho^{(2)}}{\kappa^2\left(1-\frac{1}{2}\right)}$$

$$\frac{\rho^{(2)}}{\kappa^2} + \frac{\rho^{(2)}}{\kappa^2}$$

?

2 gravitons

Se anche $e^{H_0}=\frac{1}{2}$

$$\frac{1}{m^2} = -\frac{e^{H_0}-1}{\Box}$$

$$\frac{1}{M^2} = \frac{e^{H_2}-1}{\Box}$$

$$\frac{\Box}{(e^{H_0}-1)\Box} = \frac{\Box}{\Box(e^{H_0}-1)}$$

$$\Box\varphi + m^2\varphi = 0$$

$$\Box\varphi + \frac{\Box}{e^{\Box}-1}\varphi \qquad \Box\left(\frac{e^{\Box}-1}{e^{\Box}-1}\right)$$

$$\mathcal{L} = \frac{M_p^2}{2}\Big[\tilde{R} - \frac{3}{2}(\tilde{\partial}\varphi)^2 - \frac{3}{2}e^{-2\varphi}(e^{\varphi}-1)m^2(\Box)(e^{\varphi}-1)$$

$$(\partial\varphi)^2 + \varphi m^2\varphi$$

$$+ \tilde{C}\,\frac{1}{2M^2(\Box)}\,\tilde{C}\Big]$$

$\underline{\gamma_C = 0}$:

$$\Rightarrow \frac{M_p^2}{2}\Big[R - \frac{3}{2}(\partial\varphi)^2 - \frac{3}{2}\underbrace{e^{-2\varphi}(e^{\varphi}-1)}_{e^{-\varphi}(e^{-\varphi}-1)}m^2(\Box)(e^{\varphi}-1)\Big]$$

$$\approx \Big[R - \frac{3}{2}(\partial_\mu\varphi\partial^\mu\varphi) - \frac{3}{2}(1-2\varphi)(\varphi+\varphi-\varphi)m^2(\Box)(\varphi+\varphi-\varphi)\Big]$$

$$\Big[R - \frac{3}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{3}{2}\varphi m^2(\Box)\varphi\Big\}$$

$$\Big[R + \frac{3}{2}\varphi\Box\varphi + \frac{3}{2}\varphi\frac{\Box}{e^{H_0}-1}\varphi\Big]$$

$$\Big[R + \frac{3}{2}\varphi\Big(\Box + \frac{\Box}{e^{H_0}-1}\Big)\varphi\Big] \qquad \varphi\Box\Big(\frac{e^{H}-1+1}{e^{H}-1}\Big)\varphi$$

$$\Big[R + \frac{3}{2}\varphi\Big[\frac{(e^{H_0}-1)\Box + \Box}{e^{H_0}-1}\Big]\varphi\Big] \qquad \frac{\varphi\Box\left[e^{H}(1-\frac{\Box}{M^2})\right]\varphi}{e^{H}(1-\frac{\Box}{M^2})-1}$$

$$\varphi\Big(\frac{\Box\, e^{H_0}}{e^{H_0}-1}\Big)\varphi \qquad \frac{H(0)=0}{H_0(p)}$$

$$G_\varphi(p) = \frac{e^{H_0(p)}-1}{(-p^2)\,e^{H_0(p)}}$$

$$\frac{\delta S}{\delta\varphi} = 0 \Rightarrow \Big|\ \varphi\Big(\frac{\Box}{1-e^{-H_0}}\Big)\varphi \qquad H_0(0) > 0 \Rightarrow \text{ok ghostphoton}$$

$\underline{L.W}$: $\varphi\Box\frac{(1+z^2)}{1+z^2/1}\varphi = \varphi\Big(\frac{1}{\Box}+\Box\Big)\varphi$

Starobinsky:

$$\frac{1+\Box^2}{\Box}$$

Notice $\sqrt{g}\, C_{abc}{}^{d}\, \gamma_{(0)}\, C_{\alpha\rho\gamma}{}^{\delta} \left(s^{a\alpha} s^{b\beta} s^{c\gamma} s_{d\delta} \right.$

$\delta_\mu = x^\nu \tilde{g}_{\mu\nu} \Rightarrow$ $s^{\alpha\gamma} \sqrt{\tilde{g}}\, x^\alpha x^\beta x^\gamma x^\delta\, \tilde{g}^{a\alpha} \tilde{g}^{b\beta} \tilde{g}^{c\gamma} \tilde{g}_{d\delta} \cdot$

$\cdot \tilde{C}_{abc}{}^{d}\, \gamma_{(0)}\, \tilde{C}_{\alpha\rho\gamma}{}^{\delta}$

$\gamma_c' \neq 0$ $= \tilde{g}^{a\alpha} \tilde{g}^{b\beta} \tilde{g}^{c\gamma} \tilde{g}_{d\delta}\, \tilde{C}_{abc}{}^{d}\, \gamma_{(0)}\, \tilde{C}_{\alpha\rho\gamma}{}^{\delta}$ quindi è invariante.

$\cdot \sqrt{g}\, \tilde{C}^{\alpha\rho\gamma}{}_{\delta}\, \gamma_{(0)}\, \tilde{C}_{\alpha\rho\gamma}{}^{\delta} \rightsquigarrow \tilde{g}^{\mu\nu} \sqrt{g}\left(\tilde{R}_{\mu\nu} - \gamma_{(0)} \tilde{R}^{\mu\nu} - \frac{1}{3} \tilde{R}\, \gamma_{(0)}\, \tilde{R} \right)$

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{g} \left[\tilde{R} - \frac{3}{2} \left(\tilde{\nabla} \gamma \right)^2 - \frac{3}{2} e^{-2\gamma} \left(e^{\gamma} - 1 \right) M^2(\tilde{\Box}) \left(e^{\gamma} - 1 \right) \right]$$

$$- \sqrt{g}\, \tilde{\Pi}^{\mu\nu} \tilde{G}_{\mu\nu} + \sqrt{g}\, \frac{1}{4} \left(\tilde{\Pi}_{\mu\nu} M^2(\Box) \tilde{\Pi}^{\mu\nu} - \tilde{\Pi} M^2(\Box) \tilde{\Pi} \right) \Big]$$

$e^{2\gamma}\sqrt{g}$; $e^{-2\gamma} \tilde{g}^{\mu\nu}$; $(e^{-\gamma} \tilde{g}^{\mu\nu} \nabla_\mu \nabla_\nu)$; $e^{2\gamma} \tilde{g}_{\mu\nu}$ file: superspace sul gravity

$$\frac{\delta S}{\delta \tilde{\Pi}^{\mu\nu}} = 0 \rightarrow \tilde{G}_{\mu\nu} = e^{2\gamma} \frac{1}{2} M^2(\Box) \left(\tilde{\Pi}_{\mu\nu} - g_{\mu\nu} \tilde{\Pi} \right), \quad \tilde{\nabla}^\mu \tilde{G}_{\mu\nu} = 0$$

$$\tilde{\Pi}_{\mu\nu} = e^{-2\gamma} 2\, M^{-2}(\Box) \left(\tilde{R}_{\mu\nu} - \frac{1}{6} g_{\mu\nu} \tilde{R} \right)$$

$$= e^{-2\gamma} 2 \left(\frac{e^{M_2} - 1}{\Box} \right) \left(\tilde{R}_{\mu\nu} - \frac{1}{6} g_{\mu\nu} \tilde{R} \right)$$

$-\frac{1}{6} - \frac{1}{3} = -\frac{3}{6} - \frac{2}{6} = -\frac{1}{c}$

$\tilde{\Pi}^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} \tilde{\Pi}_{\rho\sigma}$ $\quad \tilde{\Pi} = g^{\mu\nu} \tilde{\Pi}_{\mu\nu} = e^{-2\gamma} 2c\, M^{-2}(\Box) \tilde{R} - e^{-\gamma} \frac{1}{6} c\, \tilde{R} = e^{-\gamma} 2 M^{-2} \tilde{R} \left(1 - \frac{3}{2} \right) = -\frac{e^{-\gamma}}{2} M^{-2} \tilde{R}$

$$S_{RC} = \frac{M_P^2}{2} \int d^4x \sqrt{g} \left\{ \left[\left(1 + \frac{1}{2} \tilde{\Pi} \right) \tilde{g}^{\mu\nu} - \tilde{\Pi}^{\mu\nu} \right] \tilde{R}_{\mu\nu} + \frac{1}{4} \left(\tilde{\Pi}_{\mu\nu} M^2 \tilde{\Pi}^{\mu\nu} - \tilde{\Pi} M^2 \tilde{\Pi} \right) \right\}$$

$\sqrt{\bar{g}}\, \bar{g}^{\mu\nu}$

$$\tilde{g}_{\mu\nu} = \sqrt{\frac{1}{\det A}}\, A_\mu{}^\alpha\, \bar{g}_{\alpha\nu}, \qquad A_\mu{}^\alpha = \left(1 + \frac{1}{2} \tilde{\Pi} \right) \delta_\mu{}^\alpha - \tilde{\Pi}_\mu{}^\alpha,$$

$$\tilde{R}_{\mu\nu} = \bar{R}_{\mu\nu} - \bar{\nabla}_\rho C^\rho_{\alpha\nu} + \bar{\nabla}_\alpha C^\alpha_{\mu\nu} + C^\alpha_{\mu\nu} C^\beta_{\alpha\beta} - C^\alpha_{\mu\beta} C^\beta_{\nu\alpha},$$

$$C^\alpha_{\mu\nu} = \frac{1}{2} \left(X^{-1} \right)^{\alpha\beta} \left(\bar{\nabla}_\mu X_{\nu\beta} + \bar{\nabla}_\nu X_{\mu\beta} - \bar{\nabla}_\beta X_{\mu\nu} \right),$$

$$X_{\mu\nu} = \tilde{g}_{\mu\nu} = \sqrt{\frac{1}{\det A}}\, A_\mu{}^\alpha\, \bar{g}_{\alpha\nu}.$$

$$\Gamma^{(2)}_{k(c_3)} = \int d^4x \sqrt{\bar{g}} \Big[\frac{1}{4} \big(\bar{\tilde{\nabla}}^2 \tilde{\bar{\Pi}}^{\mu\nu}\big)\big(\bar{\tilde{\nabla}}_\alpha \tilde{\bar{\Pi}}_{\mu\nu}\big) - \frac{1}{2}\big(\bar{\tilde{\nabla}}^\alpha \tilde{\bar{\Pi}}^{\mu\nu}\big)\big(\bar{\tilde{\nabla}}_\mu \tilde{\bar{\Pi}}_{\nu\alpha}\big)$$

$$+ \frac{1}{2}\big(\bar{\tilde{\nabla}}_\mu \bar{\Pi}^{\mu\nu}\big)\big(\bar{\tilde{\nabla}}_\nu \bar{\Pi}\big) - \frac{1}{4}\big(\bar{\tilde{\nabla}}^\alpha \bar{\Pi}\big)\big(\bar{\tilde{\nabla}}_\alpha \bar{\Pi}\big)$$

$$- \frac{1}{4}\Big(\tilde{\bar{\Pi}}^{\mu\nu} \underset{\mu}{M^2}(\Box) \tilde{\bar{\Pi}}_{\mu\nu} - \tilde{\bar{\Pi}} M^2(\Box) \tilde{\bar{\Pi}}\Big) \Big]$$

$$\frac{e^{H_2}-1}{\Box}$$

$\Box \approx \bar{\nabla}\bar{\nabla}$ at the second order in the fields.

Scelta:
$\bar{\nabla}_\mu \tilde{\bar{\Pi}}^{\mu\nu} = 0$

↑

Verificare la simmetria.

$$- \frac{1}{4}\Big(\tilde{\bar{\Pi}}_{\mu\nu} \tilde{\Box} \bar{\Pi}^{\mu\nu} - \tilde{\bar{\Pi}} \tilde{\Box} \tilde{\bar{\Pi}}\Big) +$$

$$- \frac{1}{4}\Big(\tilde{\bar{\Pi}}_{\mu\nu} \frac{\tilde{\Pi}}{e^{H_2}-1} \tilde{\bar{\Pi}}^{\mu\nu} - \tilde{\bar{\Pi}} \frac{\tilde{\Box}}{e^{H_2}-1} \tilde{\bar{\Pi}}\Big) =$$

$$= -\frac{1}{4}\Big[\tilde{\bar{\Pi}}_{\mu\nu} \frac{e^{H_2}\tilde{\Box}}{e^{H_2}-1} \tilde{\bar{\Pi}}^{\mu\nu} - \bar{\Pi} \frac{e^{H_2}\tilde{\Box}}{e^{H_2}-1} \bar{\Pi}\Big]$$

Propagatore $\sim \quad \frac{1}{\Box} \big(1 - e^{-H_2(\Box)}\big)$

$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$

$$\mathrm{Res}\,(\ )\Big|_{p^2=0} = 0$$

$$G_{\mu\nu} = \frac{1}{2} M^2(\Box)\left(\Pi_{\mu\nu} - \rho_{\mu\nu}\Pi\right) \qquad \nabla^\mu\left(M^2(\Box)\,\Pi_{\mu\nu}\right) = \nabla_\nu\left(M^2(\Box)\,\Pi\right)$$

$$\nabla^\mu G_{\mu\nu} = 0 = \begin{cases} \nabla^\mu\left[M^2(\Box)\left(\Pi_{\mu\nu} - \rho_{\mu\nu}\Pi\right)\right] = 0 & \text{①} \\ \nabla^\mu\nabla^\nu\left[M^2(\Box)\,\Pi_{\mu\nu}\right] - \Box\, M^2(\Box)\,\Pi = 0 & \text{②} \end{cases}$$

$$\nabla^\nu\left(M^2(\Box)\,\rho_{\mu\nu}\,\Pi\right)$$

$$\text{②}: \quad \nabla^\mu\nabla^\nu\rho_{\mu\nu}\,M^2(\Box)\,\Pi - \Box\, M^2(\Box)\,\Pi = 0$$

$$= \Box\, M^2(\Box)\left(\Pi - \Pi\right) \quad ?$$

$$M^2(\Box)\,\Pi_{\mu\nu} = 2G_{\mu\nu} + M^2(\Box)\,\rho_{\mu\nu}\Pi$$

Note: Se non voglio avere problemi nella $\mathcal{L}$, devo avere

$$e^{H} \underset{\Box\to 0}{\sim} 1+\Box \quad \Rightarrow \quad \text{Ex. 1)} \quad \frac{1}{m^2} = -\frac{e^{\Box}-1}{\Box}$$

$$\text{2)} \quad H(\Box) = \int_0^{\Box} \frac{1-e^{-\omega}}{\omega}\,d\omega \sim \int_0^{\Box} \frac{1-(1-\omega)+\dots}{\omega} = \int_0^{\Box} 1+\dots = \Box+\dots$$

$$\frac{z\,e^{mz}}{e^z-1} \qquad \frac{z}{e^{z(1-m)}-e^{-mz}}$$

$$n=1 \Rightarrow \frac{z}{1-e^{-z}} = \sum_{n=0}^{\infty} B_n \frac{(-z)^n}{n!}$$

Ricordiamo.

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

↑ Bernulli number

$$\frac{e^\omega - 1}{\omega\, e^\omega}$$

Linearized theory (File Stuckelberg Quadratic Gravity.nb)

$$S = \frac{M_p^2}{2}\int d^4x\left[-\frac{3}{2}(\partial\gamma)^2 + h^{\mu\nu}(\mathcal{E}h)_{\mu\nu} - f^{\mu\nu}(\mathcal{E}f)_{\mu\nu} + \right.$$

$$\left. -\frac{1}{2}\left(f_{\mu\nu}M^2 f^{\mu\nu} - f M^2 f\right)\right]$$

$$(\mathcal{E}f)_{\mu\nu} = \Box f_{\mu\nu} - 2\partial_{(\mu}\partial^\rho f_{\nu)\rho} + \partial_\mu\partial_\nu f$$

$$M^2 = \int_0^{r_*} dr\, e^{-r\Box}$$

$$M^{-2} = \frac{e^H - 1}{\Box} \qquad e^H = \frac{1}{2},\ M^2 = -\frac{1}{2\Box}$$

$$\underbrace{f^{\mu\nu}\Box f_{\mu\nu}} - f\Box f + \underbrace{f^{\mu\nu}\left(\frac{e^H-1}{\Box}\right)^{-1} f_{\mu\nu}} - f\left(\frac{e^H-1}{\Box}\right)^{-1} f$$

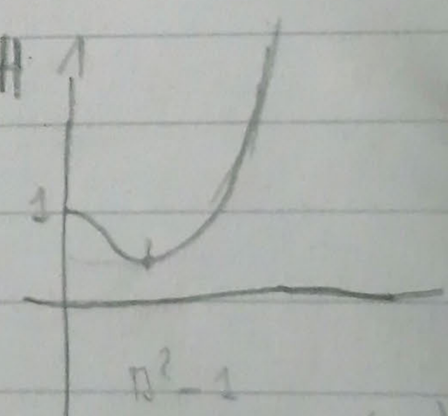
$$= f^{\mu\nu}\left(\Box + \frac{\Box}{e^H-1}\right) f_{\mu\nu} - f\left(\Box + \frac{\Box}{e^H-1}\right) f$$

$$= f^{\mu\nu}\left(\frac{\Box e^H - \cancel{\Box} + \cancel{\Box}}{e^H-1}\right) f_{\mu\nu} - f\left(\frac{\Box e^H - \cancel{\Box} + \cancel{\Box}}{e^H-1}\right) f$$

$$= f^{\mu\nu}\left(\frac{\Box\, e^H}{e^H-1}\right)\left[f_{\mu\nu} - 2\partial_{(\mu}\partial^\rho f_{\nu)\rho} f\right]$$

$$= f^{\mu\nu}\left(\frac{\Box}{1-e^{-H}}\right)\left[f_{\mu\nu} - \ldots f\right]$$

$$e^{-H} > 1 \qquad e^{+H} = 1 + \Box^2$$



$$e^H = \frac{1}{\omega^2 - 1} \Rightarrow \frac{\Box\, e^H}{e^H - 1} = \frac{\Box(1-\Box)}{-\Box + \Box^2 - 1}$$

$$= -\omega^2\left(\frac{\omega^2 - \Box}{\omega^2}\right)$$

$$= \Box - \omega^2$$

$$\boxed{e^H < 1}$$

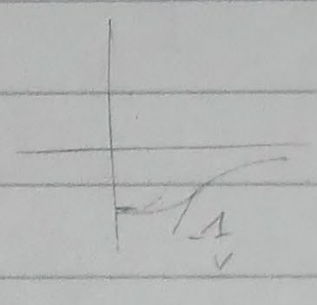
$$\mathcal{L} = \sqrt{g}\left[R + 2\,R^{\mu\nu}\,F\,\varphi_{\mu\nu} + \left(\varphi^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}\varphi\right) G(\Box)\,\varphi_{\mu\nu}\right]$$

$$2\left(R^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}R\right) F\,\varphi_{\mu\nu} + g^{\mu\nu}R\,\varphi_{\mu\nu}$$

$$R + 2G_{\mu\nu}\,F\,\varphi^{\mu\nu} + \left(\varphi^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}(\varphi - 2R)\right) G(\Box)\,\varphi_{\mu\nu}$$

$$\mathcal{L} = \sqrt{g}\left[\left(g^{\mu\nu} + 2\,F(\Box)\,\varphi^{\mu\nu}\right) R_{\mu\nu} + \left(\varphi^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}\varphi\right) G(\Box)\,\varphi_{\mu\nu}\right]$$

$$V_e \qquad e^{-2\gamma}\left(e^{\gamma} - 1\right)\,m^2(\Box)\,\left(e^{\gamma} - 1\right)$$



$$m^2(\Box) = \frac{\Box}{e^{\Box} - 1}$$

$$\left(e^{-(\gamma_0+\varphi)} - e^{-(\gamma_0+\varphi)}\right)\,m^2\left(e^{(\gamma_0+\varphi)} - 1\right)$$

$$1 - (\gamma_0+\varphi) + \ldots \qquad \left(1 + (\gamma_0+\varphi) + \tfrac{1}{2}(\varphi_0+\varphi)^2 + \ldots - 1\right)$$

φ^2 order: $e^{-2\gamma_0}\left(e^{\gamma_0} - 1\right)\,m^2(\Box)\left(\tfrac{1}{2}\,e^{\gamma_0}\,\varphi^2\right) +$

$$- e^{-2\gamma_0}\left(-2 + e^{\gamma_0}\right)\varphi\,m^2(\Box)\left(e^{\gamma_0}\,\varphi\right) +$$

$$+ \tfrac{1}{2}\,e^{-2\gamma_0}\left(-4 + e^{\gamma_0}\right)\varphi^2\,m^2(\Box)\left(e^{\gamma_0} - 1\right)$$

LW: $f^{\nu}\left(\frac{\Box\,(1+\Box^2)}{\Box^2}\ldots\right) = f^{\nu}\left(\frac{1+\Box^2}{\Box}\right)\ldots \Rightarrow G(\rho) = \frac{\rho^2}{1+\rho^4}$

EOM φ: $\quad m^2(\Box) = \frac{\Box}{e^{\Box}-1} = \frac{\Box}{\Box + \frac{1}{2}\Box^2 + \dots}$

$$\left(e^{-\varphi} - e^{-2\varphi}\right) m^2(\Box) \left(e^{\varphi} - 1\right)$$

$$\frac{\delta S}{\delta \varphi} = 0 \Rightarrow \int \left(-e^{-\varphi}\delta\varphi - (-2)e^{-2\varphi}\delta\varphi\right) m^2(\Box)\left(e^{\varphi}-1\right)$$

$$+ \left(e^{-\varphi} - e^{-2\varphi}\right) m^2(\Box)\left(e^{\varphi}\delta\varphi\right) =$$

$$= \int -\left(e^{-\varphi} - 2e^{-2\varphi}\right)\delta\varphi \; m^2(\Box)\left(e^{\varphi}-1\right) +$$

$$+ \left(m^2(\Box)\left(e^{-\varphi} - e^{-2\varphi}\right)\right) e^{\varphi}\delta\varphi$$

$$= \int \delta\varphi \left[\left(2e^{-2\varphi} - e^{-\varphi}\right) m^2(\Box)\left(e^{\varphi}-1\right) - e^{\varphi} m^2(\Box)\left(e^{-2\varphi} - e^{-\varphi}\right)\right]$$

$$\left(2(1-2\varphi) - (1-\varphi)\right) m^2(\Box)\left(\cancel{1+\varphi-1}\right) - (1+\varphi) m^2(\Box)\left(\cancel{1}-2\varphi - (\cancel{1}-\varphi)\right)$$

$$\approx m^2(\Box)\varphi + m^2(\Box)\varphi = 2m^2(\Box)\varphi$$

$$2\left(\Box\varphi + m^2(\Box)\varphi\right) = 0$$

$$\Box\varphi + \frac{\Box}{e^{\Box}-1}\varphi = 0$$

$$\frac{1}{(e^{\Box}-1)}\Box\varphi + \cancel{\Box\varphi} = 0$$

$$\frac{e^{\Box}\Box}{e^{\Box}-1}\varphi$$

$$-\frac{1}{24} - \frac{1}{24} + \frac{1}{36} + \frac{1}{36\cdot 2} + \frac{1}{36\cdot 2} = -\frac{1}{12} + \frac{2}{36} = -\frac{1}{12} + \frac{1}{18}$$

$$= -\frac{1}{36}$$

$$\frac{-1}{12}\cdot\frac{1}{2} - \frac{1}{12}\cdot\frac{1}{2} = -\frac{1}{12}$$

New basis (Back basis)

$$\mathcal{L} = -2\kappa_4^{-2}\left[R + B_{ab}\,\gamma^{(\Box)}_{B}\,B^{ab} + R\,\gamma_R\,R\right] \qquad D = 4$$

$$B_{ac} = \nabla^b \nabla^d C_{abcd} - R^{bd} C_{abcd}$$

$$B_{ab}(FRW) = 0 \qquad B_{ab}(Sch) = 0$$

$$B_{ab}(Ricci\ flat) = 0$$

$$\nabla^c B_{ac} = 0 \qquad B^a_a = 0$$

$$-\frac{1}{3} - \frac{1}{6} = -\frac{2}{6} - \frac{1}{6} = -\frac{1}{2}$$

$$B'_{ab} = \Omega^{-2} B_{ab}$$

$$B_{ab} = \frac{1}{2}\Box R_{ac} - \frac{1}{12} g_{ac}\Box R - \frac{1}{6}\nabla_c\nabla_a R + O(R^2, R\cdot C)$$

$$-\frac{1}{6}\left(\tfrac{1}{2}\Box R\, g_{ac} +$$

$$B_{ac}\,\gamma^{(\Box)}_{B}\,B^{ac} = \left(\frac{1}{2}\Box R_{ac} - \frac{1}{12} g_{ac}\Box R - \frac{1}{6}\nabla_c\nabla_a R\right)\cdot \gamma_B^{(\Box)}\cdot$$

$$\cdot\left(\frac{1}{2}\Box R^{ac} - \frac{1}{12} g^{ac}\Box R - \frac{1}{6}\nabla^c\nabla^a R\right) + O\left((R^2)^2 (R\cdot C)^2\right)$$

$$= \frac{1}{4}(\Box R_{ac})\,\gamma_B(\Box R^{ac}) - \frac{1}{24}(\Box R)\,\gamma_B(\Box R) - \frac{1}{12}(\Box R_{ac})\,\gamma_B(\nabla^c\nabla^a R)$$

$$-\frac{1}{24}(\Box R)\,\gamma_B(\Box R) + \frac{1}{12}\cdot\frac{1}{12}\cdot 4\,(\Box R)\,\gamma_a(\Box R) + \frac{1}{12}\cdot\frac{1}{6}(\Box R)\,\gamma_B(\Box R)$$

$$-\frac{1}{12}(\nabla_c\nabla_a R)\,\gamma_B(\Box R^{ac}) + \frac{1}{6}\cdot\frac{1}{12}(\Box R)\,\gamma_a(\Box R) + \frac{1}{36}(\nabla_c\nabla_a)R\,\gamma_B(\nabla^c\nabla^a R)$$

$$\left(R^{ac} - \tfrac{1}{2}R g^{ac} + \tfrac{1}{2}R g^{ac}\right) = G^{ac} + \tfrac{1}{2}R g^{ac}$$

Si sono ordinati i ∇ a destra e i $\Box$ a sinistra, ma non sono ancora contati

Al fine di ottenere il propagatore around Minkowski:

$$B_{ab}\gamma_n B^{ab} = \frac{1}{4} R_{ac}\,\Box^2\gamma_B R^{ac} - \frac{1}{36} R\,\Box^2\gamma_B R - \frac{1}{12} R\,\Box^2\gamma_B R$$

$$+ \frac{1}{36} R\,\Box^2\gamma_B R + \text{Vertexes}$$

$$= \frac{1}{4} R_{ab}\,\Box^2\gamma_0 R^{ab} - \frac{1}{12} R\,\Box^2\gamma_B R$$

$$\mathcal{L} = -2\kappa^{-2}\left[R + R_{ab}\underbrace{\left(\frac{1}{2}\Box^2\gamma_B\right)}_{\gamma_2} R^{ab} + R\underbrace{\left(\gamma_R - \frac{1}{12}\Box^2\gamma_B\right)}_{\gamma_0} R\right]$$

$$\begin{cases} 1 - \kappa^2\gamma_2 = e^{H_2} \Rightarrow \boxed{\gamma_2 = -\frac{e^{H_2}-1}{\kappa^2} = \frac{e^{H_2}-1}{\Box}} \\ 2 + \kappa^2(4\gamma_2 + 4\cdot 3\,\gamma_0) = 2e^{H_0} \end{cases}$$

$$\gamma_0 = \frac{1}{4\cdot 3}\frac{1}{\kappa^2}\left[2\left(e^{H_0}-1\right) + 4\left(e^{H_2}-1\right)\right]$$

$$\boxed{\gamma_0 = -\frac{1}{6}\frac{1}{\Box}\left[e^{H_0} - 1 + 2\left(e^{H_2}-1\right)\right]}$$

$$\frac{1}{4}\,\square^2\,\gamma_B = \frac{e^{H_2} - 1}{\square}$$

$$\gamma_B = 4\,\frac{e^{H_2} - 1}{\square^3}$$

$$\gamma_R = \gamma_0 + \frac{1}{12}\,\square^2\gamma_B = \gamma_0 + \frac{1}{12}\,\square^2\,4\left(\frac{e^{H_2} - 1}{\square^2}\right)$$

$$= \gamma_0 + \frac{1}{3}\,\frac{e^{H_2} - 1}{\square}$$

$$\Rightarrow \gamma_R = -\frac{1}{6\square}\left[e^{H_0} - 1 + 2\left(e^{H_2} - 1\right)\right] + \frac{1}{3}\,\frac{e^{H_2} - 1}{\square}$$

$$= -\frac{1}{6}\,\frac{1}{\square}\left(e^{H_0} - 1\right) - \frac{1}{3}\,\frac{e^{H_2} - 1}{\square} + \frac{1}{3}\,\frac{e^{H_2} - 1}{\square}$$

$$= -\frac{1}{6}\,\frac{1}{\square}\left(e^{H_0} - 1\right)$$

$$\mathcal{L} = -2\kappa_4^{-2}\left[R + 4B_{ab}\left(\frac{e^{H_2} - 1}{\square^3}\right)B^{ab} - \frac{1}{6}R\,\frac{e^{H_0} - 1}{\square}\,R\right]$$

$$H_2 \cong \square^M + \cdots \qquad M \geq 3 \quad \text{(dove)}$$

$D = 4$

$$P_{ab} = 2\, R_{;a;b} - 2R\,R_{ab} + \frac{1}{2} g_{ab} R^2 - 2 g_{ab} \Box R$$

$$P^a_a = 2\,\Box R - 2R^2 + 2R^2 - 8\,\Box R$$

$$= -6\,\Box R \qquad R = -\frac{1}{6}\frac{1}{\Box} P^a_a$$

Posso anche prendere Ca destra in BP posis

$$-\frac{1}{6} R \left(\frac{e^{H_0} - 1}{\Box}\right) R \longrightarrow -\frac{1}{6}\left(-\frac{1}{6}\right)^2 P \left(\frac{e^{H_0} - 1}{\Box^3}\right) P$$

$$P^a_a \to P^a_a - 6\,\overline{T}^a_a \quad (8\pi G) \Rightarrow \text{FRW è soluzione}$$

$$-6\,\Box R - 6\,T\,(8\pi G) \qquad P - R$$

$$-6\,(\Box R + T 8\pi G)$$

$$G_{\mu\nu} = T_{\mu\nu} \Rightarrow R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$$R - 2R = 8\pi G\, T$$

$$R = -8\pi G\, T$$

$$\left(R_{\mu\nu} + 8\pi\,(\nabla_\mu \phi)(\nabla_\nu \phi)\right) \frac{e^{H} - 1}{\Box} \left(R_{\mu\nu} + 8\pi\,(\nabla_\alpha \phi)(\nabla_\nu \phi)\right)$$

$$P^{\alpha\beta}$$

$$\int d^4x\, \sqrt{g}\, \delta\eta \left[\left(2e^{-2\eta} - e^{-\eta}\right) w'^2(\Omega)\left(e^{\eta}-1\right) + \right.$$

$$\left. - e^{\eta} w'^2(\Omega)\left(e^{-2\eta} - e^{-\eta}\right) \right]$$

$$\left[2e^{-2\eta} w'^2_{bulk} e^{\eta} - 2e^{-2\eta} w'^2(\Omega) - e^{-\eta} w'^2(\Omega) e^{\eta} + e^{-\eta} w'^2(\Omega) \right.$$

$$\left. + e^{+\eta} w'^2 e^{-\eta} - e^{-\eta} w'^2 e^{2\eta} \right]$$

EOM :

$$2\sqrt{g}\,\eta = 2e^{-2\eta} w'^2 e^{\eta} - e^{\eta} w'^2 e^{-2\eta}$$

$$+ e^{+\eta} w'^2 e^{-\eta} - e^{-\eta} w'^2 e^{2\eta}$$

$$- 2\left(e^{-2\eta} w'^2 + e^{2\eta} w'^2\right) + \left(e^{-\eta} + e^{\eta}\right) w'^2$$

$$\eta = \text{const} \neq 0 \Rightarrow 0 = 2\left(e^{\eta} + e^{-\eta}\right) w'^2 - \left(e^{-\eta} + e^{\eta}\right) w'^2$$

$$- 2\left(e^{-2\eta} + e^{2\eta}\right) w'^2 + \left(e^{-\eta} + e^{\eta}\right) w'^2 \neq 0$$

$$2\left(1+\eta+1-\eta\right) - \left(1-\eta+1+\eta\right) - 2\left(1-2\eta+1+2\eta\right) + \left(1-\eta+1+\eta\right) = 0$$

$$\eta = \text{const} = 0 \Rightarrow 0 = 2e^{-\eta} w'^2 - w'^2 - 2e^{-2\eta} w'^2 + e^{-\eta} w'^2$$

$$2\left(e^{-\eta} - e^{-2\eta}\right) w'^2 = 2w'^2\left(1-\eta-1+2\eta\right) = 2w'^2\eta$$

$$\eta \to -\eta :$$

$$-2\sqrt{g}\eta = 2e^{2\eta} w'^2 e^{-\eta} - e^{\eta} w'^2 e^{2\eta}$$

$$+ e^{-\eta} w'^2 e^{+\eta} - e^{\eta} w'^2 e^{-\eta}$$

$$- 2e^{2\eta} w'^2 + e^{\eta} w'^2$$

Considero l'operatore: ${}^{(4)}R \Box^{2R} R$

$$\Box^{4}R = D^i D_i R - K \nabla_m R - \nabla_m \nabla_m R + (D^i \ln N) \cdot D_i R$$

$${}^{(4)}R = K_{ij} K^{ij} - K^2 + R + \frac{2}{N} \underbrace{\nabla_\mu (n^\mu K)}_{K^2 + \nabla_n K} - \frac{2}{N} D^i D_i N$$

- $$\nabla_n R = \frac{1}{N} \left(\dot{R} - N^i \partial_i R \right)$$

$$\Box^2 R = \underbrace{D^i D_i (\Box R)}_{} - K \nabla_m (\Box R) - \nabla_m \nabla_m (\Box R) + (D^i \ln N) D_i (\Box R)$$

$$= D^i D_i (D^j D_j R) - D^i D_i (K \nabla_m R) +$$

$$- D^i D_i (\nabla_m \nabla_m R) + D^i D_i (D^j \ln N) D_j R$$

$$- K \nabla_m \left[D^i D_i R - K \nabla_m R - \nabla_m \nabla_m R + (D^i \ln N) D_i R \right]$$

$$- \nabla_m \nabla_m \left[D^i D_i R - K \nabla_m R - \nabla_m \nabla_m R + (D^i \ln N) D_i R \right]$$

$$+ (D^i \ln N) D_i \left[D^j D_j R - K \nabla_m R - \nabla_m \nabla_m R + (D^j \ln N) D_j R \right]$$

Superpronity

Bosonic Reduction

1) $\int d^8z\, E^{-1}(RR + \bar{R}\bar{R}) = \int_{M^6} \mathcal{R}^3 + \dots$

$\int d^4x\, M^2\, M M$

$\mathcal{D}i(= 0$

$\int d^6z\, E^{-1} R R \hat{\varphi}^3$

$L = 1$

$\delta^4(\sum) \int d^4\theta \int d^4p\, (D\bar{D})^{V-2} H_E$

2) $\int E^{-1}(R + \bar{R})\, d^8z$

$\int d^8z\, E^{-1} R\bar{R}\, \Box^{\gamma-1} (D^2 R)(\bar{D}^2 \bar{R})$

$D^2 \Box H \quad \bar{D}^2 \Box H$

del propagator

$R\bar{R} \int p^{2\gamma-2}\, p^4\, \frac{1}{p^{2\gamma+4}}\, d^4p\, (D\bar{D})^2 \delta^4(\theta_1 - \theta')$

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$R\bar{R} \int d^4p\, \frac{1}{p^2} = \Lambda_{ult-ff}^2\, R\bar{R}$

$\int d^8z\, E^{-1} R\bar{R}\, \Delta_R^{\gamma}\, R\bar{R}$

$\Box H \Box H$

$\delta^4(p) \int d^4\theta\, R\bar{R} \int d^4p\, \frac{1}{p^{2\gamma+4}}\, p^4\, p^{2\gamma-2}\, \underbrace{\Delta_R\, \delta^4(\theta\theta')}_{16} = \delta^4(p) \int d^4\theta\, \Lambda_{ult-ff}^2\, R\bar{R}$

$$\frac{\partial}{\partial x}\left[\frac{\ln(1+x)}{x}\right] = \frac{\frac{1}{1+x}}{1} \to ?$$

$$\frac{\partial \bar{R}}{M^2}\int_0^{2\pi} d\theta \int_0^{\infty} d\rho\, \rho \frac{1}{\rho^2} \frac{\bar{R}\bar{R}}{\Lambda^{-2\bar{z}}}\int d^4\bar{k}\, \zeta^4(\bar{k}) \int d^4\bar{\ell}\, \frac{1}{\bar{\ell}^2(\bar{\ell}+\bar{k})^2}$$

$$\int d^4\rho\, \frac{\rho^{2\bar{z}}}{\rho^{2\bar{z}+\epsilon}} = \frac{\rho^{4}_{\perp}}{\epsilon}$$

$$\frac{\bar{R}\bar{R}}{\Lambda^{-2\bar{z}}}\int d^4\bar{k}\, \zeta^4(\bar{k}) \int d^4\bar{\ell}\, \frac{1}{\bar{\ell}^2(\bar{\ell}+\bar{k})^2} \to \frac{\bar{R}\bar{R}}{\Lambda^{-2\bar{z}}}\int d^4\bar{k}\, \zeta^4(\bar{k}) \int d^4\bar{\ell}\, \frac{1}{\bar{\ell}^2(\bar{\ell}+\bar{k})^2}$$

$$\Lambda^{-2\bar{z}}\int d^8\bar{z}\, \underbrace{\partial\bar{\partial}[(\partial\bar{\partial}H_\ell)(\partial\bar{\partial}H)]}_{\partial\bar{\partial}(\partial\bar{\partial})^{\ell-1}}\Lambda^{2\ell+1}\underbrace{\partial\bar{\partial}[(\partial\bar{\partial}H)(\partial\bar{\partial}H)]}_{(\partial\bar{\partial})^{\ell}\bar{\ell}} \to \delta^4(\theta)$$

$$\frac{\partial \bar{R}}{M^2}\int_0^{2\pi} d\theta \int_0^{\infty} d\rho\, \rho \frac{1}{\rho^2} \frac{\bar{R}\bar{R}}{\Lambda^{-2\bar{z}}}\int d^4\bar{k}\, \zeta^4(\bar{k}) \int d^4\bar{\ell}\, \frac{1}{\bar{\ell}^2(\bar{\ell}+\bar{k})^2} \quad \frac{\rho^{2\bar{z}+\epsilon}}{\rho^{2\bar{z}+\epsilon}} \quad \frac{1}{\bar{\ell}^2} \quad \frac{(\partial\bar{\partial})^{-\epsilon}}{\Lambda^{2\bar{z}}}$$

$$\int d^4\rho\, \frac{\rho^{2\bar{z}}}{\rho^{2\bar{z}+\epsilon}} = \frac{\rho^{4}_{\perp}}{\epsilon}$$

$$\frac{\bar{R}\bar{R}}{\Lambda^{-2\bar{z}}}\int d^4\bar{k}\, \zeta^4(\bar{k}) \int d^4\bar{\ell}\, \frac{1}{\bar{\ell}^2(\bar{\ell}+\bar{k})^2} \to \frac{\bar{R}\bar{R}}{\Lambda^{-2\bar{z}}}\int d^4\bar{k}\, \zeta^4(\bar{k}) \int d^4\bar{\ell}\, \frac{1}{\bar{\ell}^2(\bar{\ell}+\bar{k})^2}$$

$$\Lambda^{-2\bar{z}}\int d^8\bar{z}\, \underbrace{\partial\bar{\partial}[(\partial\bar{\partial}H_\ell)(\partial\bar{\partial}H)]}_{\partial\bar{\partial}(\partial\bar{\partial})^{\ell-1}}\Lambda^{2\ell+1}\underbrace{\partial\bar{\partial}[(\partial\bar{\partial}H)(\partial\bar{\partial}H)]}_{(\partial\bar{\partial})^{\ell}\bar{\ell}} \to \delta^4(\theta)$$

3 Vertex

$$\bar{k}^2 \equiv \Lambda^2$$

$$M^2 \quad \frac{1}{M^2} \quad H^{2\ell+\epsilon} \quad \frac{1}{M^2} \quad H \quad M \quad \frac{1}{M}$$

$$\frac{1}{\kappa^2}\int d^8z\, R\, \frac{H}{\Delta}\, \frac{\bar{R}}{\Delta}\, \frac{H^{\ell-1}}{\Delta} \Rightarrow \frac{1}{\kappa^2}\int d^8z\, R \left(\frac{\Delta}{\Lambda^2}\right)^{2\ell+1} \frac{1}{\Delta}\, \bar{R}$$

$$\frac{1}{M^2}\, \frac{1}{M^2}\, H\, \frac{1}{M^2}\, H \qquad \frac{1}{\kappa^2}\cdot\Lambda^{-2(\ell+1)}\int d^8z\, R\, \Lambda^{2\ell+1}\, \bar{R}$$

$$\underbrace{(\partial\bar{\partial})(\partial\bar{\partial}H)}\quad\underbrace{(\partial\bar{\partial})(\partial\bar{\partial}H)}$$

$$\Gamma = S_{HD} + S_{SH} + \frac{i}{2} \ln \det H_{HD} - i \ln \det M - \frac{i}{2} \ln \det C$$

↑ gravity ghosts ↑ 3rd ghost.

Contributo della prima $\int \sqrt{-g}\, dx$, $\frac{1}{2} - 1 - \frac{1}{2} = -1$

Quindi ho bisogno di almeno due scalari uno è lo scalare di Stora-Zinnjustin e l'altro?

standard model:

gauge bosons: $SU(3) \times SU(2) \times U(1)$

$8 \quad , \quad 3 \quad , \quad 1 \quad = 12$

$A^a \quad , \quad Z, W^+, W^- \quad , \quad A$

$$\frac{1}{2}(12) - 12 = -6$$

Fermions: $-(3 \cdot 2 + 3 \cdot 2 \cdot 3) = -24$

6 $\quad$ 18

$\times$ $\quad$ $\times 2$

$\frac{i}{12}$ ← antiparticle $\frac{2}{36}$

Totale: $(-1) + (-6) + (-24)$

↑ Ghosts ↑ gauge bosons ← Fermions

$z = a + ib \qquad Im(z) = b = -Re(iz)$

$iz = ia - b$

Power Counting and Killers in Supersymmetry

$$\frac{1}{\kappa^2}\int d^8z\; R\; \frac{e^{H}-1}{\Delta}\;\bar{R} \;\rightsquigarrow\; \frac{1}{\kappa^2}\int d^8z\, R\left(\frac{\Delta}{\Lambda^2}\right)^{\gamma+1}\frac{1}{\Delta}\,\bar{R}$$

$$= \frac{1}{\kappa^2}\,\bar{\Lambda}^{2(\gamma+1)}\int d^4x\, d^4\theta\, R\,\Delta^{\gamma}\bar{R}$$

$R \sim (\mathcal{D}\bar{\mathcal{D}})(\mathcal{D}\mathcal{D}H)$

$[\ ,\ ] = M^{-2}$

$\langle HH\rangle = P = \kappa^2 \int d^4p\, \frac{1}{p^2}\, e^{-H}\, \delta^4(\theta-\theta')$ ($\kappa^2 \to M^{-2}$)

$\left[\int d^2\theta\, d^2\bar{\theta}\right] = M^2$

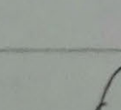
$[\delta^4(\theta)] = M^{-2}$

$\frac{1}{\kappa^2}\int d^8z\,(\mathcal{D}\bar{\mathcal{D}}H)\, e^{H}\,(\mathcal{D}\bar{\mathcal{D}}H)$

$[H] = \frac{1}{M}$

$\Delta = \Box(\mathcal{D}\bar{\mathcal{D}}H) + \dots$

$M^2 \quad \frac{1}{M^2} \quad M\,\frac{1}{M} \quad \cdots \quad M\,\frac{1}{M}$



$= \frac{1}{\kappa^2}\,\Lambda^{-2(\gamma+1)}\int d^4x \int d^4\theta\, [(\mathcal{D}\bar{\mathcal{D}})(\mathcal{D}\bar{\mathcal{D}}H)]\,(\mathcal{D}\bar{\mathcal{D}}H)\,\Delta^{\gamma}(\mathcal{D}\bar{\mathcal{D}}H)\,[(\mathcal{D}\bar{\mathcal{D}})(\mathcal{D}\bar{\mathcal{D}}H)]$

$\frac{1}{\kappa^2}\,\Lambda^{-2\gamma+2}\,\delta^4(K)\int d^4\theta \underbrace{[(\mathcal{D}\bar{\mathcal{D}})(\mathcal{D}\bar{\mathcal{D}}H)]}_{R}\int d^4p\,\frac{1}{p^2}\,\frac{p^{2\gamma}}{\left(\frac{p^{2\gamma+2}}{\Lambda^{2\gamma+2}}\right)}\,\underbrace{(\bar{\mathcal{D}}\bar{\mathcal{D}})^2\delta^4(\theta)}_{16}$

$\cong \delta^4(K)\int d^4\theta\; R\bar{R}\int d^4p\,\frac{1}{p^4}$ ($\frac{1}{\epsilon}$)

$\int d^4p\, p^{2\gamma}\frac{1}{p^{2\gamma+4}} = \frac{1}{\epsilon}$

$\bar{R}$

$(\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}H)\Delta^{\gamma}(\mathcal{D}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}H)$ con $N=2$ SUSY

$\Downarrow R$

deve eliminare un $d^4\theta$ in più quindi il nostro termine è $\frac{1}{\epsilon}$ e la divergenza è logaritmica.

cut-off scheme:

$\delta^4(K)\int d^4\theta\, R\int d^4p\,\frac{1}{p^{2\gamma+4}}\,p^{2\gamma}\,(\mathcal{D}\bar{\mathcal{D}})(\mathcal{D}\bar{\mathcal{D}}H)(\mathcal{D}\bar{\mathcal{D}}H)$

$\int d^8z\, R$

$\frac{1}{M^2}\, H$

$\int d^4p\,\frac{1}{p^4}\,p \to [\cdot] = M$

$= 0$

Per cancellare $\frac{1}{\epsilon} R\bar{R}$:

$$\mathcal{L}_{Kähler} = S_{K_R} \int d^8 t \; E^{-1} \; R\bar{R} \; \Delta^{\gamma-1} \; R\bar{R}$$

$$(D\bar{D})^2 (D\bar{D}H)$$

$$(D\bar{D}) \; (D\bar{D}H) \cdot (D\bar{D}) \; (D\bar{D}H)$$

P $\qquad$ P

$R\bar{R}$ —○— $= R\bar{R} \underbrace{\int d^4x \int d^4\theta}_{d^8z} \int d^4p \; \frac{1}{p^{2\gamma+4}} \; p^{2\gamma-2} \; p^2 \; \underbrace{(D\bar{D})^2 J(0)}_{16}$

$$\frac{1}{\epsilon} \int d^8z \; R\bar{R}$$

Per cancellare $\frac{1}{\epsilon} \int d^8z \; E^{-1} \; G_{\alpha\dot{\alpha}} \; G^{\alpha\dot{\alpha}}$ $\qquad$ $G \sim (D\bar{D})(D\bar{D}H)$

$$\mathcal{L}_{Kähler} = S_{K_G} \int d^8t \; E^{-1} \; GG \; \Delta_G^{\gamma-1} \; GG$$

$$\underbrace{(D\bar{D})(D\bar{D}H) \cdot (D\bar{D})(D\bar{D}H)}$$

GG —○— $= S_{K_G} GG \int d^4x \, d^4\theta \int d^4p \; \frac{1}{p^{2\gamma+4}} \; p^{2\gamma-2} \; p^2 \; \underbrace{(D\bar{D})^2 J(0)}_{16}$

Anche per cancellare $\frac{1}{\epsilon} \int WW \; E^{-1} \; d^8z$

In cut-off scheme.

Per cancellare $\int E^{-1}(R+\bar{R})$

$$S_{K_{R+\bar{R}}} \int d^8z \; R \; \Delta_R^{\gamma} \; R\bar{R}$$

$$\uparrow \quad \frac{1}{M^2} \quad M \; M^{2\gamma} \quad \sim (D\bar{D})(D\bar{D}H)(D\bar{D})(D\bar{D}H)$$

$$M^{-2\gamma-1}$$

$$R \;\; \text{---}\!\bigcirc\!\text{---} = R \int^{\Lambda} \frac{d^4k}{(k^2)} \int d^4\theta \int d^4p \, R^{2\gamma}(k^2) \frac{1}{p^2} \frac{p^{2\gamma+4}}{\Lambda^{2\gamma+2}} \, p^2 \, \underbrace{(D\bar{D})^2 \delta^4(\theta)}_{16} \cdot S_{K_{R+\bar{R}}}$$

$$\Lambda^2_{cut-off} \int d^8z \; R \cdot S_{K_{R+\bar{R}}} \; \Lambda^{2\gamma}$$

$$\frac{1}{M^2} \quad M \quad \frac{1}{M^{2\gamma+1}} \quad M^{2\gamma}$$

$$\frac{1}{M^2} \quad M \quad \frac{1}{M}$$

Per cancellare $\int E^{-1} \; 1 \qquad E\text{-}M$

$$\int d^8z \, E^{-1} R \, \Delta^{\gamma} R$$

$$\int d^4x \int d^4\theta \, E^{-1} \, D\bar{D}(D\bar{D}H) \, \Delta^{\gamma} \, D\bar{D}(D\bar{D}H) \quad \leftarrow \text{ed il propagatore?}$$

$$\int d^4p \, \frac{1}{p^{2\gamma+4}} \, p^{2\gamma+2} = \Lambda^2_{cut-off} E^{-1}$$

$$\xi = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\left[\alpha + \beta R + \sum_{n=0}^{\infty} C_n R \Box^n R\right]$$

$$= \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\left[\alpha + \beta R + c_0 R^2 + C_1 R \Box R + \dots\right]$$

$$\left[\alpha + \beta\lambda_1 + c_0\lambda_1^2 + C_1\lambda_1\lambda_2 + \right.$$

$$\left. + \mu(R-\lambda_1) + \mu_1(\Box\lambda_1 - \lambda_2)\right]$$

$$\frac{\delta\xi}{\delta\mu} = 0 \Rightarrow \quad \lambda_1 = R$$

$$\frac{\delta\xi}{\delta\mu_1} = 0 \Rightarrow \quad \lambda_2 = \Box\lambda_1$$

$$\alpha + \beta R + c_0 R^2 + C_1 R\Box R + C_2 R\Box^2 R + C_3 R\Box^3 R + \dots$$
$$+ C_4 \underset{(\Box^2R)(\Box^2R)}{R\Box^4 R} + C_5 \underset{(\Box^2R)(\Box^3R)}{R\Box^5 R} \qquad \Box R\Box^2 R$$

$$= \alpha + \beta\lambda_1 + c_0\lambda_1^2 + C_1\lambda_1\lambda_2 + C_2\lambda_2^2 + C_3\lambda_2\lambda_3 + C_4\lambda_3^2 + C_5\lambda_3\lambda_4$$

$$+ \mu(R-\lambda_1) + \mu_1(\Box\lambda_1 - \lambda_2) + \mu_2(\Box\lambda_2 - \lambda_3) + \mu_3(\Box\lambda_3 - \lambda_4)$$

$$\frac{\delta\xi}{\delta\mu} = 0 \Rightarrow \lambda_1 = R, \qquad \frac{\delta\xi}{\delta\mu_1} = 0 \Rightarrow \lambda_2 = \Box\lambda_1$$

$$\frac{\delta\xi}{\delta\mu_2} = 0 \Rightarrow \lambda_3 = \Box\lambda_2$$

$$\sum_{m=1}^{\infty} C_{2m-2}\lambda_m^2 = c_0\lambda_1^2 + C_2\lambda_2^2 + C_4\lambda_3^2$$
$$\sum_{m=1}^{\infty} C_{2m-1}\lambda_m\lambda_{m+1}$$

$$\mathcal{L} = \alpha + \beta\lambda_1 + \sum_{m=1}^{\infty}\lambda_m\left(C_{2m-2}\lambda_m + C_{2m-1}\lambda_{m+1}\right) + \sum_{m=1}^{\infty}\mu_m\left(\Box\lambda_m - \lambda_{m+1}\right)$$
$$+ \mu(R-\lambda_1)$$

$$\mathcal{L} = \mu R + \overset{1}{\overset{\|}{\beta}} \lambda_1 - \mu \lambda_1 - \sum_{n=1}^{\infty} \mu_n \lambda_{n+1} + \sum_{n=1}^{\infty} \lambda_n \left(\zeta_{n-2} \lambda_n + \zeta_{n-1} \lambda_{n+1} \right)$$

$$+ \sum_{n=1}^{\infty} \mu_n \square \lambda_n$$

$$\lambda_i = \chi_i + \psi_i \quad , \quad \mu_i = \chi_i - \psi_i$$

$$\sum_{i=1}^{n} (\chi_i - \psi_i) \square (\chi_i + \psi_i) =$$

$$= \sum_{n=1}^{\infty} (\chi_i \square \chi_i + \chi_i \square \psi_i - \psi_i \square \chi_i - \psi_i \square \psi_i)$$

$$= \sum_{n=1}^{\infty} (\chi_i \square \chi_i - \psi_i \square \psi_i)$$

$$= \sum_{n=1}^{\infty} \left[-(\nabla \chi_i)^2 + (\nabla \psi_i)^2 \right]$$

$$V = \lambda_1 - \mu \lambda_1 - \sum_{n=1}^{\infty} \left(\mu_n \lambda_{n+1} + \lambda_n^2 \zeta_{2n-2} + \lambda_n \lambda_{n+2} \zeta_{2n-1} \right)$$

$$= (1-\mu)(\chi_1 + \psi_1) - \sum_{n=1}^{\infty} \Big[(\chi_n - \psi_n)(\chi_{n+1} + \psi_{n+1}) +$$

$$+ (\chi_n + \psi_n)^2 \zeta_{2n-2} +$$

$$(\chi_n + \psi_n)(\chi_{n+1} + \psi_{n+1})^2 \zeta_{2n-1} \Big]$$

$$\mathcal{L} = \mu R - \sum_{n=1}^{\infty} \left[(\nabla \chi_i)^2 - (\nabla \psi)^2 \right] + V(\psi_i, \chi_i, \mu)$$

$$x = \ln \mu \quad , \quad \bar{g}_{\mu\nu} = e^{x} g_{\mu\nu}$$

$$(G_{\mu\nu} - \bar{T}_{\mu\nu}) \; \gamma^{(n)} \; (G_{\mu\nu} - T_{\mu\nu})$$

$$(\nabla_\mu \phi)(\nabla_\nu \phi)$$

$$(\nabla_\mu \phi \nabla_\nu \phi) \; \gamma^{\mu} [(\nabla^\nu \phi)(\nabla^\nu \phi)]$$

$$\bar{T}_{\mu\nu} \qquad \delta(\nabla^\nu \nabla^\nu \phi)$$

REDSHIFT

$$ds^2 = S(r) \left[-\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^{(2)} \right]$$

$$\frac{d\tau_{r_2}}{d\tau_{r_1}} = \frac{S(r_2)}{S(r_1)} \left(\frac{1 - \frac{2m}{r_2}}{1 - \frac{2m}{r_1}} \right) \xrightarrow{r_2 \to \infty} \frac{1}{S(r_1)} \frac{1}{1 - \frac{2m}{r_1}}$$

$$= \frac{1 + \frac{L^4}{r^4}}{1 - \frac{2m}{r_1}} = \frac{1 + \frac{(2m)^4}{r_1^4}}{1 - \frac{2m}{r_1}}$$

$$r_1 = x \cdot 2m$$

$$= \frac{1 + \frac{1}{x^4}}{1 - \frac{1}{x}} \qquad x = 2 \quad = \frac{1 + \frac{1}{2^4}}{1 - \frac{1}{2}}$$

$$x = 1 + \epsilon \Rightarrow$$

$$= \frac{1 + \frac{1}{(1+\epsilon)^4}}{1 - \frac{1}{1+\epsilon}} \simeq \frac{2}{\epsilon}$$

Doppler shift.

$$\beta = 0 \Rightarrow \mathcal{O}^{-1} = \frac{P^{(2)}}{k^2} - \frac{P^{(0)}}{k^2\left(2 - k^2 k_4^2 \underbrace{(n-1)}_{3}\alpha(k^2)\right)}$$

$$\alpha(k^2) = \frac{\gamma + \epsilon \Box}{k_4^2}$$

$$= - \frac{P^{(0)}}{k^2\left(2 - 3k^2(\gamma + \epsilon \Box)\right)}$$

$$- \frac{P^{(0)}}{k^2\left(2 - 3\gamma k^2 + 3\epsilon k^4\right)}$$

$\gamma \uparrow \frac{1}{\mu^2}$ $\quad \epsilon \uparrow \frac{1}{m^4}$

$$= - \frac{1}{k^2\left(1 - \underbrace{\frac{3}{2}}_{2}\gamma k^2 + \underbrace{\frac{3}{2}}_{\beta}\epsilon k^4\right)}$$

$$1 - 2k^2 + \beta k^4 = (k^2 - m_1^2)(k^2 - m_2^2) \cdot \frac{1}{m_1^2 m_2^2}$$

$$= k^4 - m_2^2 k^2 - m_1^2 k^2 + m_1^2 m_2^2$$

$$= k^4 - \underbrace{(m_1^2 + m_2^2)}_{\alpha = \frac{3}{2}\gamma} k^2 + \underbrace{m_1^2 m_2^2}_{\beta = \frac{3}{2}\epsilon}$$

$$= - \frac{1 \quad m_1^2 m_2^2}{k^2(k^2 - m_1^2)(k^2 - m_2^2)}$$

$$\Big|_{\mathrm{Res}(m_1^2)} = - \frac{\frac{3}{2}\gamma}{m_1^2(m_1^2 - m_2^2)}$$

$$\Big|_{\mathrm{Res}(m_2^2)} = - \frac{m_1^2}{m_2^2(m_2^2 - m_1^2)}$$

$$S(r) = Erf\left(\frac{r}{L}\right) = \frac{2r}{\sqrt{\pi}L}\left(1 - \frac{r}{2GM}\right) e^{\frac{r}{2GM}} = T^2 - X^2$$

$$ds^2 = \frac{32M^3G^3 e^{-r/2GM}}{r}\left(-dT^2 + dx^2\right) + r^2 d\Omega^2$$

↳ $r \gg 0$

$$1 - \frac{r}{2GM} \simeq T^2 - X^2$$

$$r \to 0 \Rightarrow ds^2 \simeq \frac{32\pi^3G^3}{r}\left(-dT^2 + dx^2\right) + \qquad 1 - (T^2 - X^2) \simeq \frac{r}{2GM}$$

$$\frac{r}{L}$$

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$$\frac{r}{2GM} \approx 1 + x^2 - T^2$$

$$ds^2 = \frac{\sqrt{\pi}}{2} Erf\left(\frac{r}{L}\right)\left[\frac{32M^3G^3}{r} e^{-r/2GM}\left(-dT^2 + dx^2\right) + r^2 d\Omega^2\right]$$

$$\simeq \frac{r}{L} \cdot \frac{32M^3G^3}{r}\left(-dT^2 + dx^2\right) + \frac{r^3}{L} d\Omega^2$$

$$= \frac{16M^2G^2 \cdot 2M}{2M}\left(-dT^2 + dx^2\right) + \frac{1}{L} \cdot \left(x^2 - T^2 + 1\right)^3 (2GM)^3 d\Omega^2$$

É simples.

$$\left(R_{\mu\nu} + \alpha F_{\mu\nu}\right) \frac{e^{H} - 1}{D} \left(R^{\mu\nu} + \alpha \tilde{F}^{\mu\nu}\right)$$

$$R_{\mu\nu}\left(\frac{e^{H} - 1}{H}\right) R^{\mu\nu} + \alpha^2 F_{\mu\nu} e^{H} F^{\mu\nu} - \alpha^2 F_{\mu\nu} F^{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \qquad G^{\mu}_{\mu} = R - 2R = -R$$

Lee-Wick Resonances !! 750 Gev

$$\frac{c}{(k+p)^2 - \underbrace{(m^* + i\Gamma m)^2}_{M^2}} + \frac{c^*}{(k+p)^2 - \underbrace{(m + i\Gamma m)^2}_{M^{*2}}}$$

$$|A|^2 = \left|\left(\frac{c}{z - M^2} + \frac{c^*}{z - M^{*2}}\right)\right|^2 =$$

$$= \left(\frac{c}{z - M^2} + \frac{c^*}{z - M^{*2}}\right)\left(\frac{c^*}{z - M^{*2}} + \frac{c}{z - M^2}\right)$$

$$\frac{|c|^2}{(z - M^2)(z - M^{*2})} + \frac{c^2}{(z - M^2)(z - M^2)} + \frac{c^{*2}}{(z - M^{*2})(z - M^{*2})}$$

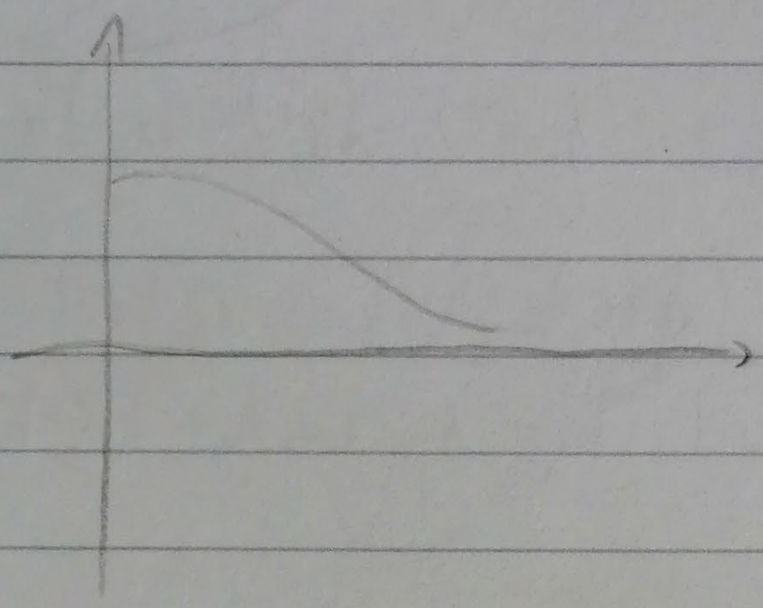
$$+ \frac{|c|^2}{(z - M^{*2})(z - M^2)} = 2\frac{|c|^2}{|z - M^2|^2} +$$

$$\langle T_V(F_x^{(2)}) T_V(F_y^{(1)}) \rangle = \sum_i c_i \, e^{-m_i |x-y|^2}$$

$A^2 dA \quad A^2 dA$

(RR)

$e^{-H} h_4 \; e^{-H} h_4$ (x2)



$$\square h = T \qquad \square \phi = \frac{1}{\phi^4}$$

$$e^{H} \square h' = T \Rightarrow e^{H} \square h' = \square h$$

$$e^{H} h' = h$$

$$h' = e^{-H} h$$

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$$\int d\vartheta \Theta = 1$$

$$\int d\vartheta = 0$$

$$\int \frac{df(\theta)}{d\theta} d\theta = 0$$

$$k^2 \tilde{\phi}(k) = \frac{1}{\phi^4} \Rightarrow \tilde{\phi}(k) = \frac{1}{k^2}$$

$$\hat{f}_{\mu\nu} \equiv \phi^2 \hat{\rho}_{\mu\nu}$$

$$\hat{f}'_{\mu\nu} = \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} \hat{f}_{\mu\nu} \Rightarrow \phi'^2 \hat{f}'_{\mu\nu} = \phi^2 \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} \hat{f}_{\mu\nu}$$

$$\phi' = \phi$$

$$\hat{f}'_{\mu\nu} = \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} \hat{f}_{\mu\nu}$$

[Notice: $\frac{e^{-H}}{p^2} = \frac{1}{p^2} + \frac{e^{-H}-1}{p^2} = \frac{1}{p^2} - \frac{1-e^{-H}}{p^2}$]

$$G(p) = \frac{1-e^{-H(p^2)}}{p^2+i\epsilon} = (1-e^{-H(p^2)})\frac{p^2-i\epsilon}{p^4+\epsilon^2}$$

$$= \frac{1-e^{-H(p^2)}}{p^4+\epsilon^2} - \underbrace{(1-e^{-H(p^4)})}_{f}\underbrace{\frac{-i\epsilon}{p^4+\epsilon^2}}_{\delta(p^2)}$$

$f(p^2)\ \delta(p^2) \Rightarrow f(0) = 0$

Now in Gravity we are forced by unitarity to have a confining RT spectrum.

$$G(p) \underset{p\sim 0}{\sim} \frac{\chi-(\chi-H)}{p^2} \sim \frac{H(p)}{p^2} \sim p^{2n} \quad n>0$$

Stuckelberg:

$$e^{H_0} \longrightarrow \frac{-1}{m^2}e^{H_0}(\Box - m^2)$$

$$\frac{1-e^{-H_0}\left(\frac{\Box}{m^2}+1\right)}{\Box} = \frac{1+\frac{\Box}{m^2}-1}{\Box} = \frac{1}{m^2}$$

In questo caso ho massa propagante e "massless" non propagante. Quindi seguendo Efimov ho salvo gli Stuckelberg confinano con RT spectrum.

$\Lambda \to 0$ deconfining phase.

$r_{conf} \lesssim \frac{1}{\Lambda}$ quindi abbiamo una specie di legge di particelle confinate in un raggio $\frac{1}{\Lambda}$

Conformal Weyl collapse

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

$$R - \frac{1}{2} \cdot 4 \cdot R = 8\pi T = 0$$

$$\Rightarrow R = 0$$

$$\mathcal{L} = \int\int (\phi^2 R + \partial_\mu \phi \partial^\mu \phi) + \mathcal{L}_m$$

$$\Updownarrow$$

$$T^\alpha{}_\alpha = 0$$

$$M(r) = \frac{r^3}{r^3 + \rho^3}$$

$$\frac{1 - 2M(r)}{r} = 1 - \frac{2r^2}{r^3 + \rho^3}$$

$$\frac{dM}{dr} = 4\pi\rho = 4\pi \frac{3\rho^3 r^2}{(\rho^3 + r^3)^2}$$

$$e^{-H} \delta = \delta_{eff}$$

$$\frac{1 - \left(1 - \frac{\rho^2}{\Lambda^2}\right)}{\rho^2} = \frac{1}{\Lambda^2}$$

$$A_{vco} = 4\pi\, S(r)\, r^2$$

$$A_H = 4\pi\, S(2m)\,(2m)^2$$

$$\frac{dM}{dv} = \sigma\, A_H\, T_H^4 = \sigma\, 4\pi\, S(2m)\,(2m)^2 \left(\frac{1}{8\pi m}\right)^4$$

$$= \sigma\, 4\pi\, S(2m)\,(2m)^2 \left(\frac{1}{4\pi}\right)^4 \left(\frac{1}{2m}\right)^{4-2}$$

$$= \frac{\sigma}{(4\pi)^3}\, S(2m)\, \frac{1}{(2m)^2}$$

$\boxed{2m = y}$

$$\boxed{\frac{dy}{dv} = \frac{2\sigma}{(4\pi)^3}\, \frac{S(y)}{y^2}}$$

Bock basis in any D

$$B_{\alpha\alpha} = \nabla^b \nabla^d C_{\alpha b c d} - \frac{1}{2} R^{bd} C_{\alpha b c d}$$

$$C_{\alpha b c d;}{}^{db} = \nabla^b \nabla^d C_{\alpha b c d}$$

pg 7 file: Weyl tensor Bock

$$= \underbrace{\frac{D-3}{D-2}}_{x} R_{\alpha c;}{}^{b}{}_{b} - \underbrace{\frac{D-3}{2(D-1)}}_{y} R_{;\alpha c} +$$

$$\underbrace{-\frac{D-3}{2(D-1)(D-2)}}_{z} g_{\alpha c} R_{;}{}^{b}{}_{b} + O(R^2)$$

$$R_{\alpha c} = G_{\alpha c} + \tfrac{1}{2} g_{\alpha c} R$$

$$\mathcal{L} = -2\kappa_D^{-2}\left[R + B_{\alpha c}\gamma^{\alpha}_{\beta} B^{\beta\alpha c} + R\gamma^{\alpha}_{5} R \right]$$

$$B_{\alpha c}\gamma^{\alpha}_{\beta} B^{\beta\alpha c} = \left(x \Box R_{\alpha c} + y \nabla_c \nabla_\alpha R + z g_{\alpha c} \Box R\right) \gamma^{\alpha}_{\beta}$$

$$\left(x \Box R^{\alpha c} + y \nabla^c \nabla^\alpha R + z g^{\alpha c} \Box R\right) + O(R^3)$$

$$= \underbrace{x^2 (\Box R_{\alpha c})\gamma^{\alpha}_{\beta}(\Box R^{\beta\alpha c})}_{\Box(G_{\alpha c} + \frac{1}{2} g_{\alpha c} R)} + xy (\Box R_{\alpha c})\gamma^{\alpha}_{\beta}(\nabla^c \nabla^\alpha R) + xz (\Box R)\gamma^{\alpha}_{\beta}(\Box R)$$

$$+ yx (\nabla_c \nabla_\alpha R)\gamma^{\alpha}_{\beta}(\Box R^{\beta\alpha c}) + y^2 (\nabla_c \nabla_\alpha R)\gamma^{\alpha}_{\beta}(\nabla^c \nabla^\alpha R) + yz (\Box R)\gamma^{\alpha}_{\beta}(\Box R) +$$

$$+ zx (\Box R)\gamma^{\alpha}_{\beta}(\Box R) + zy (\Box R)\gamma^{\alpha}_{\beta}(\Box R) + z^2 D (\Box R)\gamma^{\alpha}_{\beta}(\Box R) + O(R^3) =$$

$$= \underbrace{x^2 R_{\alpha c}(\gamma^{\alpha}_{\beta}\Box^2) R^{\beta\alpha c}}_{} + \frac{xy}{2} R(\gamma^{\alpha}_{\beta}\Box^2)R + xz\, R(\gamma^{\alpha}_{\beta}\Box^2)R$$

$$+ \frac{yx}{2} R(\gamma^{\alpha}_{\beta}\Box^2)R + y^2 R(\gamma^{\alpha}_{\beta}\Box^2)R + yz\, R(\gamma^{\alpha}_{\beta}\Box^2)R$$

$$+ zx\, R(\gamma^{\alpha}_{\beta}\Box^2)R + zy\, R(\gamma^{\alpha}_{\beta}\Box^2)R + D\cdot z^2\, R(\gamma^{\alpha}_{\beta}\Box^2)R + O(R^3) =$$

$$= x^2 \; R_{ac}\left(\gamma_B \Box^2\right) R^{ac} + \left[\frac{xy}{z} + xz + \frac{yx}{z} + y^2 + yz + zx + zy + Dz^2\right]$$

$$\cdot R\left(\gamma_B \Box^2\right) R =$$

$$= \left(\frac{D-3}{D-2}\right)^2 R_{cb}\left(\gamma_B \Box^2\right) R^{cb} + \left(xy + 2xz + 2yz + y^2 + D\cdot z^2\right) R \gamma_B \Box^2 R$$

$$= \left(\frac{D-3}{D-2}\right)^2 R_{ab}\left(\gamma_B \Box^2\right) R^{ab} - \frac{D(D-3)}{4(D-1)(D-2)^2} R\left(\gamma_B \Box^2\right) R + O(R^3)$$

$$\mathcal{L} = -2\kappa_D^{-2}\left\{R + R_{ab}\underbrace{\left[\left(\frac{D-3}{D-2}\right)^2 \gamma_B \Box^2\right]}_{\gamma_2} R^{ab} + R\underbrace{\left(\gamma_S - \frac{D(D-3)}{4(D-1)(D-2)^2}\gamma_B \Box^2\right)}_{\gamma_0} R\right\}$$

$\gamma_4 = 0$

$$\gamma_2 = \frac{e^{H_2} - 1}{\Box} = \left(\frac{D-3}{D-2}\right)^2 \gamma_B \Box^2 \Rightarrow \boxed{\gamma_B = \left(\frac{D-2}{D-3}\right)^2 \frac{e^{H_2} - 1}{\Box^3}}$$

$$\gamma_0 = -\frac{1}{4(D-1)} \frac{1}{\Box}\left[(D-2)\left(e^{H_0} - 1\right) + D\left(e^{H_2} - 1\right)\right] = \gamma_S' - \frac{D(D-3)}{4(D-1)(D-2)^2}\frac{\gamma_B \cdot \Box^2}{}$$

$$\gamma_S' = \gamma_0' + \frac{D(D-3)}{4(D-1)(D-2)^2}\frac{(D-2)^2}{(D-3)^2}\frac{e^{H_2} - 1}{\Box}$$

$$= \gamma_0 + \frac{D}{4(D-1)(D-3)}\frac{e^{H_2} - 1}{\Box} = -\frac{1}{4(D-1)}\frac{1}{\Box}(D-2)\left(e^{H_0} - 1\right) +$$

$$- \frac{D}{\Gamma(D-1)} \frac{1}{\Pi} \left(e^{H_2} - 1\right) + \frac{D}{\Gamma(D-1)(D-2)} \frac{1}{\Pi} \left(e^{H_2} - 1\right)$$

$$= - \frac{(D-2)}{\Gamma(D-1)} \frac{e^{H_0} - 1}{\Pi} - \frac{D}{\Gamma(D-1)} \left(1 - \frac{1}{D-3}\right) \frac{e^{H_2} - 1}{\Pi}$$

$$= - \frac{D-2}{\Gamma(D-1)} \frac{e^{H_0} - 1}{\Pi} - \frac{D}{\Gamma(D-1)} \frac{D-4}{D-3} \frac{e^{H_2} - 1}{\Pi}$$

$$= - \frac{1}{\Gamma(D-1)} \left[(D-2)\left(e^{H_0} - 1\right) + \frac{D(D-4)}{D-3}\left(e^{H_2} - 1\right) \right] \frac{1}{\Pi}$$

$$= - \frac{1}{\Gamma(D-1)} \left(\frac{(D-2)(D-3)\left(e^{H_0} - 1\right) + D(D-4)\left(e^{H_2} - 1\right)}{D-3} \right) \frac{1}{\Pi}$$

For $H_0 = H_2 \equiv H \Rightarrow - \frac{1}{\Gamma(D-1)} \left(\frac{(D-2)(D-3) + D(D-4)}{D-3} \right) \cdot \frac{e^{H} - 1}{\Pi}$

For $D = 4 \Rightarrow - \frac{1}{4 \cdot 3} \left(\frac{2 \cdot 1 + 0}{1} \right) \frac{e^{H} - 1}{\Pi}$

$$= - \frac{1}{6} \frac{e^{H} - 1}{\Pi}$$

For Y_D B^{ab} $\Pi \gamma H$

Π^2 $\frac{e^H}{\Pi^\gamma}$ $\bar{\Pi}^2$ $\rightarrow \Pi e^H \sim p^2 p^{2\alpha+2}$

R Y_s R

Π $\frac{e^H}{\Pi}$ Ω

$p^2 p^{2\alpha+2} = e^{\alpha H}$

Notice $B_{ab} = \frac{\delta(\sqrt{-g}\, C_{\mu\nu\sigma}\, C^{\mu\nu\sigma})}{\delta g^{ab}}$

$$S = -2\kappa_5^{-2} \int d^4x \sqrt{-g} \Big[R + \frac{\delta(\sqrt{-g}\, C^2)}{\delta g_{ab}} \gamma_{B(1)} \frac{\delta(\sqrt{-g}\, C^2)}{\delta g_{cd}} g^{ac} g^{bd}$$

$$+ R\, \gamma_{R(1)}\, R + V(B) \Big]$$

?

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{bc} u^b u^c = - \partial^\alpha \ln \phi - \underbrace{\frac{\phi_r}{\phi}}_{r-eq.} u^\alpha u^r$$

$$u^0 = u^\varphi = 0$$

$$\partial_r \ln \phi \; u^r u^r$$

$$\frac{du^r}{d\tau} + \Gamma^r_{bc} u^b u^c = - \partial^r \ln \phi$$

$$\phi = M_p S^{-\frac{1}{2}}$$

$$\partial_r \ln \phi = \partial_r \ln S^{-\frac{1}{2}} = -\frac{1}{2} \partial_r \ln S = -\frac{1}{2} \frac{1}{S} S'$$

$$\frac{du^r}{d\tau} + \Gamma^r_{rc} u^r u^c + \Gamma^r_{tc} u^t u^c = \underbrace{- \partial^r \ln \phi}$$

$$\frac{1}{2} \frac{S'}{S}$$

$$\frac{du^r}{d\tau} + \Gamma^r_{rr} (u^r)^2 + \Gamma^r_{rt} u^r u^t +$$

$$+ \Gamma^r_{tt} (u^t)^2 + \Gamma^r_{tr} u^r u^t = \frac{1}{2} \frac{S'}{S}$$

$$\frac{du^r}{d\tau} + \Gamma^r_{rr} (u^r)^2 + 2 \Gamma^r_{rt} u^r u^t + \Gamma^r_{tt} (u^t)^2 = \frac{1}{2} \frac{S'}{S}$$

$$\frac{f'(r)}{2f(r)} \qquad \frac{\rho'(r)}{2\rho(r)}$$

$$0 = \frac{du^t}{d\tau} + \Gamma^t_{ab} u^a u^b \Rightarrow \frac{du^t}{d\tau} + \Gamma^t_{tb} u^t u^b +$$

$$+ \Gamma^t_{rb} u^r u^b = 0$$

$$\frac{du^t}{d\tau} + \Gamma^t_{tr} u^t u^r + \Gamma^t_{rt} u^r u^t = 0$$

$$\frac{du^t}{d\tau} + 2 \Gamma^t_{tr} u^r u^t = 0$$

$$\frac{du^t}{d\tau} + 2 \cdot \frac{1}{2} \left(\frac{2m}{r^2 \left(1 - \frac{2m}{r}\right)} + \frac{S'}{S} \right) u^r u^t = 0$$

$$\frac{du^r}{d\tau} + \frac{f'(r)}{2f(r)}(u^r)^2 + \frac{g'(r)}{2f(r)}(u^t)^2 = \frac{1}{2}\frac{S'}{S}$$

$\frac{E^2}{g(r)^2}$ $\quad \frac{1}{2}\frac{d}{dr}\ln S(r)$

$(-+++)$

$u^\mu u_\mu = -1 \qquad u^t = \dot{t}$

$u^t u_t + u^r u_r = -1$

$(u^t)^2 \hat{g}_{tt} + (u^r)^2 \hat{g}_{rr} = -1$

$(u^t)^2 (-g(r)) + (u^r)^2 f(r) = -1$

$(u^t)^2 = +\frac{1}{g} + (u^r)^2 \frac{f}{g}$

CANUTO
$u^\mu u_\mu = 1$
$(1---)$

$\frac{du^r}{d\tau} + \frac{f'}{2f}u^{r2} + \frac{g'}{2f}\cdot\frac{1}{g}\left(1 + f u^{r2}\right)$

$\dot{u}^r + \frac{f'}{2f}u^{r2} + \frac{g'}{2g}u^{r2} + \frac{g'}{2fg}$

$\frac{du^r}{d\tau} + \frac{1}{2}\left(\frac{f'}{f} + \frac{g'}{g}\right)u^{r2} + \frac{g'}{2fg} = \frac{1}{2}\frac{S'}{S}$

$u^r = \frac{dr}{d\tau}$

$u^r du^r = \left[\frac{1}{2}\frac{S'}{S} - \frac{g'}{2fg} - \frac{1}{2}\left(\frac{f'}{f} + \frac{g'}{g}\right)u^{r2}\right]dr$

$\frac{1}{2}\frac{S'}{S}u^{r2}$ $\qquad \frac{2S'}{S}$

$f = \hat{g}_{rr} = S\left(1 - \frac{2M}{r}\right)^{-1}$

$g = -\hat{g}_{tt} = -S\left(1 - \frac{2M}{r}\right)$

$u^r du^r = 0 - \frac{M}{r^2}dr$

$u^r du^r = -M\frac{r^{-2+1}}{-2+1} + \text{cost}, \quad \frac{1}{2}u^{r2} = M\frac{1}{r} + c$

$\dot{r}^2 = \frac{2M}{r}$

$\left(\frac{1}{2}\frac{S'}{S} - \frac{g'}{2fg}\right)dr$

$\left(\frac{1}{S^2}\frac{dS}{dr}\right)dr$

$\frac{1}{2}\frac{2rS}{S} + \frac{2MS - (2M-r)rS'}{2r^2S^2} - \frac{1}{2}\frac{S'}{S}u^{r2}$

$\frac{1}{2}\frac{2rS}{S} + \frac{M}{r^2S} + \frac{S'}{S^2}\frac{1}{2}\left(1 - \frac{2M}{r}\right) - \frac{S'}{S}u^{r2}$

$\frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{S'}{S} + \frac{M}{r^2S} + \frac{1}{2}\frac{S'}{S^2}\left(1 - \frac{2M}{r}\right) - \frac{S'}{S}u^{r2} = \frac{1}{2}\frac{S'}{S}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{1}{S}\right) + \frac{M}{r^2S}$

(+ - - -)

$$\Gamma(t,t,1) \equiv \Gamma^{t}_{tt}$$

$$\frac{d^2t}{dq^2} + \frac{1}{w}\frac{dw}{dr}\frac{dt}{dq}\frac{dr}{dq} = 0$$

$$0 = \frac{d}{dq}\left(\ell_p\left(\frac{dt}{dq}\right) + \ell_p w\right)$$

$$\frac{1}{w}\frac{dw}{dr}\cdot\frac{dr}{dq}$$

$$\ell_p\left(\frac{dt}{dq}\cdot w\right) = \text{const}$$

$$\Rightarrow \frac{dt}{dq}\cdot w = \text{const}$$

$$\dot{t}\, w = \text{const}$$

$$\dot{t}\cdot S(r)\left(1-\frac{2M}{r}\right) = \text{const}$$

$$\hat{g}_{tt}$$

$$\dot{t}\cdot\hat{g}_{tt} = E \quad \Rightarrow \quad \dot{t} = \frac{E}{\hat{g}_{tt}}$$

$$\frac{du^t}{d\tau} + \cancel{2}\frac{\cancel{f}'(r)}{\cancel{2}f(r)}u^r u^t = 0$$

$$\frac{d^2t}{d\tau^2} + \frac{f'(r)}{f(r)}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0 \quad \Leftarrow \quad \frac{d}{d\tau}\left(g_{tt}\left(\frac{dt}{d\tau}\right) + g_{rr}\right)$$

Verifica $\left(\frac{dt}{d\tau}\right)^{-1}\cdot\frac{d^2t}{d\tau^2} + \frac{1}{g}\cdot\frac{dg}{dr}\cdot\frac{dr}{d\tau} = 0 \quad \Rightarrow \quad g_{tt}\left(\frac{dt}{d\tau}\cdot g\right) = \text{const}$

$u^t g = \text{const} = E$

$\hat{g}_{tt}$

(+ − − −)

$$S = -\int \phi c^2 d\tau = -c\int \sqrt{-\phi^2 \hat{g}_{\mu\nu} \frac{dx^\mu}{dq} \frac{dx^\nu}{dq}}\, dq$$

$$c\left(\frac{d\tau}{dq}\right)^2 = -\phi^2 \hat{g}_{\mu\nu} \frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = \left(1-\frac{r_s}{r}\right)\frac{dt^2}{dq^2} - \frac{1}{1-\frac{r_s}{r}}\left(\frac{dr}{dq}\right)^2$$

$$= -\phi^2 \frac{d\hat{\tau}^2}{dq^2} =$$

$$\delta\left(\frac{d\tau}{dq}\right)^2 = -\delta\left(\phi^2 \frac{d\hat{\tau}^2}{dq^2}\right) =$$

$$= -(\delta\phi^2)\cdot\left(\frac{d\hat{\tau}^2}{dq^2}\right) - 2\phi^2 \delta\left(\frac{dt}{dq}\right)\left(\frac{dt}{dq}\right) =$$

$$\frac{\partial \phi^2}{\partial x^\mu}\epsilon$$

$$\frac{d\tau^2}{dq^2} = \phi^2 \hat{g}_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = \phi^2 \frac{d\hat{s}^2}{dq^2}$$

$$\tau = \int \hat{g}_{tt}\, \dot{t}\, \phi^2$$

$$\dot{t} = \frac{\text{Ξ}}{(\hat{g}_{tt}\phi^2)}$$

$$4\phi \cancel{\phi} = \frac{d\tau^2}{d\hat{s}^2} \Leftarrow d\tau^2 = d\hat{s}^2$$

$$\boxed{\frac{d\hat{s}^2}{d\tau^2} = \phi^2} \Rightarrow \mathcal{L} = -c\int dq\, \phi^2$$

$$L = -\sqrt{-\phi^2 \hat{g}_{tt}\dot{t}^2} \Rightarrow \frac{\partial L}{\partial \dot{i}} = -\frac{N(t\phi^2)\hat{g}_{tt}\dot{i}}{\sqrt{-\phi^2 \hat{g}_{tt}\dot{i}^2}}$$

$$\hat{g}_{tt}\dot{t}^2 + \hat{g}_{rr}\dot{r}^2 = -\phi^2$$

$$\hat{g}_{tt}\cdot\frac{E^2}{\hat{g}_{tt}^2} + \hat{g}_{rr}\dot{r}^2 = -\phi^2$$

$$\hat{g}_{tt}\hat{g}_{rr}\dot{r}^2 + E^2 = -\hat{g}_{tt}\phi^2$$

$$+\dot{r}^2 + \hat{g}_{tt}\phi^2 = -E^2$$

$$\dot{r}^2 + S(r)\left(1-\frac{r_s}{r}\right) = S(r)^2 E^2 - \phi^2$$

$$\text{const} = \hat{g}_{tt}\dot{t}$$

$$\hat{B}_{ab} = \Omega^{-2} B_{ab} \qquad \hat{\phi} = \Omega^{-1} \phi$$

$$\hat{g}_{ab} = \Omega^2 g_{ab}$$

$$\hat{g}^{ab} = \Omega^{-2} g^{ab}$$

$$\hat{B}_{ab} \hat{B}_{cd} \hat{g}^{ac} \hat{g}^{bd} = \Omega^{-8} B_{ab} B^{ab}$$

$$\hat{C}_{abcd} = \Omega^2 C_{abcd}, \qquad \hat{C}_{abcd} B_{fg} B_{hi} C$$

$$\Omega^2 \; \Omega^{-2} \; \Omega^{-2} \; \Omega^2$$

$$\hat{g}^{ab} \hat{B}_{bc} \Omega^{-4} B^{c}$$

$$R^{e}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{e}{}_{\nu\sigma} - \partial_\nu \Gamma^{e}{}_{\mu\sigma}$$

$$R^{e}{}_{\sigma e\nu} = \partial_e \Gamma^{e}{}_{\nu\sigma} - \partial_\nu \Gamma^{e}{}_{e\sigma}$$

$$R^{e}{}_{\sigma\mu}{}^{\sigma} = g^{\sigma\nu} (\partial_\mu \Gamma^{e}{}_{\nu\sigma} - \partial_\nu \Gamma^{e}{}_{\mu\sigma})$$

$$= \partial_\mu \Gamma^{e}{}_{\nu}{}^{\sigma}{}_{\sigma} - \partial_\nu \Gamma^{e}{}_{\mu}{}^{\sigma}{}_{\sigma}$$

$$\Rightarrow R^{\sigma}{}_{\nu} = \partial_\nu \Gamma^{\sigma}{}_{\mu}{}^{\mu} - \partial^{\mu} \Gamma^{\sigma}{}_{\nu\mu}$$

1 - 3

$$g^{ec} R_{@b@d} = R_{bd} = g^{ac}(-) R_{bacd} = g^{ac}(-1)^2 R_{badc}$$

$$= g^{ac} R_{b@dc}$$

2 - 4

$$- g^{ac} R_{abdc}$$

1 - 4

$$(-+++)$$

$$q = -\int \sqrt{-g}\, dx^0 dx^1 = -\int \sqrt{-g^{(2)}} = -\int \sqrt{g}\, d^2z$$

$$= -\int dz$$

$$dz^2 = -ds^2 = -(h_{00}\, dt^2 + h_{ij}\, dx^i dx^j)$$

$$= -(-dt^2 + dx^{i2})$$

$$= \int_0^{} dt^2$$

$$_2 + \Phi =$$

GEODESICS IN

Affine gauge

Riemann Notation (1)

$\lambda = f$

GENERAL SECTION

$$S = -\int \sqrt{-\lambda^2 \phi^2 \hat{g}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau$$

$$-\int m \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} = \qquad m^2 \kappa_D^2 = \dagger = \frac{m}{M_P}$$

$$-\int m \sqrt{-\phi^2 \kappa_D^2 \hat{g}\, dx dx} =$$

$[\lambda] = M^0$

$[\phi] = M$

$\phi = \bar{\phi} \kappa_D$

$$\mathcal{L} = -\sqrt{-\lambda^2 \phi^2 \dot{x}^2} = -\sqrt{-\lambda^2 \phi^2 (\hat{g}_{tt} \dot{t}^2 + g_{rr} \dot{r}^2)}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -\frac{1}{\not{\mathcal{L}} \sqrt{-\lambda^2 \phi^2 \dot{x}^2}} \cdot (-\lambda^2 \phi^2 \hat{g}_{tt} \not{\dot{t}}) = -\frac{\lambda^2 \phi^2 \hat{g}_{tt} \dot{t}}{\mathcal{L}} = -E \quad \text{const}$$

Affine gauge:

$$\mathcal{L} = -\frac{\lambda^2 \phi^2}{M_P} \Rightarrow \quad + \hat{g}_{tt} \dot{t} = -E$$

$[E] = M^0$

if → $\boxed{\dot{x}^2 = -\lambda^2 \phi^2 / M_P^2}$

$$\mathcal{L} = -\sqrt{-\lambda^2 \phi^2 \frac{(-\lambda^2 \phi^2)}{M_P^2}} = -\frac{\lambda^2 \phi^2}{M_P}$$

$$\hat{g}_{tt} \dot{t}^2 + \hat{g}_{rr} \dot{r}^2 = -\lambda^2 \phi^2 / M_P^2$$

$$\not{\hat{g}}_{tt} \frac{E^2}{\not{\hat{g}}_{tt}} + \hat{g}_{rr} \dot{r}^2 = -\frac{\lambda^2 \phi^2}{M_P^2}$$

$\phi = S^{-1/2} M_P$

$$+ \hat{g}_{rr} \hat{g}_{tt} \dot{r}^2 + E^2 = -\lambda^2 \hat{g}_{tt} \phi^2 \frac{1}{M_P^2}$$

$$-S^2(r) \dot{r}^2 + \lambda^2 \phi^2 \hat{g}_{tt}/M_P^2 = -E^2$$

$$+ \lambda^2 S^{-1}(r) \frac{M_P^2}{M_P^2} \left(1 + \frac{2M}{r}\right) S(r) = -E^2$$

$$S^2(r) \dot{r}^2 + \lambda^2 \left(1 - \frac{2M}{r}\right) S^{-1}(r) \cdot S(r) = +E^2$$

$$S^2(r) \dot{r}^2 + \lambda^2 \left(1 - \frac{2M}{r}\right) = E^2$$

$$S^2(r) \dot{r}^2 = \frac{2M}{r}$$

$\frac{dr}{d\tau} < 0$

$$\int S(r) \sqrt{r}\, dr = -\sqrt{2M}\, \tau + \zeta$$

$$-\frac{2L^4}{5r^{5/2}} + \frac{2}{3} r^{3/2} = -\sqrt{2M}\, \tau + \zeta$$

$$\sqrt{2M}\, \tau = \frac{2L^4}{5}\left(\frac{1}{r^{5/2}} - \frac{1}{r_0^{5/2}}\right) + \frac{2}{3}\left(r^{3/2} - r_0^{3/2}\right)$$

$E = \lambda = 1$

$\Rightarrow \dot{r} = 0$ for $r \to \infty$

Geodesics in Proper time gauge

$\frac{d\lambda^\mu}{d\tau^2} = -1$ ← proper time gauge

$d\hat{s}^2 = \hat{g}_{\mu\nu}\, d\lambda^\mu\, d\lambda^\nu$

$\dot{\lambda}^2 = -1 \Rightarrow \mathcal{L} = -\sqrt{\lambda^2\phi^2} = -\lambda\phi$

$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -+\lambda\phi\,\hat{g}_{tt}\,\dot{t} = -E$

$\dot{t} = * \frac{-E}{\lambda\phi\,\hat{g}_{tt}} = \frac{-E}{\lambda S^{-\frac{1}{2}} M_p\, \hat{g}_{tt}}$

$\dot{t}^2 = \frac{E^2}{\lambda^2\phi^2\hat{g}_{tt}^2}$

$\hat{g}_{tt}\,\dot{t}^2 + \hat{g}_{rr}\,\dot{r}^2 = -1$

$\hat{g}_{tt}\frac{E^2}{\lambda^2\phi^2\hat{g}_{tt}^2} + \hat{g}_{rr}\,\dot{r}^2 = -1$

$\frac{E^2}{\lambda^2\phi^2} + \hat{g}_{rr}\hat{g}_{tt}\,\dot{r}^2 + \hat{g}_{tt} = 0$

$\Gamma t\ \hat{g}_{tt}\,\dot{t}^2 + \hat{g}_{rr}\,\dot{r}^2 = 0$

$-\frac{E^2}{\lambda^2\phi^2 S(r)\left(1-\frac{2M}{r}\right)} + \frac{S(r)}{1-\frac{2M}{r}}\,\dot{r}^2 = 0$

$-\frac{E^2}{\lambda^2} + S(r)\,\dot{r}^2 = 0$

$\phi^2 = S^{-1} M_p^2 = S^{-1} K_4^{-2}$

$-S(r)\,\dot{r}^2 - \left(1-\frac{2M}{r}\right)S(r) + \frac{E^2}{\lambda^2}\frac{S(r)}{M_p^2} = 0$

$S(r) = \left(1 + \frac{L^4}{r^4}\right)$

$\phi^{-2} = S K_4^2$

$S(r)\,\dot{r}^2 + \left(1-\frac{2M}{r}\right)S(r) - \frac{E^2}{\lambda^2}\frac{S(r)}{M_p^2} = 0$

Se $E = \lambda M_p \Rightarrow S(r)\,\dot{r}^2 - \frac{2M}{r}S(r) = 0$

$\frac{\lambda}{M^2} = \dot{r}^2 S(r)$

$\frac{2p}{dp} > 0$

$M^2 = \left(\frac{2p}{\lambda p}\right)^2 \cdot \lambda\left(\frac{b^4}{L^4}+1\right)$

$2p\,|M^2| = -\lambda p\,\sqrt{\lambda\left(\frac{b^4}{L^4}+1\right)}$

$\left(1-\frac{E^2}{2M\lambda}\right)S(r)$

$0 < 1 - \frac{2M\lambda}{\lambda^3} \overset{r\to\infty}{=} 0 = \frac{2M\lambda}{\lambda^3} - 1 + \dot{r}^2 \quad \infty \leftarrow \lambda$ if $E \neq \lambda M_p \Rightarrow$ for $r\to\infty$

$$\mathcal{L} = -M_p \Rightarrow -\frac{\lambda^2 \phi^2 \hat{\rho}_{tt}\, \dot{t}}{M_p} = E \qquad [E] = M$$

$$-\lambda^2 \phi^2 \dot{x}^2 = \lambda^2 M_p^2 \qquad \frac{ds^2}{d\tilde{\tau}^2} = \lambda^2 M_p^2$$

$$\dot{x}^2 = -\frac{M_p^2 \lambda^2}{\lambda^2 \phi^2} \qquad \dot{x}^2 = -\frac{M_p^2}{\phi^2}$$

$$\hat{\rho}_{tt}\, \dot{t}^2 + \hat{\rho}_{rr}\, \dot{r}^2 = -\frac{M_p^2 \lambda^2}{\lambda^2 \phi^2}$$

$$\hat{\rho}_{tt}\, \frac{E^2 M_p^2}{\lambda^4 \phi^4 \hat{\rho}_{tt}^2} + \hat{\rho}_{rr}\, \dot{r}^2 = -\frac{M_p^2 \lambda^2}{\lambda^2 \phi^2}$$

$$\hat{\rho}_{tt}\, \hat{\rho}_{rr}\, \dot{r}^2 + \frac{E^2 M_p^2}{\lambda^4 \phi^4} + \frac{\lambda^2 M_p^2}{\lambda^2 \phi^2}\, \hat{\rho}_{tt} = 0$$

$$-S^2(r)\, \dot{r}^2 + \frac{\lambda^2 M_p^2}{\lambda^2} \cdot \frac{S(r)}{M_p^2} \left(-1 + \frac{2M}{r}\right) S(r) + \frac{E^2 M_p^2}{\lambda^4} \cdot \frac{1}{M_p^4}\, S^2(r) = 0$$

$E = \lambda^2 M_p$

$$\boxed{\dot{r}^2 = \frac{2M}{r}}$$

---

kerr: $\dfrac{E^2}{f^2 \phi^2} + \hat{\rho}_{rr} \hat{\rho}_{tt}\, \dot{r}^2 + \hat{\rho}_{tt} = 0$ $\qquad \hat{\rho}_{rr} = S(r,\theta) \cdot \dfrac{\Sigma}{\Delta}$ $\qquad S = \left(1 + \dfrac{L^2}{\Sigma}\right)^{-\frac{1}{4}}$

$\hat{\rho}_{tt} = -S(r,\theta)\left(1 - \dfrac{r_s r}{\Sigma}\right)$ $\qquad = \left(1 + \dfrac{L^2}{r^2 + a^2}\right)^{-\frac{1}{4}}$

$$-S^4 \cdot \frac{\Sigma}{\Delta}\left(1 - \frac{r\, r_s}{\Sigma}\right)\dot{r}^2 - S^4\left(1 - \frac{r r_s}{\Sigma}\right) + \frac{E^2}{f^2} S^4 = 0$$

$\Sigma = r^2 + a^2 \cos^2\theta \overset{\theta = \pi/2}{=} r^2$ $\qquad r \to \infty$

$\Delta = r^2 - r_s r + a^2$ $\qquad S\dot{r}^2\left(\frac{-r r_s}{a^2}\right) = \frac{r_s}{r}$

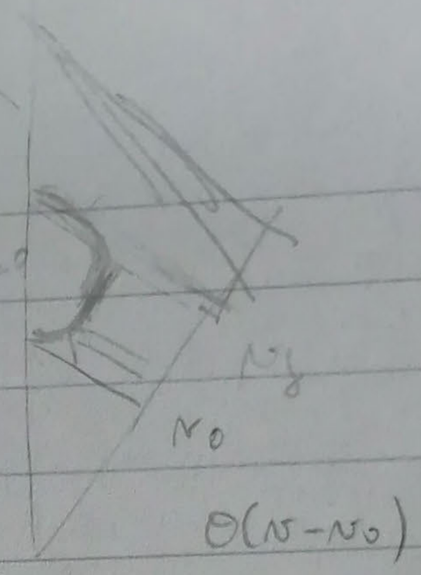
$E = f K_t \Rightarrow$

$$S\, \frac{1}{\Delta}(r^2 + a - r r_s)\dot{r}^2 + 1 - \frac{r r_s}{r^2} - 1 = 0$$

$$S \cdot \dot{r}^2\left(\frac{r^2 - r r_s}{\Delta}\right) = \frac{r_s}{r} \Rightarrow S\dot{r}^2 \frac{(r^2 - r r_s)}{(r^2 - r r_s + a^2)} = \frac{r_s}{r} \to S\dot{r}^2\left(\frac{r^2 - r r_s}{a^2}\right) \approx \frac{r_s}{r}$$

$S\dot{r}^2 = -\dfrac{a^2}{r^2}$ $\qquad \left(\dfrac{L^2}{r^2}\right)^2 \dot{r} = a\sqrt{\dfrac{}{r^2}}$

$$\hat{g}_{tt}\,\dot{t}^2 + \hat{g}_{rr}\,\dot{r}^2 = -\lambda\phi^2 e^2$$



$$\dot{t} = -\frac{E\,e}{\hat{g}_{tt}}$$

$$\frac{\hat{g}_{tt}\,E^2 e^2}{\hat{g}_{tt}^2} + \hat{g}_{rr}\,\dot{r}^2 = -\lambda\phi^2 e^2$$

$$E^2 e^2 + \hat{g}_{rr}\,\hat{g}_{tt}\,\dot{r}^2 + \lambda\phi^2 e^2 \hat{g}_{tt} = 0$$

$$-s^2$$

$$-s^2\dot{r}^2 + e^2\left(E^2 + \lambda\phi^2\hat{g}_{tt}\right) = 0$$

$$e^2 = \frac{\phi^2}{\lambda M_p^2} \qquad \lambda M_p^2 s^{-1} s\left(-1 + \frac{2M}{r}\right)$$

$$s^2\dot{r}^2 + e^2\lambda M_p^2\left(1 - \frac{2M}{r}\right) + e^2E^2 = 0$$

$$e = \frac{1}{\phi} = \frac{1}{\sqrt{\lambda}M_p s^{-\frac{1}{2}}} \qquad \left(e = \frac{1}{\sqrt{\lambda}M_p s^{-1}}\right)$$

poss o?

$$s^2\dot{r}^2 + s^{+1}\left(1 - \frac{2M}{r}\right) - \frac{1}{\lambda M_p^2}\,s\,E^2 = 0 \qquad E = \sqrt{\lambda}\,M_p$$

$$s\dot{r}^2 + \left(1 - \frac{2M}{r}\right) = 1$$

$$s\dot{r}^2 = \frac{2M}{r}$$

$$s = \left(1 + \frac{L}{r}\right)^q$$

$$\frac{dp}{dz} = -\sqrt{s^{-1}\frac{2M}{r}} = -\sqrt{\left(1+\frac{L}{r}\right)^{-q}\frac{2M}{r}} = -\sqrt{\frac{2M}{r}}\frac{1}{\left(1+\frac{L}{r}\right)^2}$$

$$\sqrt{\frac{r}{2M}}\frac{(r+L)^2}{r^2}\,dr = -dz \qquad \text{for } \frac{L}{r} > -1$$

$$\tau = -\frac{1}{3}\sqrt{\frac{2r}{M}}\frac{(r^2+6rL-3L^2)}{r}$$

OPPURE $\quad s(r) = \left(1+\frac{L^2}{r^2}\right)^2$ $\qquad$ MEGLIO

$$\frac{dp}{dz} = -\sqrt{\frac{2M}{r}}\left(\frac{r^2+L^2}{r^2}\right)^{-1}$$

$$-\sqrt{\frac{r}{2M}}\frac{r^2+L^2}{r^2}\,dr = -dz$$

$$\tau = -\frac{1}{3}\sqrt{\frac{2r}{M}}\frac{r^2-3L^2}{r} + \text{cost}$$

$r\sim 0 \Rightarrow \sqrt{\frac{2}{M}}L\frac{1}{\sqrt{r}} \simeq \tau$

$\frac{2}{M}L^4\frac{1}{r} \simeq \tau^2$

$\boxed{r \simeq \frac{2L^4}{M\tau^2}}$

$\tau = 0$ for $r = r_0 \Rightarrow$ cost $= +\frac{1}{3}\sqrt{\frac{2r_0}{M}}\frac{r_0^2-3L^2}{r_0}$

$$\tau = -\frac{1}{3}\sqrt{\frac{2r}{M}}\frac{r^2-3L^2}{r} + \frac{1}{3}\sqrt{\frac{2r_0}{M}}\frac{r_0^2-3L^2}{r_0}$$

$$\tau = \;\;\;\

CANUTO Geodesics

$$\left(1-\frac{2M}{r}\right)\dot{t}^2 - \frac{1}{1-\frac{2M}{r}}\dot{r}^2 = \epsilon\beta^2$$

$$\frac{1}{\beta^2} = \frac{1}{\phi^2} = S \qquad \phi^2 = S^{-1}$$

$$S\left(1-\frac{2M}{r}\right)\dot{t}^2 - \frac{S}{1-\frac{2M}{r}}\dot{r}^2 = \epsilon$$

$$\beta^{-1}\dot{t}\,B = E$$

$$\phi^{-1}\dot{t}\,B = E \qquad \dot{t} = \frac{E}{B}\phi$$

$$\dot{t}^2 = \frac{E^2}{B^2}\phi^2 = \frac{E^2}{B^2 S^2} = \frac{E^2}{\hat{g}_{tt}^2}$$

$$\hat{\rho}_{tt}\dot{t}^2 - \hat{\rho}_{rr}\dot{r}^2 = \epsilon$$

$$\frac{\hat{\rho}_{tt}E^2}{\hat{\rho}_{tt}^2} - \hat{\rho}_{rr}\dot{r}^2 = \epsilon.$$

$$E^2 - \underbrace{\hat{\rho}_{rr}\hat{\rho}_{tt}}_{S^2}\dot{r}^2 = \epsilon\,\hat{\rho}_{tt} \qquad S\left(1-\frac{2M}{r}\right)$$

$(-+++)$ $\quad ds^2 = S(r)\left[-\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^{(2)}\right]$ (?)

$\downarrow N_0$

$g_{\mu\nu} = \beta^2 \hat{g}_{\mu\nu} = S\,\hat{g}_{\mu\nu}$ $\quad \beta = \sqrt{S}$ $\quad S = \left(1+\frac{L^2}{r^2}\right)^2$

si $\rightarrow$ $\boxed{\beta = S^{-\frac{1}{2}}}$

(1) $\boxed{\frac{2\ddot{\beta}}{\beta}}$ $\quad \dot{\beta} = \frac{-2L^2}{r^3}\dot{r}$, $\ddot{\beta} = \frac{-2L^2}{r^3}\ddot{r} - 2\dot{r}L^2\left(\frac{-3r^2\dot{r}}{r^6}\right)$ $\quad \beta = 1 + \frac{L^2}{r^2}$

$\ddot{\beta} = -\frac{2L^2}{r^3}\left(\ddot{r} - \frac{3\dot{r}^2}{r}\right)$ $\quad \dot{\beta} = \frac{-2L^2 r\,\dot{r}}{r^4}$

$\downarrow$

$\frac{2\ddot{\beta}}{\beta} = \frac{6L^2 M r(3L^2+7r^2)}{(L^2+r^2)^4} \rightarrow 0^+$ $\quad = -\frac{2L^2}{r^3}\dot{r}$

(2) $\boxed{-\frac{4\dot{\beta}^2}{\beta^2}} = -4\cdot\left(\frac{-2L^2\dot{r}}{(L^2+r^2)r}\right)^2 = -\frac{16L^4\dot{r}^2}{(L^2+r^2)^2 r^2}$ $\quad 0^-$

$= \frac{-16L^4}{(L^2+r^2)^2 r^2}\cdot\frac{2M}{r}\cdot\left(\frac{r^2}{r^2+L^2}\right)^2 = -\frac{32L^4 Mr}{(L^2+r^2)^4}$

(3) $\boxed{\frac{\beta_\mu \beta^\mu}{\beta^2}} = \partial_r\beta\,\partial^r\beta\frac{1}{\beta^2} = g^{rr}(\partial_r\beta)^2\frac{1}{\beta^2} = \left(1-\frac{2M}{r}\right)S^{-1}(r)\left(\frac{-2L^2}{r^3}\right)^2\frac{1}{\beta^2}$

$= \left(1-\frac{2M}{r}\right)\frac{r^4}{(r^2+L^2)^2}\frac{4L^4}{r^6}\frac{1}{\beta^2}$

$= \left(1-\frac{2M}{r}\right)\frac{4L^4}{(r^2+L^2)^2 r^2}\left(\frac{r^2}{r^2+L^2}\right)^2 \rightarrow -\frac{2M\,4L^4 r^4}{r^5 L^4} \rightarrow -r$

(4) $\boxed{\frac{\beta^\mu{}_{;\mu}}{\beta^3}} = \frac{\Box\beta}{\beta^3}$

$\Box\phi = \frac{1}{\sqrt{-g}}\partial^\mu\left(\sqrt{-g}\,g_{\mu\nu}\partial^\nu\phi\right) = \frac{1}{\sqrt{-g}}\partial^r\left(g_{rr}\partial^r\phi\right)$

$= \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}\,g^{rr}\partial_r\phi\right) = \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}\left(1-\frac{2M}{r}\right)\cdot S'(r)\cdot\left(\frac{-2L^2}{r^3}\right)\phi\right)$

$\sqrt{-g} = \left(r^4\sin^2\theta\; S^4(r)\right)^{\frac{1}{2}} = r^2\sin\theta\, S^2 = r^2\left(\frac{L^2+r^2}{r^2}\right)^4\sin\theta$

$= \frac{(L^2+r^2)^4}{r^6}\sin\theta$

?

$$\square\beta = \frac{2L^2\left[(-6M+r)r^2 + L^2(7r-16M)\right]}{r^3(L^2+r^2)^2}$$

$$\frac{\square\beta}{\beta} = \frac{r^2}{r^2+L^2}\,\frac{2L^2\left[(-6M+r)r^2+L^2(7r-16M)\right]}{r^3(L^2+r^2)^2} \longrightarrow -\frac{32M}{rL^2}$$

Notice

(5)

$$\frac{-2\beta_\mu \dot{u}^\mu}{\beta} = -2\left(\frac{-2L^2}{r^2}\right)\dot{u}^r\,\frac{r^2}{(r^2+L^2)}$$

$$= \frac{4L^2\,\dot{u}^r}{r(r^2+L^2)}$$

$$u^r = \frac{dr}{d\tau}$$

$$u^r \equiv \dot{r} = -\sqrt{\frac{2M}{r}}\,\frac{r^2}{r^2+L^2}$$

$$\dot{u}^r = \ddot{r} = -\frac{Mr^2(r^2-3L^2)}{(r^2+L^2)^3}$$

"$\frac{\partial \dot{r}}{\partial r}\left(\frac{dr}{d\tau}\right)_{\to \dot{r}}$"

$$\Rightarrow \boxed{\frac{-2\beta_\mu \dot{u}^\mu}{\beta}} = -\frac{4L^2Mr(r^2-3L^2)}{(r^2+L^2)^4} \xrightarrow{r\to 0} 0^+$$

Total contribution:

$$(\dot{u}^\mu)_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,\dot{u}^\mu\right)$$

(?

$$\dot{u}^\theta, \dot{u}^\varphi = 0 \Rightarrow$$

$$= \frac{1}{\sqrt{g}}\left[\partial_t\left(\sqrt{g}\,\dot{u}^t\right) + \partial_r\left(\sqrt{g}\,\dot{u}^r\right)\right]$$

$$\underbrace{f(r)}_{=0}$$

$$\dot{u}^t = \frac{d\,t}{d\tau}\Big|_0 = \frac{d}{d\tau}\left[\frac{E}{\lambda m_p\, s^{-\frac{1}{2}}(r)\, s(r)\left(-1+\frac{2M}{r}\right)}\right]$$

$$= -\frac{d}{dr}\left(\frac{E}{\lambda m_p}\,\frac{1}{s^{\frac{1}{2}}(r)\left(1-\frac{2M}{r}\right)}\right)\dot{r}$$

$$= -\frac{E}{\lambda m_p}\,\frac{d}{dr}\left(\frac{1}{s^{\frac{1}{2}}(r)\left(1-\frac{2M}{r}\right)}\right)\dot{r}$$

$$= -\frac{E}{\lambda m_p}\, 2\sqrt{\frac{2M}{r}}\,\frac{r^4\left(L^2(3M-r)+Mr^2\right)}{(r-2M)^2\,(L^2+r^2)^3} = f(r)$$

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \tau}\cdot\frac{\partial \tau}{\partial t}$$

$$= \left(\frac{\partial v}{\partial r}\right)\left(\frac{\partial r}{\partial \tau}\right)\cdot\left(\frac{\partial \tau}{\partial t}\right) = \frac{\partial v}{\partial r}\cdot\left(\frac{\partial r}{\partial t}\right) = 0$$

$$\Rightarrow -(\dot{u}^r)_{;\mu} = -\frac{1}{\sqrt{-g}}\,\partial_r\left(\sqrt{-g}\,\dot{u}^r\right)$$

$$= -\frac{v^2}{(r^2+L^2)^4}\cdot\partial_r\left(\frac{(r^2+L^2)^4}{r^2}\cdot(-1)\,\frac{Mr^2\left(r^2-3L^2\right)}{(r^2+L^2)^3}\right)$$

$$= -\frac{4Mr^3}{L^2+r^2}\left(L^2-r^2\right) \longrightarrow 0^-$$

$$\dot{r} = \sqrt{\frac{2M}{r}}\, s^{-1/2}(r) \qquad \beta = s^{+\frac{1}{2}}$$

$$\ddot{r} = -\frac{m\,(s(r) + r\,s'(r))}{r^2\, s^2(r)}$$

$$\frac{\ddot{\beta}}{\beta} = \frac{e}{\beta}\frac{d}{dt}\left(\beta^{-1}\dot{r}\right) = \frac{2}{\beta}\left(\beta''\dot{r}^2 + \beta'\ddot{r}\right)$$

$$\frac{E}{m} \rightarrow \frac{E}{\sqrt{f}\,M_p\,\phi} = \frac{\mathcal{E}}{\phi}$$

$$\phi^* = s^{-\frac{1}{2}} M_p$$

$$\hat{g}^*_{\mu\nu} = s\,\hat{g}_{\mu\nu} \qquad \phi^{-2}\hat{g}^*_{\mu\nu}$$

Schw

$$s = \left(1 + \frac{L^2}{\rho^2}\right)^2$$

$$\frac{ds^{*2}}{dt^2} = -1$$

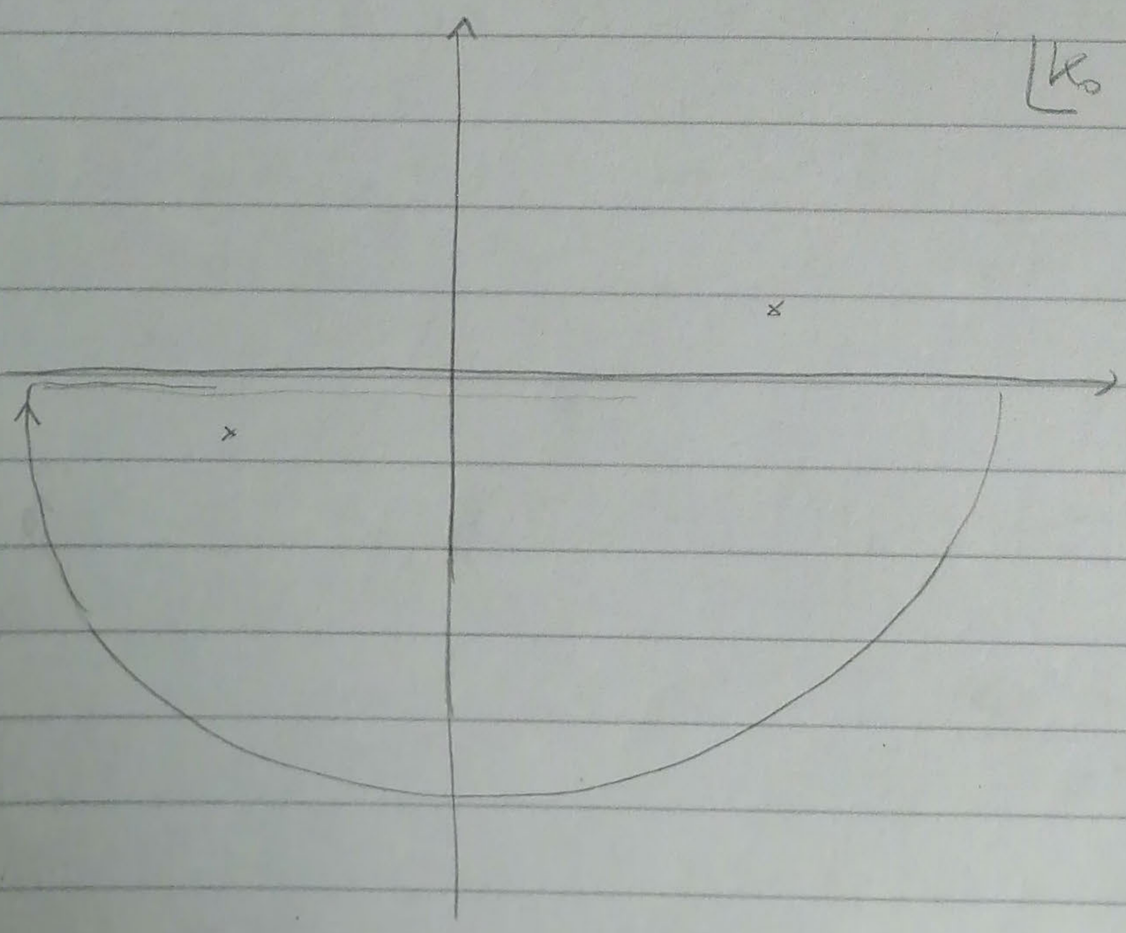
CARONE - PAPER

$$\frac{1}{1+\lambda\Sigma} = \frac{1}{|1+\lambda\Sigma|^2} \cdot 1+\lambda\Sigma^*$$

$$\Downarrow$$

$$Im\left(\frac{1}{1+\lambda\Sigma}\right) = Im\left(\frac{1}{|1+\lambda\Sigma|^2} \cdot 1+\lambda\Sigma^*\right)$$

$$= -\frac{\lambda\, Im(\Sigma)}{|1+\lambda\Sigma|^2}$$



Radial geodesic for photons

(?)

$ds^2 = ds^2 = 0$

Sch. coordinates: $\left(\theta = \frac{\pi}{2},\ \varphi = const\right)$

$$-\left(1-\frac{2m}{r}\right)dt^2 + \frac{1}{1-\frac{2m}{r}}dr^2 = 0$$

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2m}{r}\right)^2 \qquad \dot{r} = -\left(1-\frac{2m}{r}\right)$$

$$\ddot{r} = 2m\left(-\frac{\dot{r}}{r^2}\right) = +\frac{2m}{r^2}\left(1-\frac{2m}{r}\right)$$

$$\frac{dr}{dt} = \pm\left(1-\frac{2m}{r}\right)$$

K-S. coordinates: $\left(\theta = \frac{\pi}{2} \quad \varphi = const\right)$

$$ds^2 = 0 \iff dS^2 = 0 \implies -dT^2 + dX^2 = 0$$

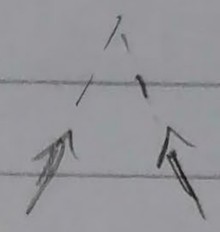
$$\frac{dX}{dT} = \pm 1 \implies \dot{X} \equiv \frac{dX}{dT} = 1$$

① $\frac{2\ddot{\beta}}{\beta} = \frac{2}{\beta}\left[-\frac{2L^2}{r^3}\left(\ddot{r} - \frac{3\dot{r}^2}{r}\right)\right] = \frac{2L^2}{\beta}\frac{(8m-3r)(2m-r)}{r^4}\frac{2r^2}{r^2+L^2}$

$= \frac{4L^2}{r^4(r^2+L^2)}\cdot(8m-3r)(2m-r) \longrightarrow 0^+$

② $-\frac{4\dot{\beta}^2}{\beta^2} =$

$$\dot\Theta + \frac{1}{3}\Theta^2 = c$$



$$c \cong \pm\frac{A}{r^4} \simeq \pm A' r^2$$

$$r \to \infty \Rightarrow c \to \infty$$

$$\frac{d\Theta}{dr} = c - \frac{1}{3}\Theta^2$$

$$\frac{d\Theta}{c - \frac{1}{3}\Theta^2} = d\tau$$

$$-R_{\mu\nu} u^\mu u^\nu = -\Lambda g_{\mu\nu} u^\mu u^\nu = \Lambda$$

$$\sqrt{\frac{3}{c}}\ \text{Arctanh}\left(\frac{\Theta}{\sqrt{3c}}\right) = \tau + \text{const}$$

$$\Theta(\tau) = \Theta_0 \Rightarrow \text{const} = \sqrt{\frac{3}{c}}\ \text{Arctanh}\left(\frac{\Theta_0}{\sqrt{3c}}\right) = \text{const}$$

$$\tau = \sqrt{\frac{3}{c}}\left[\text{Arctanh}\left(\frac{\Theta}{\sqrt{3c}}\right) - \text{Arctanh}\left(\frac{\Theta_0}{\sqrt{3c}}\right)\right]$$

$$\Theta = \sqrt{3c}\ \text{Tanh}\left(\sqrt{\frac{c}{3}}\,\tau\right) + \text{const}'$$

$$\Theta(\tau = 0) = \Theta_0$$

$$\Theta(0) = \sqrt{3c}\ \text{Tanh}\sqrt{\frac{c}{3}}\ (\tau = 0) + \text{const}'$$

$$\boxed{\Theta(\tau) = \sqrt{3c}\ \text{Tanh}\left(\sqrt{\frac{c}{3}}\,\tau\right) + \Theta_0}$$

$$\dot{\vartheta} + \frac{1}{3}\Theta^2 = 1$$

$$\Theta = \sqrt{3}\ \mathrm{Tanh}\left(\sqrt{3^{-1}}\,\tau\right) + \vartheta_0$$

$$\Theta(0) = \vartheta_0$$

$$\frac{d\Theta}{dv}\frac{dv}{d\tau} + \frac{1}{3}\Theta^2 = \frac{2}{r}$$

?

$$\Theta'\left(-\sqrt{\frac{2M}{r}}\,\frac{v^2}{v^4+L^2}\right) + \frac{1}{3}\Theta^2 - \frac{2}{r} = 0$$

$$-\Theta'\sqrt{2M}\,\frac{r^{3/2}}{L^2} + \frac{1}{3}\Theta^2 - \frac{2}{r} = 0$$

$$\Theta' - \underbrace{\frac{L^2}{3\sqrt{2M}}}_{c}\, r^{-3/2}\Theta^2 + b r^{-\frac{3}{2}} = 0$$

$\phi = \phi(x), \phi(x)$

$$\int d^D x \int_0^{r_*} dr \left[\frac{R}{r^2} - \frac{2\Lambda}{r^*} + - \left(2R_{,\nu} - \phi^{,\nu} + \frac{1}{D-2} g^{,\nu} \phi \right) e^{-r\Box} \phi_{,\nu} \right]$$

$$\int d^D x \int dr \left[\frac{R}{r^2} - \frac{2\Lambda}{r^*} - \int_0^{r} \left(2R^{\mu\nu} - \phi^{,\mu\nu}(s) + \frac{1}{D-2} g^{,\nu} \phi_s \right) \cdot \phi_{\mu\nu}(s) \, ds \right]$$

$$l = \int_0^r ds \; \chi_{\mu\nu}(r-s) \, (\Box + \partial_s) \, \phi^{,\nu}(s) \, ds \Big]$$

$$- \lambda_{\mu\nu} \left[(\Box + \partial_r) \chi^{\mu\nu} - R^{\mu\nu} \right]$$

$$\frac{\int \int}{\int \chi_{\mu\nu}} = 0 \Rightarrow (\Box + \partial_s) \phi_{\mu\nu}(s) = 0$$

$$\frac{\int \int}{\int \phi^{,\nu}} = 0 \Rightarrow -(\Box + \partial_s) \chi_{,\nu}(r-s) + 2R_{,\nu} - 2\phi_{,\nu} + \frac{1}{D-2} g_\mu \phi$$

$$\frac{e^{-r_* \Box} - 1}{\Box} = - \int_0^{r_*} dr \; e^{-r\Box} = + \frac{1}{\Box} e^{-r\Box} \Big|_0^{r_*}$$

$$H(\Box) \qquad H = \int_0^z dw \, \frac{1 - e^{-w^2}}{w} = \int_0^z dw \int_0^1 ds \; e^{-s w^2} \cdot w$$

$$\frac{e^{-1}}{\Box} = \int_0^z dw \int_0^1 ds \; w \left(-\frac{1}{s w^2} \right) e^{-s w^2} \Big|_0^1$$

$$\Rightarrow H = \int_0^z dw \int_0^1 ds \; e^{-s w^2} \, w$$

Unruh

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$$\dot{F}(\Delta E) = -\int_{-\infty}^{+\infty} d\Delta\tau \; e^{-i\Delta E \Delta\tau/\hbar} \underbrace{\langle 0_M | \phi(x(\tau_1)) \; \phi x(\tau_2) | 0_M \rangle}_{G_+}$$

$$-\int_{-\infty}^{+\infty} d\Delta\tau \; e^{-i\Delta E \Delta\tau/\hbar} \; \frac{\hbar}{4\pi\left[(\Delta t - i\epsilon)^2 - (\Delta\vec{x})^2\right]}$$

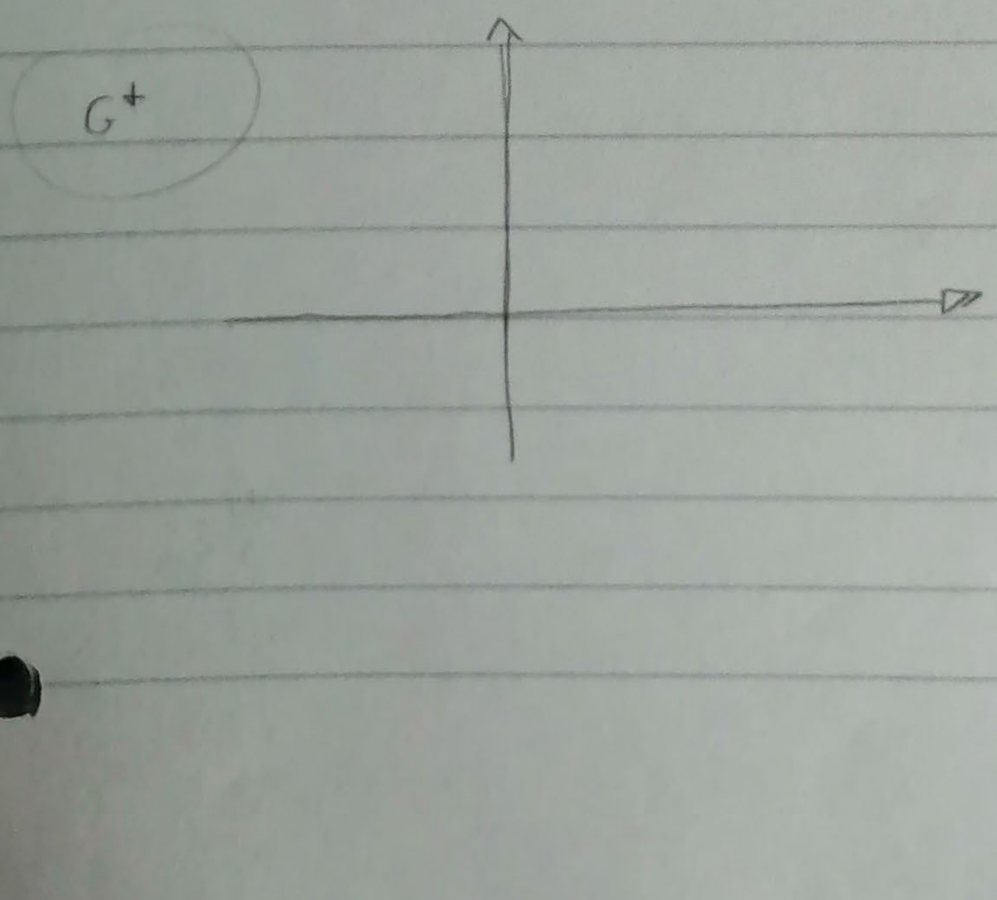
$$t = \frac{1}{a}\sinh(a\tau) \qquad \Delta t = \frac{1}{a}\sinh a\tau$$

Note

$$G(p) = \frac{1}{p^2}$$

$$\dot{F}(\Delta E) = -\int_{-\infty}^{+\infty} d\Delta\tau \; e^{-i\Delta E \Delta\tau/\hbar} \; \frac{1}{\frac{4}{a^2}\sinh\left(\frac{a}{2}\Delta\tau\right)}$$

$$G(p) = \frac{e^{-H(p)}}{p^2}$$



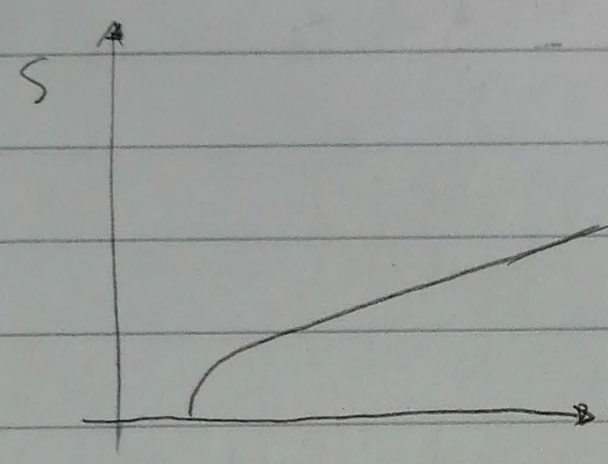
conformal gravity entropy

$$T = \frac{1}{8\pi m} \qquad A = 4\pi R^2 = 4\pi\left(r^2 + \frac{L^4}{r^2}\right)$$

$$= 4\pi\left((2m)^2 + \frac{L^4}{(2m)^2}\right)$$

$$S = \int \frac{dm}{T} = 8\pi \int m\, dm = 4\pi m^2 = \frac{1}{8}\left(A + \sqrt{A^2 - 64\pi^2 L^4}\right)$$

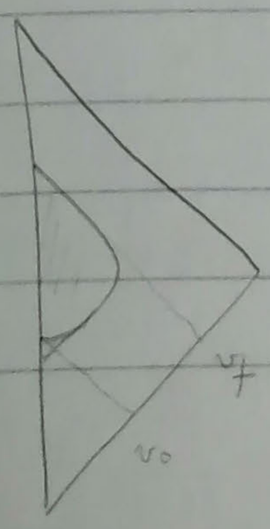
$$= \frac{1}{8}\left(A + A\left(1 - \frac{64 L^4 \pi^2}{A^2}\right)^{\frac{1}{2}}\right)$$



$$\frac{dM}{dt} = \sigma A T^4 = \sigma\, 4\pi\left((2m)^2 + \frac{L^4}{(2m)^2}\right)\cdot\frac{1}{(8\pi m)^4}$$

$$\underset{m\to 0}{\simeq} \frac{1}{m^6}$$

$$t = -\frac{1}{\sigma}\,\frac{L^4 + 80 m^4}{20480\,\pi^3 m^5} + \text{const.}$$



$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}}$$

Se $r<0 \Rightarrow 1-\frac{2m}{r}>0$

$ds^2=0 \Rightarrow \left|\frac{dr}{dt}\right| = \left|1-\frac{2m}{r}\right|$ $\qquad \frac{dr}{dt}>0 \Rightarrow \frac{dr}{dt} = 1-\frac{2m}{r}$

$$1-\frac{2m}{r} = \begin{cases} 1-\frac{2m}{r} & \text{se } r>2m \\ \frac{2m}{r}-1 & \text{se } r<2m \end{cases}$$

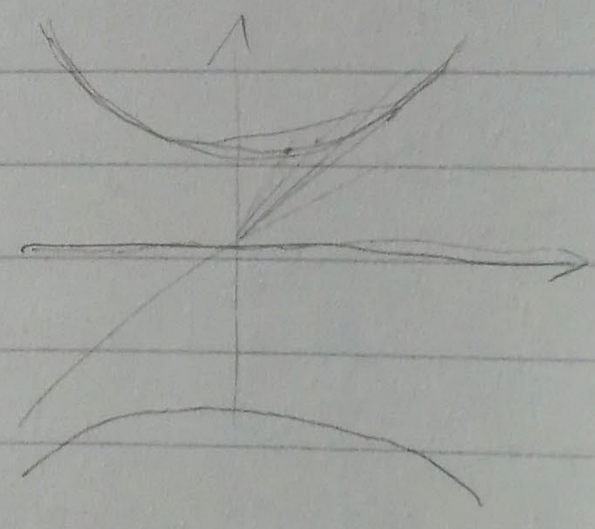
$$\left|\frac{dr}{dt}\right| = \begin{cases} -\frac{dr}{dt} & \text{se } \frac{dr}{dt}<0 \qquad \frac{dT}{dR}<0 \\ \end{cases}$$

$$-\frac{dr}{dt} = \frac{2m}{r}-1$$

$$\frac{dr}{dt} = 1-\frac{2m}{r} \qquad \frac{dr}{d\rho} = \frac{r-2m}{r}$$

$$\frac{r}{r-2m}\,dr = d\rho \qquad \frac{d\rho}{dt} = \frac{1}{\frac{dt}{d\rho}} \qquad \frac{d^2\rho}{dt^2} = -\frac{\frac{d^2t}{d\rho^2}}{\left(\frac{dt}{d\rho}\right)^3}$$

$$= \frac{2m}{r^2}\left(1-\frac{2m}{r}\right)$$



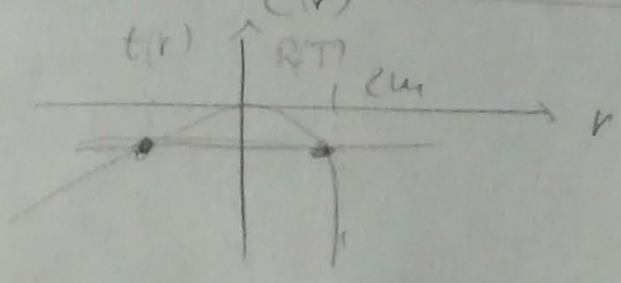
$$r + 2m\lg(2m-r) = t + \text{cost} \Rightarrow T + 2m\lg(2m-T) = R + \text{cost}$$

$$R(0) = 2m\lg 2m + \text{cost}$$

$$1 + \frac{2m}{2m-r}\cdot(-1) = \frac{2m-r-2m}{2m-r} = \frac{r}{r-2m}$$

$$r + 2m\lg(2m-r) - 2m\lg 2m = t \qquad t(r)$$

$$r + 2m\lg\left(1-\frac{r}{2m}\right) = t(r)$$



1) $\underline{f(R)}$ nonlocal gravity; or:

2) $\underline{\text{UV completion of } f(R) \text{ gravity}}$

$f(R(g'))$

3) UV completion of gravitational theories.

$$S(g') = S'(g)$$

$$= S(g) + E\,F\,E$$

$$f[R(g')] = f[R(g)] + E_{f(R)} \cdot F \cdot E_{f(R)}$$

$$= f_{\text{Nonlocal}}(R) \equiv S'(g)$$

Interazione:

$$R^2 \; e^{H} \; R^2$$

$$p^4 \quad p^{2k+2} \quad p^4$$

$$R \Box R \; e^{H} \; \Box R$$

$$p^4 \quad p^{2k+2} \quad p^4$$

(Formulae viste su Wikipedia; formule in Exact solutions viste)

EOM: $f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f'(R) \equiv E_{\mu\nu}$

$f(R(g'))$

$$f(R) + \left[f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f'(R) \right] \cdot F^{\mu\nu\sigma\rho} \cdot \left[f'(R) R_{\sigma\rho} - \frac{1}{2} f(R) g_{\sigma\rho} - (\nabla_\sigma \nabla_\rho - g_{\sigma\rho} \Box) f'(R) \right]$$

(?)

non è lineare

Ex:

$f(R) = R + \epsilon R^2 \qquad f'(R) = 1 + 2\epsilon R$

$$R + \epsilon R^2 + \left[R_{\mu\nu} + 2\epsilon R R_{\mu\nu} - \frac{1}{2}(R + \epsilon R^2) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) 2\epsilon R \right]$$

$p^2 + p^4$ $\qquad p^4$

$F^{\mu\nu\sigma\rho}$

$$\left[R_{\sigma\rho} + 2\epsilon R R_{\sigma\rho} - \frac{1}{2}(R + \epsilon R^2) g_{\sigma\rho} - (\nabla_\sigma \nabla_\rho - g_{\sigma\rho} \Box) 2\epsilon R \right]$$

1) Devo ricostruire: $R + G_{\mu\nu}\gamma_G R^{\mu\nu} + R\gamma_{S'}R$ + Vertexes

$D = 4 \qquad \gamma_G = \frac{e^{H_2} - 1}{\Box}, \qquad \gamma_{S'} = \frac{1}{6}\frac{e^{H_2} - e^{H_0}}{\Box}$

2) Poté che $f(R)$ contiene 1-graviton ed 1 scalar $\Rightarrow$ la azione della teoria nonlocale deve contenere la stessa azione perturbativa.

$\Downarrow$ KIM

$$f_{NL}(R) = R + \epsilon R^2 + R_{\mu\nu} F^{\mu\nu\rho\sigma} R_{\rho\sigma} +$$

$$+ R_{\mu\nu} F^{\mu\nu\rho\sigma}\left(-\tfrac{1}{2} R g_{\rho\sigma}\right) +$$

$$+ R_{\mu\nu} F^{\mu\nu\rho\sigma}\left(-\nabla_\rho \nabla_\sigma + g_{\rho\sigma}\Box\right) R \, z\epsilon$$

$$-\tfrac{1}{2} R g_{\mu\nu} F^{\mu\nu\rho\sigma} R_{\rho\sigma}$$

$$-\tfrac{1}{2} R g_{\mu\nu} F^{\mu\nu\rho\sigma}\left(-\tfrac{1}{2} R\right) g_{\rho\sigma}$$

$$-\tfrac{1}{2} R g_{\mu\nu} F^{\mu\nu\rho\sigma}\left(-\nabla_\rho \nabla_\sigma + g_{\rho\sigma}\Box\right) R \, z\epsilon +$$

$$+ z\epsilon\left[\left(-\nabla_\mu \nabla_\nu + g_{\mu\nu}\Box\right) R\right] F^{\mu\nu\rho\sigma} R_{\rho\sigma}$$

$$+ z\epsilon\left[\left(-\nabla_\mu \nabla_\nu + g_{\mu\nu}\Box\right) R\right] F^{\mu\nu\rho\sigma}\left(-\tfrac{1}{2} R\right) g_{\rho\sigma}$$

$$+ z\epsilon\left[\left(-\nabla_\mu \nabla_\nu + g_{\mu\nu}\Box\right) R\right] F^{\mu\nu\rho\sigma} z\epsilon\left(-\nabla_\rho \nabla_\sigma + \Box g_{\rho\sigma}\right) R$$

$$f^{\mu\nu\epsilon\sigma} = \rho^{\mu\nu}\rho^{\epsilon\sigma} f_1 + \rho^{\mu\epsilon}\rho^{\nu\sigma} f_2$$

$$f_{NL}(R) = R + \epsilon R^2 + R\, f_1\, R + R_{\mu\nu} f_2 R^{\mu\nu} +$$

$$+ R\, f_1 \left(\tfrac{1}{2} R\right) \cdot \epsilon^2 + R\, f_2 \left(\tfrac{1}{2} R\right)$$

$$+ R\, f_1 \left(-\Box + \epsilon \Box\right) R\, 2\epsilon + 2\epsilon R_{\mu\nu} f_2 \left(-\nabla^\mu \nabla^\nu + \rho^{\mu\nu}\Box\right) R$$

$$\| \quad G_{\mu\nu} + \tfrac{1}{2} R \rho_{\mu\nu}$$

$$\rho^\mu G_{\mu\nu} = 0$$

$$2\epsilon \cdot \tfrac{1}{2} R\, f_2 (-\Box) R$$

$$-\tfrac{1}{2} R\, \epsilon^2 f_1 \quad R - \tfrac{1}{2} R\, f_2 R +$$

$$+ \tfrac{1}{\epsilon} R\, \epsilon\, f_1 R\, \epsilon + \tfrac{1}{\epsilon} R\, f_2\, \epsilon R +$$

$$-\tfrac{1}{2} R\, \epsilon^2 f_1 \left(-\Box + \epsilon \Box\right) R\, 2\epsilon \quad -\tfrac{1}{2} R\, f_2 \left(-\Box + \epsilon\Box\right) R\, 2\epsilon$$

$3\Box$

$$+ 2\epsilon \left[\left(-\Box + \epsilon\Box\right) R\right] f_1\, R + 2\epsilon \left(-\nabla_\mu \nabla_\nu + \rho_{\mu\nu}\Box\right) R\, f_2\, R^{\mu\nu} +$$

2ϵ $\qquad G^{\mu\nu} + \tfrac{1}{2} R \rho^{\mu\nu}$

$$+ 2\epsilon \left[\left(-\Box + \epsilon\Box\right) R\right] f_1 \left(\tfrac{1}{2} R\right) \cdot \epsilon + 2\epsilon \left(-\nabla_\mu \nabla_\nu + \rho_{\mu\nu}\Box\right) R\, f_2 \left(\tfrac{1}{2} R\right) \rho^{\mu\nu}$$

$(-\Box + \epsilon\Box)$ $\qquad \leftarrow$

$$+ (2\epsilon)^2 \left[\left(-\Box + \epsilon\Box\right) R\right] f_1 \left(-\Box + \epsilon\Box\right) R + (2\epsilon)^2 \left[\left(-\nabla_\mu \nabla_\nu + \rho_{\mu\nu}\Box\right) R\right] f_2 \left(-\nabla^\mu \nabla^\nu + \rho^{\mu\nu}\Box\right) R$$

ϵ^2

$$= R + \epsilon R^2 + \underbrace{R f_2 R} + R_{\mu\nu} f_2 R^{\mu\nu}$$

$$\underbrace{-2 R f_1 R} \underbrace{- \frac{1}{2} R f_2 R}$$

$$\underbrace{+6\epsilon R f_1 \Box R} \underbrace{- \epsilon R f_2 \Box R} \underbrace{+ 2\epsilon R f_2 \Box R}$$

$$\underbrace{-2 R f_2 R}, \underbrace{- \frac{1}{2} R f_2 R}$$

$$\underbrace{+6 R f_2 R}, \underbrace{+ R f_2 R},$$

$$\underbrace{-12\epsilon R f_2 \Box R} \quad -3\epsilon R f_2 \Box R$$

$$\underbrace{+6\epsilon R f_2 \Box R}, \underbrace{-\epsilon R f_2 \Box R}, \underbrace{+2\epsilon R f_2 \Box R}$$

$$\underbrace{-12\epsilon R f_2 \Box R}, \underbrace{-3\epsilon R f_2 \Box R},$$

$$\underbrace{+36\epsilon^2 R f_2 \Box^2 R}, \underbrace{+4\epsilon^2 [(\nabla_\mu \nabla_\nu - \Box g_{\mu\nu}) R] f_2 (\nabla^\mu \nabla^\nu - \Box g^{\mu\nu}) R}$$

$$4\epsilon^2 (\nabla_\mu \nabla_\nu R) f_2 (\nabla^\mu \nabla^\nu R)$$

$$-4\epsilon^2 (\nabla_\mu \nabla_\nu R) f_2 \Box g^{\mu\nu} R$$

$$-4\epsilon^2 R f_2 \Box^2 R +$$

$$4\epsilon^2 R f_2 \Box^2 R \; 4$$

$$= 4\epsilon^2 [\, R f_2 \Box^2 R - R f_2 \Box^2 R - R f_2 \Box^2 R +$$

$$4 R f_2 \Box^2 R \,]$$

$$= R + \epsilon R^2 + R\Big(f_1 - 2f_1 - \tfrac{1}{2} f_2 + 6\epsilon f_1 \Box + \epsilon f_2 \Box +$$

$$-\tfrac{1}{2} f_2$$

$$-8 f_1 + 4 f_1 + f_2 - 12\epsilon f_1 \Box - 3\epsilon f_2 \Box$$

$$+ 6\epsilon f_1 \Box + \epsilon f_2 \Box$$

$$-12\epsilon f_1 \Box - 3\epsilon f_2 \Box$$

$$+ 36\epsilon^2 f_1 \Box^2 + 12\epsilon^2 f_2 \Box^2 \Big) R$$

$$+ R_{\mu\nu} f_2 R^{\mu\nu} =$$

$$\boxed{f_2 = \gamma_2}$$

- $R\Big\{ f_1 \big(1 - 2 + 6\epsilon\Box - 2 + 4 - 12\epsilon\Box + 6\epsilon\Box - 12\epsilon\Box + 36\epsilon^2\Box^2 \big)$

$$+ f_2 \big(0 + \epsilon\Box - 3\epsilon\Box + \epsilon\Box - 3\epsilon\Box + 12\epsilon^2\Box^2 \big) + \epsilon^4 \Big\} R =$$

$$= R\Big[f_1 \big(1 - 12\epsilon\Box + 36\epsilon^2\Box^2 \big) + f_2 \big(-6\epsilon\Box + 12\epsilon^2\Box^2 \big) + \epsilon^4 \Big] R$$

$$f_1 \big(36\epsilon^2\Box^2 - 12\epsilon\Box + 1 \big) + f_2 \big(12\epsilon^2\Box^2 - 6\epsilon\Box \big) + \epsilon^4 = \gamma_0$$

- $f_1 \big[12(3\epsilon^2\Box^2 - \epsilon\Box) + 1 \big] + f_2\, 6\Box(3\epsilon^2\Box - \epsilon) + \epsilon^4 = \gamma_0$

$$f_1 = \frac{\gamma_0 - \epsilon^4 - \gamma_2 6\epsilon\Box(3\epsilon\Box - 1)}{\big[12\epsilon\Box(3\epsilon\Box - 1) + 1 \big]}$$

$$\frac{12\epsilon\square(3\epsilon\square^{-1})+1}{2\chi\epsilon\square^{2}(3\epsilon\square^{-1})+2\square}$$

$$\sim \frac{-H \cancel{\chi\chi\chi} - 2\epsilon\square + H\,8\epsilon\square(3\epsilon\square^{-1})}{2\chi\epsilon\square^{2}(3\epsilon\square^{-1})+2\square}$$

$$= \frac{H\left[8\epsilon\square(3\epsilon\square^{-1}) - 4\right] - 8\epsilon\square}{8\square\left[4\cancel{2}\epsilon\square(3\epsilon\square^{-1})+2\right]}$$

$$= -\frac{1}{2}\frac{(e^{H}-1)}{\square} - \epsilon + (e^{H}-1)\,4\epsilon\,\beta\epsilon\square^{-1})$$

$$= \frac{12\epsilon\square(3\epsilon\square^{-1})+1}{}$$

$$= -e^{H} + 1 - 2\epsilon\square + (e^{H}-1)\,8\epsilon\square(3\epsilon\square^{-1})$$

$$f_2 = \frac{\gamma_0 - \epsilon\cancel{W} - 2\chi\epsilon\square(3\epsilon\square^{-1})}{}$$

$$\gamma_0 = \frac{(\beta-2)(e^{H_0}-1)+6(e^{H_2}-1)}{\epsilon(\beta-1)\square} = \frac{2(e^{H_0}-1)+4(e^{H_2}-1)}{6\square}$$

$$\beta=4 \qquad \frac{5.7\square}{4.3\square}$$

$$H_2=H_0\equiv H$$

$$= -\frac{1}{2}\frac{e^{H}-1}{\square}$$

$$\gamma_2 = \frac{e^{H_2}-1}{\square}$$

$H_0 \neq H_2$

$$- \frac{(e^{H_0}-1)+2(e^{H_2}-1)}{6\pi} - \epsilon - (e^{H_2}-1)\, 4\epsilon(3\epsilon\pi-1)$$

$f_1 =$

$$12\epsilon\pi(3\epsilon\pi-1)+1$$

$$= - \frac{(e^{H_0}-1)+2(e^{H_2}-1) - 6\pi\epsilon - (e^{H_2}-1)\, 24\epsilon\pi(3\epsilon\pi-1)}{6\pi[12\epsilon\pi(3\epsilon\pi-1)+1]}$$

$e^{H_0} \equiv e^{\tilde{H}_0(1+2\epsilon\pi)}$

$$= - \frac{(e^{H_0}-1) - 6\pi\epsilon - 2(e^{H_2}-1)[12\epsilon\pi(3\epsilon\pi-1)+1]}{6\pi[12\epsilon\pi(3\epsilon\pi-1)+1]}$$

$$= - \frac{e^{\tilde{H}_0}(1+2\epsilon\pi)+1-6\pi\epsilon}{6\pi[12\epsilon\pi(3\epsilon\pi-1)+1]} - \frac{2(e^{H_2}-1)}{6\pi}$$

$\epsilon = \frac{1}{6\pi^2}$

$$- \frac{(1+\tilde{H}_0+\cdots)(1+2\epsilon\pi)+1-6\pi\epsilon}{6\pi[12\epsilon\pi(3\epsilon\pi-1)+1]} - \frac{e^{H_2}-1}{3\pi}$$

$$= - \frac{1-2\epsilon\pi-\tilde{H}_0-\tilde{H}_0^2+\cdots+1-6\pi\epsilon}{6\pi[12\epsilon\pi(3\epsilon\pi-1)+1]} - \frac{e^{H_2}-1}{3\pi}$$

$\alpha = -6 \Rightarrow 1-6\epsilon\nu = 1+6\epsilon k^2$

$k^2 = -\frac{1}{6\epsilon}$

$-k_0^2+\tilde{k}^2 = -\frac{1}{6\epsilon}$ $\quad \epsilon > 0$

$$= - \frac{(e^{\tilde{H}_0}-1)}{6(1-\frac{\pi}{M^2})\pi} - \frac{2(e^{H_2}-1)}{\frac{4}{3}\pi}$$

$$= - \frac{(e^{\tilde{H}_0}-1)}{6\pi(1-\frac{\pi}{M^2})} - \frac{e^{H_2}-1}{3\pi}$$

$P_0 = D^\delta(1-\frac{\pi}{M^2})$, $\quad P_2 = \pi^{\delta+1}$.

$$\Rightarrow - \frac{(1+\pi^\delta(1-\frac{\pi}{n^2})+\cdots-1)}{6\pi(1-\frac{\pi}{n^2})}$$

$$f_1 = -\frac{\left(e^{\tilde{H}_0} - 1\right)}{6\Box\left(1-\frac{\Box}{\Lambda^2}\right)} - \frac{e^{H_1} - 1}{3\Box}$$

$$e^{\tilde{H}_0}, \quad \hat{H}_0 = \frac{1}{2}\left[\Gamma\left(0, P_\gamma^{(0)}(\Box)^2\right) + \gamma_E + \ln\left(P_\gamma^{(0)}(\Box)\right)^2\right]$$

$$\rightarrow \ln\left(P_\gamma^{(0)}(\Box)\right)$$

$$e^{\tilde{H}_0} \rightarrow P_\gamma^{(0)}(\Box) \qquad P_\gamma^{(2)} = \Box^{\alpha+1}$$

$$f_1 \rightarrow -\frac{P_\gamma^{(0)}(\Box)}{6\Box\left(1-\frac{\Box}{\Lambda^2}\right)} - \frac{P_\gamma^{(2)}(\Box)}{3\Box} \qquad f_2 \rightarrow \Box^{\alpha}, \quad \alpha = 2\gamma$$

$$P_\gamma^{(0)}(\Box) = \left[\frac{\Lambda}{M}\left(\frac{\Box}{\Lambda^2}\right)^{\gamma-x}\frac{1}{\Lambda^8}\left(\Box^2 + M^2\Box + M^4\right)^2(\Box - M^2)^3\frac{1}{\Lambda^6}\right]^2 \left(\Box^2 + M^2\Box + M^4\right)\frac{1}{\Lambda^4} P_\gamma^{(2)}(\Box)$$

$$f_1 \rightarrow -\left[\frac{\Lambda}{M}\left(\frac{\Box}{\Lambda^2}\right)^{\gamma-x}\frac{1}{\Lambda^{14}}\left(\Box^2 + M^2\Box + M^4\right)^2(\Box - M^2)^3\right]^2 \left(\Box^2+M^2\Box+M^4\right)\frac{1}{\Lambda^4}\frac{1}{6\Box}\frac{M^2}{(M^2-\Box)}$$

$$= -\Lambda^2\left(\frac{\Box}{\Lambda^2}\right)^{2\gamma-2x}\frac{1}{\Lambda^{28}}\left(\Box^2 + M^2\Box + M^4\right)^5\frac{1}{\Lambda^4}(\Box - M^2)^6\frac{1}{6\Box}\frac{1}{M^2-M}$$

$$= \frac{1}{6}\frac{1}{\Lambda^{26}}\frac{\Box^{2\gamma-2x-1}}{\Lambda^{4\gamma-4x}}\left(\Box^2 + M^2\Box + M^4\right)^5(\Box - M^2)^5\frac{1}{\Lambda^4} =$$

$$= \frac{1}{6\Lambda^{30}}\frac{\Box^{2\gamma-2x-1}}{\Lambda^{4\gamma-4x}}\left(\Box^3 - M^6\right)^5 = \frac{1}{6\Lambda^{30}}\frac{\Box^{2\gamma-15}}{\Lambda^{4\gamma+36}}\left(\Box^3 - M^6\right)^5$$

$$\begin{pmatrix} 2\gamma - 2x - 1 > 15 \\ 2\gamma + 18 - 1 > 15 \\ 2\gamma + 17 \end{pmatrix}$$

$x = 7 \in \mathbb{N}$ ok

$$-2x - 1 = -15$$
$$2x + 1 = 15$$
$$2x = 14$$
$$x = 7$$

↓ UV

$\Box^{2\gamma}$

$$\frac{3\dot{\beta}}{\beta} = \frac{3}{\beta}\frac{d\beta}{dr}\cdot\dot{r} \qquad \beta = s^{-\frac{1}{2}}$$

$$= \frac{3}{\beta}\frac{d\beta}{dr}\cdot\left(-\sqrt{\frac{2m}{r}}\right)s^{-1} \qquad = \frac{1}{1+\frac{L^2}{r^2}}$$

- $\frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}\,u^r\right) = \frac{1}{r^2s^2}\partial_r\left(r^2s^2\dot{r}\right)$

$$= -\frac{1}{r^2s^2}\partial_r\left(r^2s^2\sqrt{\frac{2m}{r}}\,s^{\nearrow}\right)$$

$$= -\frac{1}{r^2s^2}\partial_r\left(r^{\frac{5}{2}}s\sqrt{2m}\right)$$

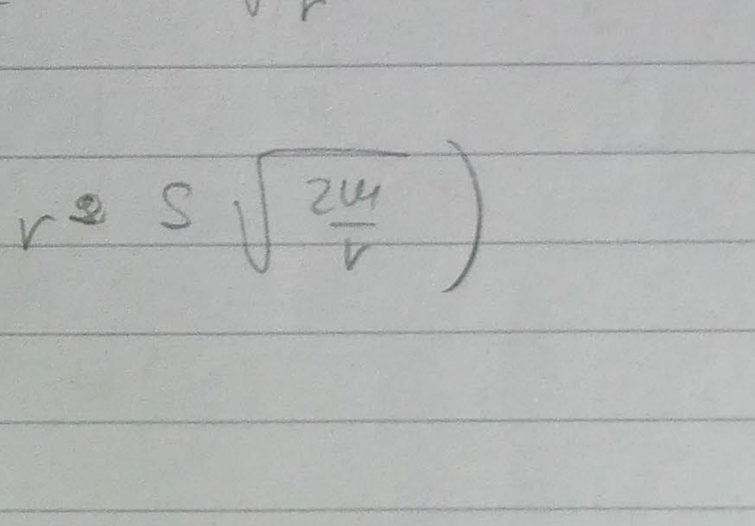
- Photons :

$$\frac{1}{r^2s^2}\partial_r\left(r^2s^2\dot{r}\right) \longrightarrow \frac{1}{T^2s^2(T)}\partial_T\left(T^2s^2(T)\,\dot{T}\right)$$

$$\left(\frac{2m}{T}-1\right)d\beta^2 - \frac{dT^2}{\left(\frac{2m}{T}-1\right)} = 0$$

$$\left(\frac{dT}{d\beta}\right)^2 = \left(\frac{2m}{T}-1\right)^2$$

- $\dot{T} = \pm\left(\frac{2m}{T}-1\right)$



Cut-off scale : $2\kappa_4^{-2}\int\sqrt{-g}\,(R-2\Lambda)\,d^4x + O(R^2)$

$= 2\int\sqrt{-g}\,(\phi^2 R + 6\partial_\mu\phi\,\partial^\mu\phi)\,d^4x + O(R^2)$

$-2\lambda\int\sqrt{g}\,\phi^4\,d^4x$

$\left[\frac{\Lambda}{\kappa_4^2}\right] = M^2, \quad [\Lambda] = M^2$

G_N, Ω_Λ, m

$I, L, \ldots$

Quantisation :

$\phi = \bar{\phi} + \psi$

$S(\psi) \Rightarrow \psi = 0$ in S'

$\kappa\frac{\partial}{\partial x} = \frac{\partial}{\partial g_{ik}}\,k$

$= \frac{\partial}{\partial g_{ik}}\left(\frac{\partial x}{\partial g_{ik}}\right)$

$S = 2\kappa_4^{-2}\int d^4x\sqrt{g}\left[(\phi^2 R + \partial_\mu\phi\,\partial^\mu\phi) - 2\Lambda\phi^4\right]$

$= 2\kappa_4^{-2}\left[\int d^4x\,\sqrt{-g}\,(R-2\Lambda)\right]_{g=\phi^2\hat{g}}$

$\int \mathcal{D}(\hat{g}(-g)^{\frac{1}{2}})\;\mathcal{D}(\phi(-g)^{\frac{1}{8}})\;e^{-S(g)}\Big|_{g=\phi^2\hat{g}}$

$\kappa_{eff}^{-2} = Z_\kappa\,\kappa_4^{-2}$

$\kappa_4^{-2} + \kappa_4^{-2}\,k^2\beta_\kappa = \kappa_4^{-2} + \kappa_0^{-2}\beta_\kappa$

$\kappa_4^{-2} = \kappa_4^{0\,-2} + \beta_\kappa(k_0^2 - k^2)$

$Z_\kappa - 1 = \frac{1}{\epsilon}\beta_\kappa = \left(\ell_P\frac{\Lambda}{\mu}\right)\beta_\kappa + k^2\beta_{2\kappa}$

$Z_\kappa = 1 + k^2\left\{\frac{\beta_\kappa}{\kappa_4^{-2}}\right.$

$\beta_\kappa < 0 \Rightarrow \kappa_4^{-2} = \kappa_{0\,4}^{-2} + (k^2 - k_0^2)(\ldots)$

$$z\,\alpha = \alpha_B$$

$$z = 1 + \frac{1}{\epsilon}\beta_\alpha \frac{1}{\alpha}$$

$$Q = \alpha_B z^{-1} = \alpha_B - \frac{1}{\epsilon}\beta_\alpha \frac{\alpha_B}{\alpha} \approx \alpha_B - \frac{1}{\epsilon}\beta_\alpha$$

$$u^\mu{}_{;\mu} = \frac{1}{\sqrt{-\hat{g}^*}}\partial_\mu\left(\sqrt{-\hat{g}^*}\,u^\mu\right)$$

$$\hat{g}^*_{\mu\nu} = S(x)\,\hat{g}_{\mu\nu}$$

$$\partial_\mu \ln(S^2) + \partial_\mu \ln\sqrt{-\hat{g}}$$

$$u^\mu \partial_\mu \ln S^2\sqrt{-\hat{g}} + \partial_\mu u^\mu$$

$$\partial_\mu(u^\mu$$

$$u^\mu{}_{;\mu} = \frac{1}{S^2(x)\sqrt{-\hat{g}}}\partial_\mu\left(S^2(x)\sqrt{-\hat{g}}\,u^\mu\right)$$

$$= \frac{1}{S^2\sqrt{-\hat{g}}}\left[\sqrt{-\hat{g}}\,u^\mu \partial_\mu S^2 + S^2(x)\,\partial_\mu\left(\sqrt{-\hat{g}}\,u^\mu\right)\right]$$

$$= u^\mu\frac{(\partial_\mu S^2)}{S^2} + \frac{1}{\sqrt{-\hat{g}}}\partial_\mu\left(\sqrt{-\hat{g}}\,u^\mu\right)$$

↳ definido por la luz

$$= + u^\mu{}_{;\mu}\Big|_{\hat{g}}$$

$$u^\mu = \frac{dx^\mu}{dz}$$

$$u^\mu \frac{1}{\sqrt{-\hat{g}}}\partial_\mu\sqrt{-\hat{g}} + \partial_\mu u^\mu$$

$$u^\mu\left[\frac{\partial_\mu S^2}{S^2} + \frac{\partial_\mu\sqrt{-\hat{g}}}{\sqrt{-\hat{g}}}\right] + \partial_\mu u^\mu = \left(\partial_\mu + \frac{(\partial_\mu S^2)}{S^2} + \frac{\partial_\mu\sqrt{-\hat{g}}}{\sqrt{-\hat{g}}}\right)u^\mu > 0$$

(*)

$$G_{1-} = (\tilde{\gamma}_2 \rho^{\mu\epsilon} \rho^{\nu\sigma} + \beta \tilde{\gamma}_0 \rho^{\mu\nu} \rho^{\epsilon\sigma}) G_{\epsilon\sigma}$$

$$(R_{\mu\nu} - \tfrac{1}{2} \rho_{\mu\nu} R)(\alpha \tilde{\gamma}_2 \rho^{\mu\epsilon} \rho^{\nu\sigma} + \beta \tilde{\gamma}_0 \rho^{\mu\nu} \rho^{\epsilon\sigma})(R_{\epsilon\sigma} - \tfrac{1}{2} \rho_{\epsilon\sigma} R)$$

$$(\alpha \tilde{\gamma}_2 R^{\epsilon\sigma} - \alpha \tilde{\gamma}_2 \tfrac{1}{2} \rho^{\epsilon\sigma} R + \beta \tilde{\gamma}_0 R \rho^{\epsilon\sigma} - \tfrac{1}{2} \tilde{\gamma}_0 R D \rho^{\epsilon\sigma})$$

$$(R_{\epsilon\sigma} - \tfrac{1}{2} \rho_{\epsilon\sigma} R) =$$

$\alpha = 1, \beta = 1$

$$= \underbrace{\tilde{\gamma}_2 R_{\mu\nu} R^{\mu\nu}} - \tfrac{1}{2} \tilde{\gamma}_2 R^2 - \tilde{\gamma}_2 \tfrac{1}{2} R^2 + \tilde{\gamma}_2 \tfrac{D}{2} \tfrac{1}{2} R^2 +$$

$$+ \tilde{\gamma}_0 R^2 - \tfrac{1}{2} D \tilde{\gamma}_0 R^2 - \tfrac{1}{2} \tilde{\gamma}_0 R D R + \tfrac{1}{4} \tilde{\gamma}_0 D \cdot D R R$$

$$= R_{\mu\nu} \tilde{\gamma}_2 R^{\mu\nu} + R \left(-\tfrac{1}{2} \tilde{\gamma}_2 - \tfrac{1}{2} \tilde{\gamma}_2 + \tfrac{D}{4} \tilde{\gamma}_2 + \tilde{\gamma}_0 - \tfrac{1}{2} D \tilde{\gamma}_0 \right.$$

$$\left. - \tfrac{1}{2} D \tilde{\gamma}_0 + \tfrac{D^2}{4} \tilde{\gamma}_0 \right) R$$

$$= R_{\mu\nu} \tilde{\gamma}_2 R^{\mu\nu} + R \left(-\tilde{\gamma}_2 + \tfrac{D}{4} \tilde{\gamma}_2 + \tilde{\gamma}_0 \underbrace{\left(1 - D + \tfrac{D^2}{4}\right)}_{1 - 4 + \frac{16}{4}} \right) R$$

$D = 4 \Rightarrow$ $\quad D$ im $D=4$ $\quad$ 1 im $D=4$

$$R_{\mu\nu} \tilde{\gamma}_2 R^{\mu\nu} + R \cdot \tilde{\gamma}_0 R$$

$$\tilde{\gamma}_0 = -\tfrac{1}{2} \tilde{\gamma}_2$$

MSS section III.

$$1 - 6\left(\Pi + \frac{\bar{R}}{3}\right) \gamma_3(\Pi) - \frac{1}{2}\gamma_{S2}\left(\Pi + \frac{2}{3}\bar{R}\right) =$$

$$= 1 - 6\left(\Pi + \frac{\bar{R}}{3}\right)\left[-\frac{1}{6}\frac{1}{\Pi + \frac{\bar{R}}{3}}\left(e^{H_0\left(\Pi+\frac{\bar{R}}{3}\right)} - 1\right) - \Pi\frac{1}{12\left(\Pi+\frac{\bar{R}}{2}\right)}\left(e^{H_{S2}\left(\Pi+\frac{\bar{R}}{2}\right)\left(\Pi+\frac{\bar{R}}{3}\right)} - 1\right)\frac{1}{\Pi+\frac{\bar{R}}{3}}\right]$$

$$-\frac{1}{2}\Pi\left\{\frac{1}{\left(\Pi - \frac{\bar{R}}{6}\right)}\left(e^{H_{S2}\left(\left(\Pi-\frac{\bar{R}}{6}\right)\left(\Pi-\frac{\bar{R}}{3}\right)\right)} - 1\right)\right\}_{\Pi \to \Pi + \frac{2}{3}\bar{R}}$$

$$= \cancel{1} + e^{H_0\left(\Pi+\frac{\bar{R}}{3}\right)} \cancel{-1} + \frac{\Pi}{\cancel{2}\left(\Pi+\frac{\bar{R}}{2}\right)}\left(e^{H_{S2}\left(\left(\Pi+\frac{\bar{R}}{2}\right)\left(\Pi+\frac{\bar{R}}{3}\right)\right)} - 1\right)$$

$$-\frac{1}{2}\Pi\frac{1}{\left(\Pi - \frac{\bar{R}}{6} + \frac{2}{3}\bar{R}\right)}\left[e^{H_{S2}\left(\Pi+\frac{\bar{R}}{2}\right)\left(\Pi+\frac{\bar{R}}{3}\right)} - 1\right]$$

$$-\frac{1}{6} + \frac{4}{6} = \frac{1}{2}$$

$$= + e^{H_0\left(\Pi+\frac{\bar{R}}{3}\right)}$$

$G(p) \sim p^{2\gamma+4}$

1) Vertex $\sim \frac{c_4 p^{2\gamma-4} R^2 \cdot R^2}{p^2}$ (p^4)

$$= \bigcirc = R\int d^4p \frac{p^{2\gamma-2}}{p^{2\gamma+4}} < \infty$$

2) Vertex $\sim \frac{c_4' p^{2\gamma-4} R R^2}{p^2}$

$$= \bigcirc = R''\int d^4p \frac{1}{p^{2\gamma+4}} p^{2\gamma-6} p^4 < \infty$$

3) $\frac{p^{2\gamma-6} R^2 R^2}{p^2}$

$$= \bigcirc = \bar{R}^2\int d^4p \frac{1}{p^{2\gamma+4}} p^{2\gamma-8} p^4 < \infty$$

$$\textcircled{O} \quad \overline{\overline{I}}\bigcirc = \sqrt{g}\,\frac{\omega_{0,\gamma-2}}{\omega_{2,\gamma}} \int d^6p\, \frac{P^{2\gamma}}{P^{2\gamma+4}} = \sqrt{g}\,\frac{\omega_{0,\gamma-1}}{\omega_{2,\gamma}} \underbrace{\int_\epsilon^{R\Omega^{\gamma-2}R} \frac{dl}{l}}_{\ln \frac{1}{\epsilon}}$$

$$\textcircled{D} \quad R\Box^{\gamma}R + R\Box^{\gamma-1}R$$

$$\overline{\overline{R}}\bigcirc = R\,\frac{\omega_{0,\gamma-1}}{\omega_{2,\gamma}} \int d^6p\, \frac{1}{p^{2\gamma+4}}\, p^{2\gamma+2+2} = R\,\frac{\omega_{0,\gamma-1}}{\omega_{2,\gamma}} \int_\epsilon^{\Lambda^2} \frac{dk}{k}$$

$$\beta_R^{(2)} = \frac{\omega_{0,\gamma-1}}{\omega_{2,\gamma}}$$

$$\xi_3 R^2 \Box^{\gamma-1} R$$

$$\overline{\overline{R\,\xi_3}}\bigcirc = R\xi_3 \int d^6p\, \frac{1}{p^{2\gamma+4}\omega_\gamma}\, p^{2\gamma-2+4} = \frac{\xi_3 R^2}{\omega_\gamma} \ln \frac{\Lambda^2}{\epsilon}$$

$$G = R - 2R = -R$$

$$G_{\mu\nu}\left(\rho^{\mu\alpha}\rho^{\nu\beta} + \rho^{\mu\nu}\rho^{\alpha\beta}\right) G_{\alpha\beta} = \left(G^{\alpha\beta} + G\rho^{\alpha\beta}\right) G_{\alpha\beta}$$

$$= G^{\mu\nu} G_{\mu\nu} + G\,G$$

$$= \left(R^{\mu\nu} - \tfrac{1}{2}\rho^{\mu\nu} R\right)\left(R_{\mu\nu} - \tfrac{1}{2}\rho_{\mu\nu} R\right) + R^2$$

$$= R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{2}R^2 - \tfrac{1}{2}R^2 + \tfrac{4}{4}R^2 + R^2$$

$$= R_{\mu\nu}R^{\mu\nu} + R^2$$

$$\phi \to R^{\frac{2-D}{2}} \phi \qquad R = (\phi K)^{\frac{2}{D-2}}$$

$$= \left[(\phi K)^{\frac{2}{D-2}}\right]^{\frac{2-D}{2}} \phi = (\phi K)^{-1} \phi = \frac{1}{K}$$

—

$$\alpha = \frac{1}{\rho^2}$$

$$\frac{d\alpha}{dt} = \frac{d\,1/\rho^2}{dt} = -\frac{2\rho}{\rho^4}\frac{d\rho}{dt} = -\frac{2}{\rho^3}\frac{d\rho}{dt}$$

$$\beta = \frac{-1}{6\pi^2}$$

$$\frac{d\alpha}{dt} = \beta \Rightarrow \frac{d\rho}{dt} = -\frac{\rho^3}{2}\beta > 0$$

—

$$r^3 + r^4 \to 0$$

$$r^3(1 + r) \to r^3$$

$$\frac{1}{r^3} + \frac{1}{r^4} = \frac{r^4 + r^3}{r^4 r^3} = \frac{r^3(1+r)}{r^3 r^4} = \frac{(1+r)}{r^4}$$

$$\simeq \frac{1}{r^4}$$

$$\frac{1}{r^4} > \frac{1}{r^3} \quad \text{for } r \to 0$$

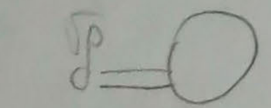
$$\mathcal{L} \sim \omega_{0,\gamma} R \Box^{\gamma} R + \omega_{2,\gamma} R_{ic} \Box^{\gamma} R_{ic}$$
$$+ \omega_{0,\gamma-1} R \Box^{\gamma-1} R + \omega_{2,\gamma-1} R_{ic} \Box^{\gamma-1} R_{ic}$$
$$+ \omega_{0,\gamma-2} R \Box^{\gamma-2} R + \omega_{2,\gamma-2} R_{ic} \Box^{\gamma-2} R_{ic}$$

$D = 4$

$$\left[\omega_{0,\gamma}\right] = M^{-2\gamma}, \quad \left[\omega_{2,\gamma}\right] = M^{-2\gamma}, \quad \left[\omega_{0,\gamma-1}\right] = M^{-2\gamma+2}, \quad \left[\omega_{2,\gamma-1}\right] = M^{-2\gamma+2}, \quad \left[\omega_{0,\gamma-2}\right] = M^{-2\gamma+4}, \quad \left[\omega_{2,\gamma-2}\right] = M^{-2\gamma+4}$$

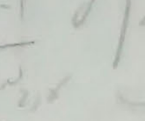
$$\left[\frac{\omega_{0,\gamma-1}}{\omega_{0,\gamma}}\right] = \frac{M^{-2\gamma+2}}{M^{-2\gamma}} = M^2, \qquad \left[\frac{\omega_{0,\gamma-2}}{\omega_{0,\gamma}}\right] \sim M^4$$

(A)



$$= \sqrt{g}\,\frac{\omega_{0,\gamma-1}}{\omega_{2,\gamma}} \int d^4p\, p^{\alpha} \frac{p^{2\gamma-2}}{p^{2\gamma+\alpha}} = K^2 \frac{\omega_{0,\gamma-1}}{\omega_{0,\gamma}}$$

(B)



$$= \frac{\omega_{0,\gamma-1}^2}{\omega_{0,\gamma}\omega_{2,\gamma}} \sqrt{g} \int d^4p \frac{p^{2\gamma+2}}{p^{2\gamma+4}} \frac{p^{2\gamma+2}}{p^{2\gamma+4}}$$

$$\sqrt{g}\,\bigcirc = \frac{\omega_{0,\gamma}}{\omega_{2,\gamma}} \int d^4p \frac{p^{2\gamma}}{p^{2\gamma+4}} = K^4 \frac{\omega_{0,\gamma}}{\omega_{2,\gamma}}$$

$$\phi^2 G_{\mu\nu} = \underbrace{2\nabla_\nu\phi\,\nabla_\mu\phi} + 2\phi\nabla_\nu\nabla_\mu\phi$$

$$-g_{\mu\nu}\left(\underbrace{2(\nabla_\alpha\phi)(\nabla^\alpha\phi)} + 2\phi\Box\phi\right)$$

$$\underbrace{-6\left(\phi_{,\mu}\phi_{,\nu}\right)} \underbrace{-\tfrac{1}{2}g_{\mu\nu}\,\partial^\alpha\phi\,\partial_\alpha\phi}$$

$$= -4\,\phi_{,\mu}\phi_{,\nu} + g_{\mu\nu}\,\phi_{,\alpha}\,\phi^{,\alpha}$$

$$-2g_{\mu\nu}\,\phi\,\Box\phi + 2\phi\,\nabla_\nu\nabla_\mu\phi$$

Teoria Misturada com Escalar.

Fonte CQG.

$$E_{\mu\nu} = (G_{\mu\nu} - T_{\mu\nu}(\phi)) \quad , \quad E = -R - \frac{D}{2}R + T = 0$$

$$\phi^2\left(\hat{R} - \frac{1}{2}D\hat{R}\right) = \Box\phi^2 - D\Box\phi^2 - \frac{4(D-1)}{D-2}\left(\partial^\alpha\phi(\partial_\alpha\phi) - \frac{D}{2}\partial^\alpha\phi(\partial_\alpha\phi)\right)$$

$$\frac{1}{2}\phi^2\hat{R}(2-D) = =$$

$$E_{\mu\nu}\,\Gamma^{(1)\,\mu\nu\rho\sigma}\,E_{\rho\sigma}$$

$$(G_{\mu\nu} - T_{\mu\nu}(\phi))\,\Gamma^{(1)\,\mu\nu\rho\sigma}\,(G_{\rho\sigma} - T_{\rho\sigma})$$

$$(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R - T_{\mu\nu})\left(2\gamma_2^{(1)}\,g^{\mu\rho}g^{\nu\sigma} + \beta\,\gamma_0^{(1)}\,g^{\mu\nu}g^{\rho\sigma}\right)(G_{\rho\sigma} - T_{\rho\sigma})$$

$R_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}R$

$\alpha\,\gamma_2^{(1)}\,R^{\mu\nu}$ ... Vod. pp. precedente ✱

Del paper Conformalcompensations.pdf

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D = 4 EOM:

$$G_{\mu\nu} \frac{1}{\kappa^2} \frac{\kappa}{3} \phi^2 + \frac{1}{6} \left(4 \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} \right)$$

$$+ \frac{1}{2} \frac{2}{3} \left(g_{\mu\nu} \phi \Box \phi - \phi \phi_{;\mu\nu} \right) = 0$$

$$\phi^2 G_{\mu\nu} = -4 \phi_{,\mu} \phi_{,\nu} + g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} +$$

$$- 2 \left(g_{\mu\nu} \phi \Box \phi - \phi \phi_{;\mu\nu} \right)$$

D = 4 Transformation of G_{ab} under $g_{ab} \to \Omega^2 g_{ab}$:

$$G_{ab} + \frac{4}{\Omega^2} \Omega_{,a} \Omega_{,b} - \frac{1}{\Omega^2} \Omega_{,c} \Omega^{,c} g_{ab} - \frac{2}{\Omega} \left(\Omega_{;ab} - g_{ab} \Box \Omega \right)$$

$$\Omega^2 G_{ab} = -4 \Omega_{,a} \Omega_{,b} + \Omega_{,c} \Omega^{,c} g_{ab} + 2 \left(\Omega \Omega_{;ab} - g_{ab} \cdot \Omega \Box \Omega \right)$$

↑ same Upadhi.

$$= \begin{cases} \nabla_\nu \nabla_\mu \left(\phi^2 \right) = \nabla_\nu \left(2 \phi \nabla_\mu \phi \right) = 2 (\nabla_\nu \phi)(\nabla_\mu \phi) + 2 \phi \nabla_\nu \nabla_\mu \phi \\ \Box \phi^2 = 2 \phi_{;\nu} \phi_{;}{}^{\nu} + 2 \phi \Box \phi \end{cases}$$

Special Relativity

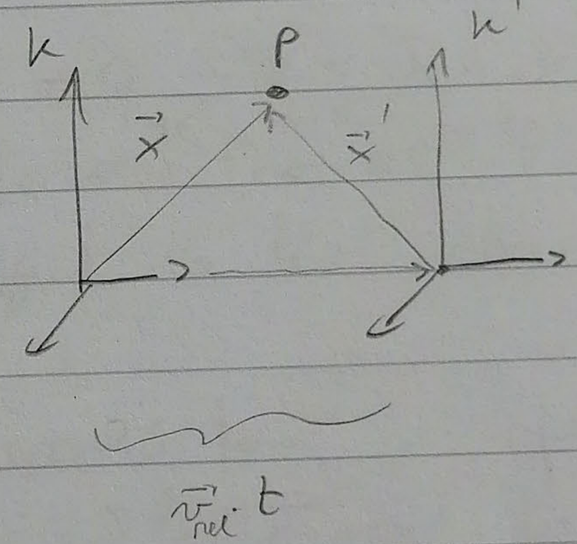
→ GALILEAN RELATIVITY: ⟶ THE LAWS OF PHYSICS MUST HAVE THE SAME FORM IN ALL INERTIAL REFERENCE SYSTEMS

$K - K'$: 2-Reference systems of coordinates $(t, \vec{x})$, $(t', \vec{x}')$

↳ They move with relative speed $\vec{v}$.

The coordinates $(t, \vec{x})$ and $(t', \vec{x}')$ are related by:

$$\text{(1)} \begin{cases} \vec{x}' = \vec{x} - \vec{v}_{rel}\, t \\ t' = t \end{cases}$$



Consider a set of particles interacting ⇒ the EOM in K' for the i-particle is:

$$\text{(2)} \quad m_i \frac{d\vec{v}_i'}{dt'} = -\vec{\nabla}_i' \sum_j V_{ij}\left(|\vec{x}_i' - \vec{x}_j'|\right)$$

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} = 1 \cdot \frac{\partial}{\partial x}$$

(1) ⇒ $\frac{d\vec{x}_i'}{dt'} = \frac{d\vec{x}}{dt'} - \vec{v}_{rel}$, $\frac{d\vec{v}_i'}{dt'} = \frac{d\vec{v}}{dt'} \overset{t'=t}{=} \frac{d\vec{v}}{dt}$, $\frac{d\vec{v}_{rel}}{dt} = 0$

$\vec{\nabla}_i' = \vec{\nabla}_i$, and $\vec{x}_i' - \vec{x}_j' = \vec{x}_i - \vec{x}_j$

⇒ $m_i \frac{d\vec{v}_i}{dt} = -\vec{\nabla}_i \sum_j V_{ij}\left(|\vec{x}_i - \vec{x}_j|\right)$ (3)

Firstly (2) and (3) have the same form in K' and K.

What about the Wave eq.? For example the Maxwell equations

Consider: $\Box' \psi = 0$

$$\left(\sum_i \frac{\partial^2}{\partial x_i'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \right) \psi = 0 \qquad (4)$$

Under (1) we find in the reference K:

$$\left[\vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial}{\partial t^2} - \frac{2}{c^2} \vec{v} \cdot \vec{\nabla} \frac{\partial}{\partial t} - \frac{1}{c^2} (\vec{v} \cdot \vec{\nabla})(\vec{v} \cdot \vec{\nabla}) \right] \psi = 0 \qquad (5)$$

SEE ~~EXOL~~ EXT. 0

The EOM for ψ is not invariant under (1).

Notice

Unlike the Schrödinger-eq:

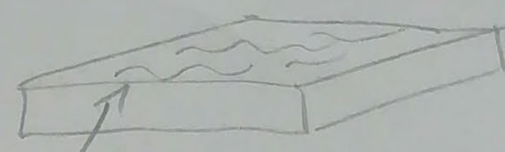
$$\frac{-\hbar^2}{2m} \vec{\nabla}'^2 \psi' + V \psi' = i\hbar \frac{\partial}{\partial t} \psi' \qquad \leftarrow \text{in } K' \qquad (6)$$

(6) has the same form in K if V is invariant under (1)

and $\psi = \psi' \underbrace{e^{\frac{im}{\hbar} \vec{v} \cdot \vec{x} - i \frac{m v^2}{2\hbar} \cdot t}}_{\text{Phase}}$. Therefore, (6) is

invariant under Galileo transformations.

sound:

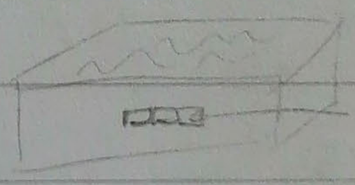


The non invariance under ① of the EOM for the sound waves is not a problem and Eq. ② is in the reference system where the waves propagates. Indeed, we know that the sound waves propagate in a medium.

However, for the light the medium is a kind of esoteric "AETHER". (Luminiferous) It does not affect matter, but only light. The Maxwell's eq. only hold in a limited class of Galileo reference frames, that is, in the coordinate frames at rest with respect to the aether.

Possible solutions of the issue:

1) Maxwell eq. are wrong. The correct theory must be GALILEO INVARIANT



2) GALILEO RELATIVITY applies to Classical mechanics, but not to the electro-magnetic phenomena and we need the aether.

3) there is a new relativity Principle valid for both classical mechanics and electromagnetism.

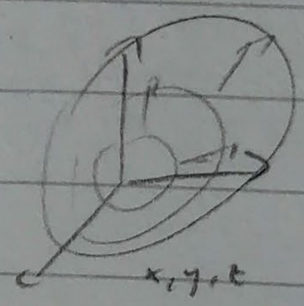
Experiments showed MAXwell eq. to be correct, but no evidence For the "aether".

Therefore, Einstein chose the third option: Relativity principle valid for the Clasical mechanics and electromagnetism.

Postulates

1) The laws of nature must be the same in all reference Frames that move of uniform motion to each other.

2) the speed of the light is the same in all reference Frames.



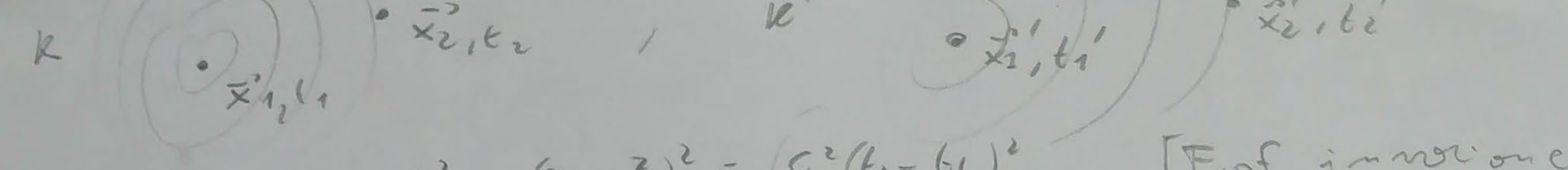
From "c" invariance it follows:

$$(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 = c^2(t_2-t_1)$$

Def: $s_{21}^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - c^2(t_2-t_1)^2 \Rightarrow$ Infinitesimal $ds^2 = c^2dt^2 - d\vec{x}^2$

If $s_{12}^2 = 0$ in K $\Rightarrow$ it is "0" $\forall$ K.
(in any reference system.

SEE EXT. 1



$$(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 = c^2(t_2-t_1)^2$$
$$\text{or } (x'_2-x'_1)^2 + (y'_2-y'_1)^2 + (z'_2-z'_1)^2 = c^2(t'_2-t'_1)^2 \Rightarrow$$

$\Rightarrow$ [Proof invariance of "c"
$s_{12}^2 = 0$ in K $\Rightarrow$
$s'^2_{12} = 0$ in K'.

Lorentz - Transformations LT

$$s_{12}^2 = \Delta x^\mu \Delta x^\nu \eta_{\mu\nu} \quad , \quad \Delta x^\mu = x_1^\mu - x_2^\mu$$

$$x'^\alpha = \Lambda^\alpha_{\ \beta} x^\beta + a^\alpha \qquad (2.1)$$

such that $\Lambda^\alpha_{\ \gamma} \Lambda^\beta_{\ \delta} \eta_{\alpha\beta} = \eta_{\gamma\delta}$ where

$$\eta_{\alpha\beta} = \begin{pmatrix} +1 & -1 & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$$

↳ MATRIX FORM:
$(\Lambda^T)_\gamma^{\ \alpha} \eta_{\alpha\beta} \Lambda^\beta_{\ \delta} = \eta_{\gamma\delta}$
or $\Lambda^T \eta \Lambda = \eta$.

Lorentz transf. leave invariant the "proper time" $d\tau$, $d\tau = \frac{ds}{c}$

defined by:

$$d\tau^2 = dt^2 - d\vec{x}^2 = +\eta_{\alpha\beta} dx^\alpha dx^\beta$$

$$d\tau'^2 = +\eta_{\alpha\beta} dx'^\alpha dx'^\beta = +\eta_{\alpha\beta} \Lambda^\alpha_{\ \gamma} \Lambda^\beta_{\ \delta} dx^\gamma dx^\delta$$

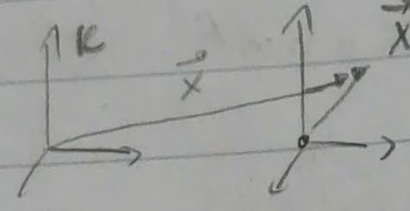
$$= +\eta_{\gamma\delta} dx^\gamma dx^\delta = +d\tau^2$$

$\Rightarrow d\tau'^2 = d\tau^2$

(2.1) are the only non-singular transformations $x \to x'$ that leave $d\tau^2$ inv. (namely exist the inverse).

$$A_\alpha B^\alpha = A'_\alpha B'^\alpha$$

Time dilatation



$\vec{x}' = 0$

K' is moving.

Particle at rest in K'.

$$ds^2 = ds'^2 \Rightarrow c^2dt^2 - d\vec{x}^2 = c^2dt'^2 - d\vec{x}'^2$$

$d\vec{x}' = 0$

velocity of K' respect to K.

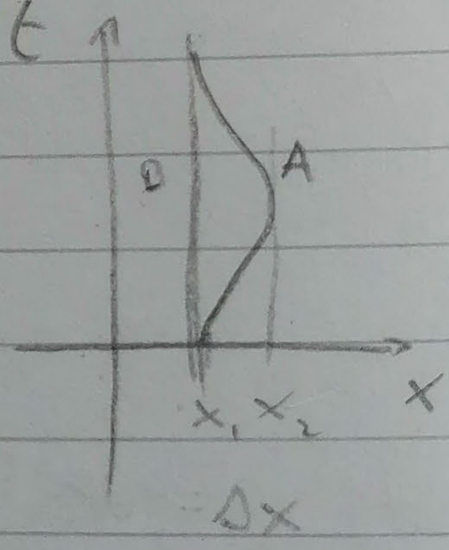
$$dt^2\left(1 - \frac{d\vec{x}^2}{c^2dt^2}\right) = dt'^2\left(1 - \frac{d\vec{x}'^2}{c^2dt'^2}\right)$$

$$\Delta t = \Delta t' \left(\frac{1}{1 - \frac{v^2}{c^2}}\right)^{1/2}$$

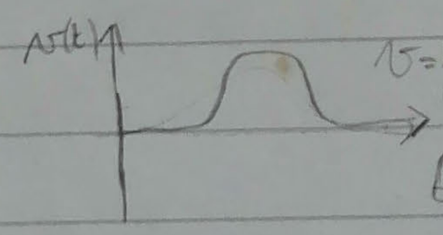
$$\Delta t' = \Delta t \sqrt{1 - \frac{v^2}{c^2}} < \Delta t$$

Close trajectory

If $v = v(t) \Rightarrow \Delta t' = \int_{t_1}^{t_2} dt \sqrt{1 - \frac{v(t)^2}{c^2}}$



EXERCISE:



$v = \frac{1}{2}[E_r f(t-10) + E_r f(-t+30)]$

$\int_0^{300} \sqrt{1 - v^2/c^2}\, dt = 291 < 300.$

Again an observer (K') looking at a clock at rest

. will see ticks separated by a space-time interval $d\vec{x} = 0$ and $dt = \Delta t$, therefore:

$$d\tau = \sqrt{dt^2 - d\vec{x}^2} = \Delta t.$$

A second observer (K), moving with velocity $\vec{v}$, will

observe that the two ticks are separated by a time interval dt' and a space interval $\vec{v}\,dt' = d\vec{x}'$ ⇒

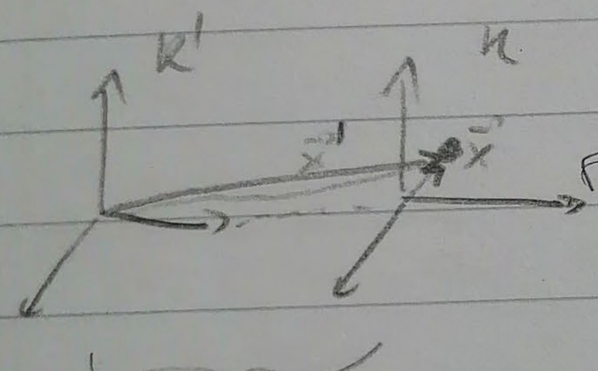
$$ds'^2 = \sqrt{c^2dt'^2 - d\vec{x}'^2} = c^2dt'\sqrt{1-\frac{v^2}{c^2}}.$$

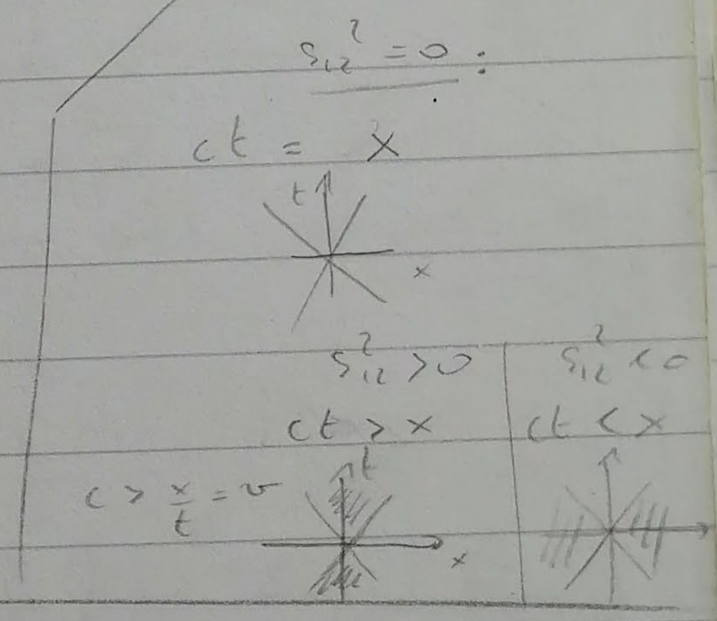
Therefore, $d\tau = d\tau' \Rightarrow dt = dt'\sqrt{1-\frac{v^2}{c^2}}$

$$dt' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,dt$$

$\frac{v^2}{c^2}$ → 1 ; dt → proper-time

$dt' > dt$





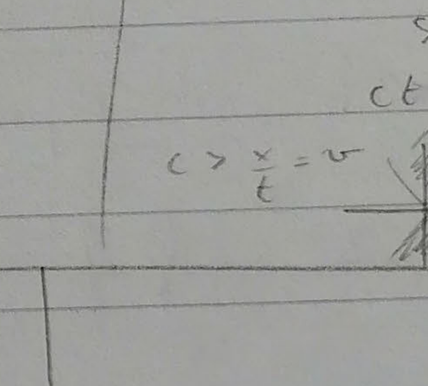
Time-like and Space-like intervals

If $\vec{x}_1, t_1$ and $\vec{x}_2, t_2$ are the coordinates of two events in K, does exist a coordinate system K' in which the two events occur at the same point in space?

$s_{12}^2 = c^2t_{12}^2 - l_{12}^2$ in K, and

$s_{12}'^2 = c^2t_{12}'^2 - l_{12}'^2$ in K'

$s_{12}^2 = s_{12}'^2$ and $l_{12}'^2 = 0$ ⇒ $s_{12}^2 = c^2t_{12}^2 - l_{12}^2 = c^2t_{12}'^2 > 0$



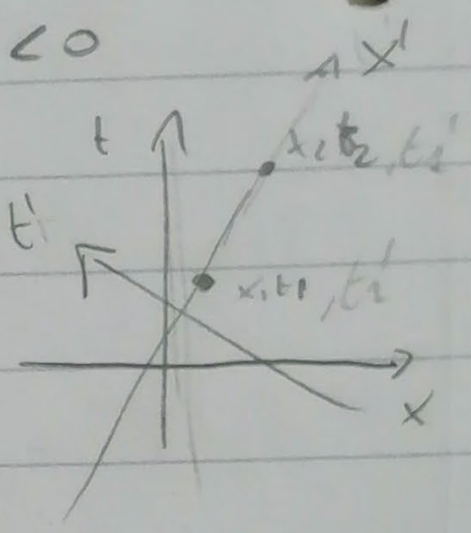
the answer is yes and the interval is timelike.

The time interval in K' is: $t'_{12} = \frac{1}{c}\sqrt{c^2t_{12}^2 - l_{12}^2}$.

Can we find a reference frame $\mathcal{R}'$ in which two events happens at the same time? $s_{12}^2 = s_{12}'^2 \Rightarrow c^2t_{12}^2 - l_{12}^2 = c^2t_{12}'^2 - l_{12}'^2$

$\Rightarrow s_{12}^2 = -l_{12}'^2 < 0$ (since $t_{12}' = 0$)

Answer: NO



Lorentz transformation (derivation)

The transformations: $x = x'\cosh\eta + ct'\sinh\eta$

$ct = x'\sinh\eta + ct'\cosh\eta$

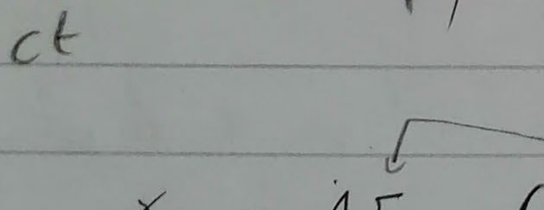
leave invariant: $\cosh^2 x - \sinh^2 x = 1$

Proof: $c^2t^2 - \vec{x}^2 = x'^2\sinh^2\eta + c^2t'^2\cosh^2\eta + 2x't'c\sinh\eta\cosh\eta$

$s^2 = c^2t^2 - \vec{x}^2$ $\quad -[x'^2\cosh^2\eta + c^2t'^2\sinh^2\eta + 2x't'c\sinh\eta\cosh\eta]$

$= -x'^2(-\sinh^2\eta + \cosh^2\eta) + c^2t'^2(\cosh^2\eta - \sinh^2\eta)$

Let us consider K' that moves respect to K: $\cosh^2\eta - \sinh^2\eta = 1$



We consider the origin of K', then $x' = 0 \Rightarrow$

$$\begin{cases} x = ct'\sinh\eta, \\ ct = ct'\cosh\eta, \end{cases}$$

or $\dfrac{x}{ct} = \tanh\eta$,

but $\dfrac{x}{ct} = \dfrac{v}{c}$ (v_{rel} = velocity of K' respect to K).

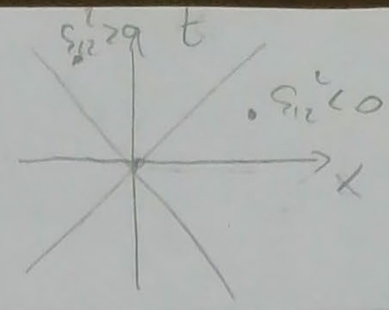
Therefore, $\tanh\eta = \dfrac{v}{c} \Rightarrow \sinh\eta = \dfrac{v/c}{\sqrt{1 - \frac{v^2}{c^2}}}$,

$$\cosh\eta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

$x'^\alpha = \Lambda^\alpha_\beta x^\beta$

$$\Lambda^\alpha_\beta = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ ← Boost →

Notice: $\frac{dx^2}{dt^2} = c^2$



$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2$

$s_{12}^2 < 0 \Rightarrow c^2 t_{12}^2 - l_{12}^2 < 0$

$\frac{l_{12}^2}{t_{12}^2} > c^2$, $\frac{t_{12}^2}{l_{12}^2} < c^2$

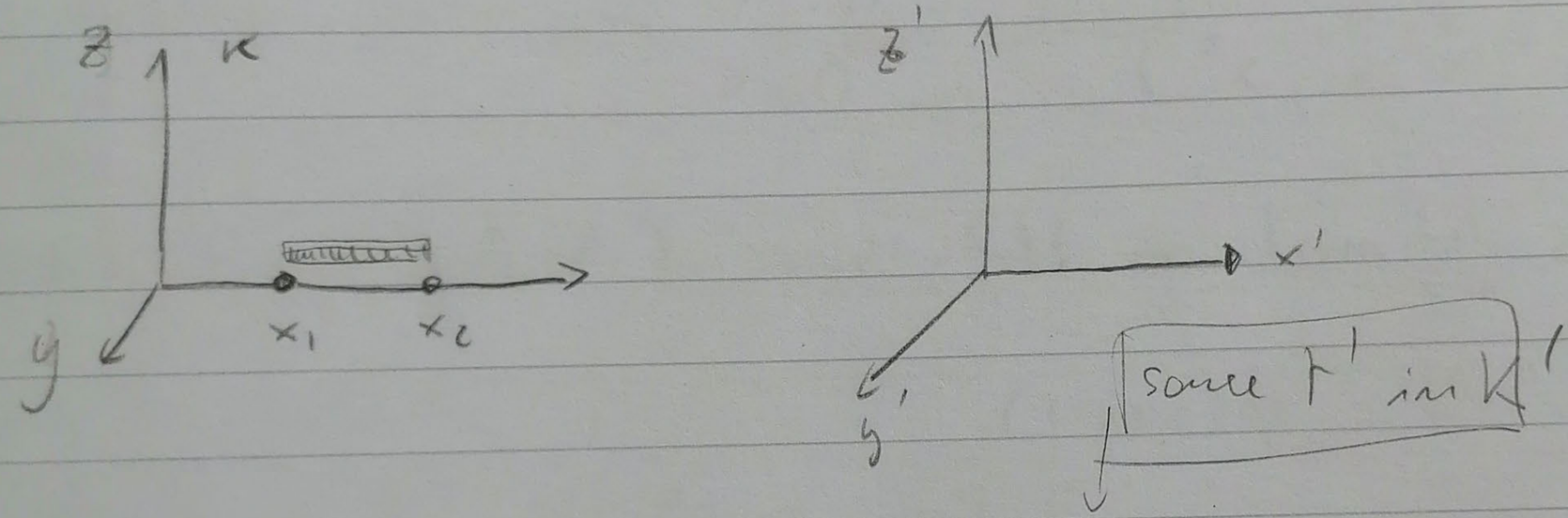
Dual $x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$, $t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}$.

large "c" $\Rightarrow$ GALILEO TRANSF ($v \ll c$)

Lorentz contraction

Rod at rest in $\bar{k}$ and parallel to the x-axis.

The length is $x_2 - x_1 = \Delta x$ in K



$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$, $x_2 = \frac{x_2' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$, $\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$

The Rod moves respect to K' (it is at rest in K)

$x_2 - x_1 = (x_2' - x_1')\gamma$ $\Rightarrow$ $\Delta x = \frac{\Delta x'}{\sqrt{1 - \frac{v^2}{c^2}}} > \Delta x'$

($x_2 - x_1 = l_0$, $x_2' - x_1' = l$, $\Delta x = l_0$, $\Delta x' = l$)

$\Rightarrow l = \sqrt{1 - v^2/c^2}\, l_0 < l_0$.

The length is max in the rest reference frame.

Rotations $\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathcal{O}_{3\times3} & \\ 0 & & & \end{pmatrix}$

MATHEMATICS OF SPECIAL RELATIVITY SPACETIME

Invariant:

$S^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$ in $D=4$.

It is invariant under $x^\alpha \to x'^\alpha = \underset{\uparrow}{\Lambda^\alpha}_\beta x^\beta$.

LT.

Let us consider a general coordinate transformation:

$x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3)$ in $D=4$.

Tensors are defined on the base of their transformations

SCALAR: $\phi'(x') = \phi(x(x'))$

Vectors: $A'^\alpha = \dfrac{\partial x'^\alpha}{\partial x^\beta} A^\beta$ CONTRAVARIANT

Ex. $\Lambda^\alpha{}_\beta$

$B'_\alpha = \dfrac{\partial x^\beta}{\partial x'^\alpha} B_\beta$ COVARIANT

$\Lambda_\alpha{}^\beta = (\Lambda^{-1})^\beta{}_\alpha$

TENSORS OF RANK 2:

$F'^{\alpha\beta} = \dfrac{\partial x'^\alpha}{\partial x^\gamma} \dfrac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$ TENSOR CONTRAVARIANT

$G'_{\alpha\beta} = \dfrac{\partial x^\gamma}{\partial x'^\alpha} \dfrac{\partial x^\delta}{\partial x^\beta} G_{\gamma\delta}$ covariant

$x'^0 = x^0 + 2x^1 \quad x'^1 = x^0 + x^1$

$\frac{\partial x'^1}{\partial x^1} = 1 \qquad \left(\frac{\partial x^0}{\partial x'^1}\right)\left(\frac{\partial x'^1}{\partial x^1}\right)$

Scalar product: $B \cdot A = B_\alpha A^\alpha$

↳ It is INVARIANT :

$\Lambda_\alpha{}^\gamma \qquad \Lambda^\beta{}_\delta$

$B' \cdot A' = B'_\alpha A'^\alpha = \frac{\partial x^\gamma}{\partial x'^\alpha} B_\gamma \frac{\partial x'^\alpha}{\partial x^\delta} A^\delta = B_\alpha A^\alpha$

$= B \cdot A$

$= B'_\alpha A'_\beta \eta^{\alpha\beta}$

$= \Lambda_\alpha{}^\gamma B_\gamma \Lambda_\beta{}^\delta A_\delta \eta^{\alpha\beta} = \eta^{\gamma\delta} B_\gamma A_\delta$

$= B_\gamma A^\gamma$

$\frac{\partial x^\gamma}{\partial x'^\alpha}$ $\frac{\partial x'^\alpha}{\partial x^\delta}$ $\Rightarrow \delta^\gamma_\delta$

$\Lambda_\alpha{}^\gamma \cdot \Lambda^\alpha{}_\delta = \delta^\gamma{}_\delta$

$\Rightarrow (\Lambda^{-1})^\gamma{}_\alpha \Lambda^\alpha{}_\delta = \delta^\gamma{}_\delta$

SPECIAL RELATIVITY : line element : $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$

$\Lambda_\alpha{}^\delta \equiv (\Lambda^{-1})^\delta{}_\alpha$

$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = dx_\alpha dx^\beta \Rightarrow A_\alpha = \eta_{\alpha\beta} A^\beta$

In general $F^{\cdots\alpha\cdots} = \eta^{\alpha\beta} F^{\cdots}{}_{\beta\cdots}$

Explicitly. If $A^\alpha = (A^0, \vec{A}) \Rightarrow A_\alpha = (A^0, -\vec{A})$

$\eta_{ii} = -1$

$B \cdot A = B_\alpha A^\alpha = B^\alpha A^\beta \eta_{\alpha\beta} = B^0 A^0 - \vec{B} \cdot \vec{A}$

Notice: $\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}$

like covariant vector $\Rightarrow D$ $\frac{\partial}{\partial x^\alpha} = \partial_\alpha \equiv (\partial_0, \partial_i)$

$\partial^\alpha = (\partial_0, -\partial_i)$

η_{ii}

Divergence:

$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \partial_0 A^0 + \partial_i A^i$

$= \partial_t A^0 + \vec{\nabla} \cdot \vec{A}$

D'ALEMBERTIAN : $\Box = \partial_\alpha \partial^\alpha = \partial_t^2 - \vec{\nabla}^2$

$\partial_i \partial_i$

Notice : $A_\alpha A^\alpha = A_0^2 - A_1^2 - A_2^2 - A_3^2$

$= A_0 A^0 + A_1 A^1 + A_2 A^2 + A_3 A^3$

$\Rightarrow = A^\alpha A^\beta \eta_{\alpha\beta}$; Just because $\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.

$\eta^{\alpha\beta} \eta_{\beta\delta} = \delta^\alpha_\delta \Rightarrow \eta^{\alpha\beta} =$ inv of $\eta_{\alpha\beta}$.

$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Scalar product again:

$A_\alpha A^\alpha = A^\beta A_\beta = A^\alpha A^\beta \eta_{\alpha\beta} = A_\alpha A_\beta \eta^{\alpha\beta}$.

If B^α transform with $\Lambda^\alpha_\beta \Rightarrow B_\alpha$ transform with $(\Lambda^{-1})^\alpha_\beta$:

Proof. $\eta_{\alpha\beta} = \eta_{\gamma\delta} \Lambda^\gamma_\alpha \Lambda^\delta_\beta$ $\quad \eta_{\alpha\beta} = (\Lambda^T)_\alpha^\gamma \eta_{\gamma\delta} \Lambda^\delta_\beta$

$B'_\alpha = \eta_{\alpha\beta} B'^\beta = \eta_{\alpha\beta} \Lambda^\beta_\gamma B^\gamma$ $\quad \eta = \Lambda^T \eta \Lambda$

$= (\eta \Lambda)_{\alpha\delta} B^\gamma$ $\quad (\Lambda^T)^{-1} \eta = \eta \Lambda$

$= [(\Lambda^T)^{-1} \eta]_{\alpha\delta} B^\delta$ $\quad [(\Lambda^T)^{-1}]^\gamma_\alpha \eta_{\gamma\delta}$

$= [(\Lambda^T)^{-1} B]_\alpha$

$= [(\Lambda^T)^{-1}]_\alpha^\gamma \underbrace{\eta_{\gamma\delta} B^\delta}_{B_\gamma}$

$= (\Lambda^{-1})^\gamma_\alpha B_\gamma$

$\left(\frac{\partial x}{\partial x'}\right)^\gamma_\alpha \equiv \frac{\partial x^\gamma}{\partial x'^\alpha}$

Four velocity:

, vector $\qquad ds^2 = c^2 dt^2 - d\vec{x}^2$

Def $\quad u^\alpha \equiv \frac{dx^\alpha}{ds}$, but $\quad ds = c\,dt\sqrt{1-\frac{v^2}{c^2}}$

scalar $\qquad x^0 = ct$

$$\Rightarrow u^\alpha = \left(\frac{c\,dt}{c\,dt\sqrt{1-v^2/c^2}}, \frac{dx^i}{c\,dt\sqrt{1-v^2/c^2}}\right)$$

$$= \left(\frac{1}{\sqrt{1-v^2/c^2}}, \frac{\vec{v}}{c\sqrt{1-v^2/c^2}}\right)$$

Let us evaluate: $\quad u_\alpha u^\alpha = \frac{dx^\alpha}{ds}\frac{dx_\alpha}{ds} = \frac{ds^2}{ds^2} = 1$

$$\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\eta_{\alpha\beta}$$

Four-acceleration

$$w^\alpha = \frac{d^2x^\alpha}{ds^2}$$

$u^\alpha u_\alpha = 1 \Rightarrow \frac{d}{ds}(u^\alpha u_\alpha) = 0 \Rightarrow 2u^\alpha w_\alpha = 0$

$\Rightarrow u^\alpha w_\alpha = 0.$

$\Lambda^{-1}\Lambda = \mathbb{1}$

$(\Lambda^{-1}\Lambda)^T = \mathbb{1}$ $\Rightarrow$ $\Lambda^{-1}\Lambda = (\Lambda^{-1}\Lambda)^T$

$\Lambda^{-1}\Lambda = \Lambda^T(\Lambda^{-1})^T$

$(\Lambda^T)^{-1}$

$(\Lambda^T)^{-1} = (\Lambda^{-1})^T$

$(AB)^T = B^T A^T$

$[(AB)^T]_{ij} = (AB)_{ji}$

$= A_{jk} B_{ki}$

$= B_{ki} A_{jk}$

$= (B^T)_{ik} (A^T)_{kj}$

Relativistic Mechanics

Point particle action: $S = -mc\int ds$

Free particle $= -mc\int c\,dt\sqrt{\frac{ds^2}{c^2 dt^2}}$

$= -mc\int c\,dt\sqrt{\frac{\eta_{\alpha\beta}}{c^2}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}$

$= -mc^2\int dt\sqrt{1-\left(\frac{d\vec{x}}{dt}\right)^2\frac{1}{c^2}}$

$= -mc^2\int dt\sqrt{1-\frac{v^2}{c^2}}$. $\quad v_i = \dot{x}_i$

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}}}$$

EOM: $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} - \frac{\partial\mathcal{L}}{\partial x_i} = 0$

Energy and momentum:

$$\vec{p} = \frac{\partial L}{\partial\vec{v}} = -mc^2\frac{1}{k\sqrt{1-\frac{v^2}{c^2}}}\cdot\left(-k\frac{\vec{v}}{c^2}\right) = \frac{m\vec{v}}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}$$

$\frac{\partial\mathcal{L}}{\partial x_i} = 0 \Rightarrow \frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = \text{const}$.

$$E = \vec{p}\cdot\vec{v} - \mathcal{L} = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}}$$

$$= \left(mv^2 + mc^2\left(1-\frac{v^2}{c^2}\right)\right)\gamma^{-1}$$

$$= mc^2\gamma^{-1} .$$

$$\simeq mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}+\dots\right) = mc^2 + \frac{1}{2}mv^2 + \dots$$

$E \neq 0$ for $\vec{v} = 0$.

From previous pg:

$$\vec{p} = m\gamma(\vec{v})\,\vec{v}$$

Force (Alonso - Finn)

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$= \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt} + \frac{m\vec{v}}{\left(1-v^2/c^2\right)^{3/2}}\left(\frac{\vec{v}}{c^2}\cdot\frac{d\vec{v}}{dt}\right)$$

$$= \gamma m\,\vec{a} + \frac{\gamma^3 m}{c^2}\left(\vec{v}\cdot\vec{a}\right)\vec{v} \quad \Rightarrow \vec{F} \not\parallel \vec{a}$$

$$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp} \quad \text{where} \quad \vec{a}_{\parallel} : \vec{a} \parallel \vec{v}, \quad \vec{v}\cdot\vec{a}_{\perp} = 0,$$
$$\vec{a}_{\perp} : \vec{a} \perp \vec{v}, \quad \vec{v}\cdot\vec{a} = \vec{a}_{\parallel}\cdot\vec{v}$$

$$\vec{F} = \underbrace{\frac{\gamma^3(\vec{v})\,m}{c^2}\left(\vec{a}_{\parallel}\cdot\vec{v}\right)\vec{v}}_{} + \gamma(\vec{v})m\left(\vec{a}_{\perp} + \vec{a}_{\parallel}\right) = \gamma^3(\vec{v})\,m\,\vec{a}_{\parallel} + \gamma(\vec{v})\,m\,\vec{a}_{\perp}$$

Therefore, $\vec{F} \not\parallel \vec{a}$.

Circular motion

$\vec{a}_{\parallel} = 0$ $\quad \left|\frac{d\vec{v}}{dt}\right| = v^2/R$

$$\vec{F} = \gamma m\frac{d\vec{v}}{dt}, \quad |\vec{F}| = \gamma m\frac{v^2}{R}.$$

Uniform - Motion

$$\gamma m\left(\frac{\gamma^2 v^2}{c^2} + 1\right) = \gamma^3 m\left(\frac{v^2}{c^2} + \frac{1 - v^2/c^2}{}\right)$$
$$\gamma^3 m\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \gamma^3 m\left(\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}\right)$$

$\vec{F} = \text{const.} \Rightarrow \vec{F} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right) \Rightarrow$

↑ Uniform ; $\vec{F}$ = const

$$\Rightarrow \vec{F}\cdot t = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

$$\text{or } \vec{v} = \frac{c\left(\vec{F}/mc\right)t}{\sqrt{1+\left(\frac{\vec{F}}{mc}\right)^2 t^2}} \longrightarrow c \quad \text{for } t \to \infty.$$

Trajectory

$$\frac{F}{mc} = \frac{a}{c} \Rightarrow \frac{L}{T^2}\frac{T}{L} = \frac{1}{T}$$

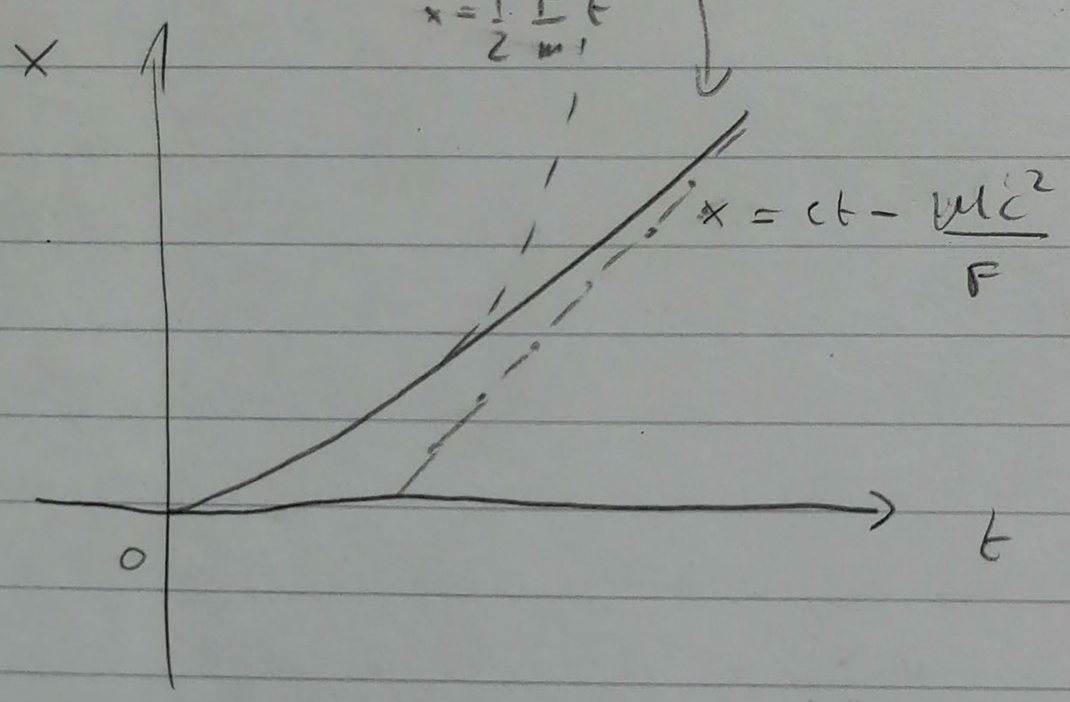
$$\vec{v} = \frac{d\vec{x}}{dt} = c\,\frac{(\vec{F}/mc)\cdot t}{\sqrt{1+\left(\frac{\vec{F}}{mc}\right)^2 t^2}}$$

for $x(0)=0$,

$$x = \frac{mc^2}{F}\left(\sqrt{1+\left(\frac{\vec{F}}{mc}\right)^2 \cdot t^2} - 1\right)$$

$$\approx \frac{1}{2}\frac{F}{m}\cdot t^2 \quad \text{for} \quad t \approx 0$$

$$\approx ct - \frac{mc^2}{F}$$



Energy from momentum:

$$E_k = \int_0^{\vec{v}} \vec{F} \cdot d\vec{r} = \int_0^{v} \frac{d}{dt}\left(m\vec{v}\gamma\right) \cdot \left(\frac{d\vec{r}}{dt}\right) dt$$

$$= \int_0^{v} \vec{v} \cdot d\left(m\vec{v}\gamma\right) \qquad \text{(*) } \vec{v}\cdot d\vec{v} = v\,dv$$

$$= \int_0^{v} d\left(\vec{v}\cdot m\vec{v}\gamma\right) - \int_0^{v} d\vec{v}\cdot\left(m\vec{v}\gamma\right) = m\gamma\vec{v}\cdot\vec{v} - \int_0^{v} m\gamma\vec{v}\cdot d\vec{v}$$

$$= \gamma m v^2 - \int_0^{v} \frac{m v\, dv}{\sqrt{1-v^2/c^2}}$$

$$= \gamma m v^2 + mc^2\sqrt{1-\frac{v^2}{c^2}} - mc^2$$

$$= \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2 \qquad = E_k = \gamma mc^2 - mc^2 = E - mc^2$$

$$E_k\left(\vec{v}=0\right) = 0$$

If we have an increasing of energy $\Rightarrow$ we have increasing of mass.

$$\boxed{E_k + mc^2 = E = \gamma mc^2}$$

(*) $v = \sqrt{\dot{x}_i^2} = \sqrt{v_i^2} = \sqrt{v_i \cdot v_i}$

$$dv = \frac{1}{2\sqrt{v_k v_k}}\, 2\, v_i\, dv_i = \frac{v_i\, dv_i}{\sqrt{v^2}} = \frac{v_i\, dv_i}{v}$$

$$\Rightarrow v\,dv = \frac{v_i\, v_i\, v_j\, dv_j}{\sqrt{v_k v_k}}$$

Lorentz transformations of energy and momentum

$$E = \gamma m c^2$$

$$\vec{p} = \gamma m \vec{v}$$

$$\frac{E^2}{c^2} - \vec{p}^{\,2} = -\gamma^2 m^2 v^2 + \gamma^2 m^2 c^2 =$$

$$= \gamma^2 m^2 \left(c^2 - v^2\right) = + \gamma^2 m^2 c^2 \left(1 - \frac{v^2}{c^2}\right)$$

$$= + \gamma^2 m^2 c^2 \frac{1}{\gamma^2}$$

$$= + m^2 c^2$$

$$\Rightarrow \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2 c^2 = \text{const.} \qquad \Rightarrow E = c\sqrt{m^2 c^2 + \vec{p}^{\,2}}$$

$\Downarrow$ I can define a 4-vector:

$$p^\alpha = \left(\frac{E}{c}, \vec{p}\right) \qquad p^\alpha p_\alpha = p^\alpha p^\beta \eta_{\alpha\beta}$$

$$= \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2 c^2$$

Therefore: $p^{\alpha'} = \Lambda^\alpha{}_\beta \, p^\alpha$

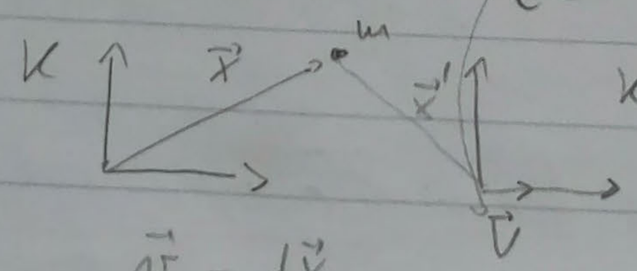
$p_y = p_z = 0$

$$\begin{cases} E' = \dfrac{E - V p_x}{\sqrt{1 - V^2/c^2}} \\ p_x' = \dfrac{p_x - V E/c^2}{\sqrt{1 - \dfrac{V^2}{c^2}}} \end{cases}$$

speed of K' respect to K along "x"

natural frame notation

$V \neq v$

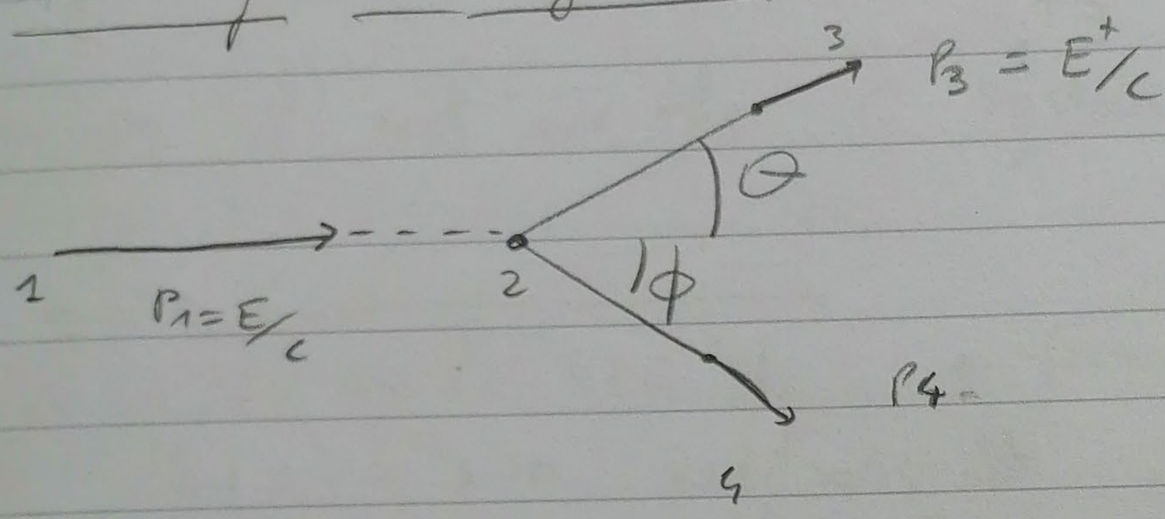


$\vec{v} = \dfrac{d\vec{x}}{dt}$, $\vec{V} = \vec{V}_{\text{relative}}$.

For massless particle the relation $\frac{E^2}{c^2} - \vec{p}^2 = 0 \Rightarrow |\vec{p}| = \frac{E}{c}$

is still valid, but is not justified here.

Scattering at high energy



$\vec{F} = 0$

$\frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = const$

$p^\alpha = const$

$|\vec{p}_1| = E/c$, $\vec{p}_2 = 0$, $|\vec{p}_3| = \frac{E^+}{c}$, $\vec{p}_4$

$E_1 = E$, $E_2 = mc^2$, $E_3 = E^+$, $E_4 = c\sqrt{m_4^2 c^2 + p_4^2}$

- Momentum conservation :

$$\vec{p}_1 = \vec{p}_3 + \vec{p}_4 \Rightarrow (A)\ \vec{p}_4^{\,2} = (\vec{p}_1 - \vec{p}_3)^2$$

$$= \vec{p}_1^{\,2} + \vec{p}_3^{\,2} - 2\vec{p}_1\cdot\vec{p}_3$$

$$= \frac{E^2}{c^2} + \frac{E^{+2}}{c^2} - \frac{2E^+E}{c^2}\cos\theta$$

Energy conservation :

$$E + mc^2 = E^+ + c\sqrt{mc^2 + \vec{p}_4^{\,2}}$$

we solve for $\vec{p}_4^{\,2}$: $\frac{1}{c^2}\left(E - E^+ + mc^2\right)^2 = mc^2 + \vec{p}_4^{\,2}$

$$(B)\ \vec{p}_4^{\,2} = \frac{E^2}{c^2} + \frac{E^{+2}}{c^2} - \frac{2EE^+}{c^2} + \frac{2Emc^2}{c^2} - \frac{2E^+mc^2}{c^2} - mc^2$$

$$= \frac{E^2}{c^2} + \frac{E^{+2}}{c^2} + \frac{2(E - E^+)mc^2}{c^2} - \frac{2EE^+}{c^2}$$

$$(A) = (B) \Rightarrow E - E^+ = \frac{E E^+}{mc^2}(1-\cos\theta)$$

Finally $$\frac{1}{E^+} - \frac{1}{E} = \frac{1}{mc^2}(1-\cos\theta)$$

Given (E, θ) we get E^+.

Energy THRESHOLD FOR the reaction $1+2 \to 3+4$

for 4 different particles, namely $m_i \neq 0 \quad i = 1,2,3,4$
$m_i \neq m_j \quad \forall i,j$.

— m_2 is at rest in the LAB :

$$E_1 = c\sqrt{m_1c^2 + p_1^2}, \quad E_2 = m_2c^2, \quad \vec{p}_2 = 0$$

the total energy and momentum are :

$$E = E_1 + m_2c^2, \quad \vec{P} = \vec{p}_1,$$

In another reference frame : $\vec{P}^2 - \frac{E^2}{c^2} = \vec{P}'^2 - \frac{E'^2}{c^2} \Leftrightarrow P^\alpha P_\alpha = P'^\alpha P'_\alpha$.

In the reference system "4", $\vec{P}' = 0 \Rightarrow \vec{P}^2 - \frac{E^2}{c^2} = -\frac{E'^2}{c^2}$

$$\Rightarrow E'^2 = -c^2\vec{P}'^2 + E^2 \Rightarrow E' = \sqrt{E^2 - c^2\vec{P}'^2}$$

$$= \sqrt{(E_1 + E_2)^2 - c^2\vec{P}_1^2}$$

$$= c\sqrt{(m_1^2 + m_2^2)c^2 + 2E_1m_2}$$

Kinetic energy E_{K_1}

Now $E_1 = E_{K1} + m_1c^2 = \gamma m_1 c^2 - m_1 c^2 + m_1 c^2 = \gamma m_1 c^2$

$\Rightarrow E' = c\sqrt{(m_1+m_2)^2c^2 + 2E_{K_1} \cdot m_2}$

The minimum energy to produce m_3, m_4 is the one for which they are at rest in the "C" system: $E' = (m_3 + m_4)c^2$

$\Rightarrow c\sqrt{(m_1+m_2)^2c^2 + 2E_{K_1} m_2} = (m_3+m_4)c^2$

$\Rightarrow E_{K_1} = \frac{c^2}{2m_2}\left[(m_3+m_4)^2 - (m_1+m_2)^2\right]$.

$\lambda = \frac{\hbar}{M}$, $r_s = \frac{2MG}{c^2}$

$\lambda = \frac{1}{M}$

$\lambda < r_s \Rightarrow$

$\frac{1}{M} < \frac{2M}{1}$

$2M > M_p$

$M > \frac{M_p}{2}$

BH Production (1 BHs)

For $m_3 = 0, m_4 >> m_1, m_2 \Rightarrow E_{K_1} \simeq \frac{c^2 (m_3+m_4)^2}{2m_2}$ ($m_3 \to 0$)

$m_2 = 0, M_4 = M_p \Rightarrow E_{K_1} = \frac{c^2 M_p^2}{2m_2} = \frac{2c^2 M_p^2}{m_2}$

$\gamma m_1 c^2 = \frac{c^2 M_p^2}{2m_2}$

$E_{K_1} = \gamma(v) m_1 c^2 - m_1 c^2 \Rightarrow \gamma = \frac{M_p^2}{2m_1 m_2} =$

$\simeq \gamma(v) m_1 c^2$

$\uparrow$

$\gamma >> 1$

$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{M_p^2}{2m_1 m_2}$

$m_1 = m_2 = m_{prot} \Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{M_p^2}{2m_{prot}^2} = \frac{(10^{19})^2 GeV^2}{1 GeV^2} = \frac{1}{2}\cdot(10^{19})^2$

$\left(\frac{v}{c}\right)^2 \simeq 1$.

$\frac{1}{1-\frac{v^2}{c^2}} = \frac{1}{2}(10^{38})^2$, $1 - \frac{v^2}{c^2} = \frac{2}{10^{72}}$

$v^2/c^2 = 1 - 2\cdot 10^{-72} \Rightarrow \frac{v}{c} \simeq 1 - \frac{1}{2}10^{-72}$

$$\epsilon'^{\alpha\beta} = \Lambda^{\alpha}_{\ \gamma}\,\Lambda^{\beta}_{\ \delta}\,\epsilon^{\gamma\delta}$$

$$= (\det \Lambda)\,\epsilon^{\alpha\beta}$$

Proof:

$$\epsilon'^{00} = \Lambda^{0}_{\ \gamma}\,\Lambda^{0}_{\ \delta}\,\epsilon^{\gamma\delta} = \Lambda^{0}_{\ 1}\,\Lambda^{0}_{\ 0}\,\underbrace{\epsilon^{10}}_{-1} + \Lambda^{0}_{\ 0}\,\Lambda^{0}_{\ 1}\,\underbrace{\epsilon^{01}}_{1} = 0$$

$$\epsilon'^{01} = \Lambda^{0}_{\ \gamma}\,\Lambda^{1}_{\ \delta}\,\epsilon^{\gamma\delta}$$

$$= \Lambda^{0}_{\ 0}\,\Lambda^{1}_{\ 1}\,\epsilon^{01} + \Lambda^{0}_{\ 1}\,\Lambda^{1}_{\ 0}\,\epsilon^{10}$$

$$= \Lambda^{0}_{\ 0}\,\Lambda^{1}_{\ 1}\,1 + \Lambda^{0}_{\ 1}\,\Lambda^{1}_{\ 0}\,(-1)$$

$$= \det \begin{pmatrix} \Lambda^{0}_{\ 0} & \Lambda^{0}_{\ 1} \\ \Lambda^{1}_{\ 0} & \Lambda^{1}_{\ 1} \end{pmatrix}.$$

Verify:

$\beta = 0 \Rightarrow \partial_\alpha F^{\beta 0} = \frac{4\pi}{c} J^0$

$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{4\pi}{c} J^0$$

$$\partial_x E^x + \partial_y E^y + \partial_z E^z = 4\pi\rho$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

Covariant Form of the Maxwell equations

Def. $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \qquad A^\alpha = (\phi, \vec{A})$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix} \qquad F^{\alpha\beta} = -F^{\beta\alpha}$$

- $F_{\alpha\beta} = \eta_{\alpha\gamma}\eta_{\beta\delta} F^{\gamma\delta}$

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{for } \alpha=0, \beta=1, \gamma=2, \delta=3 \\ & \text{and even permutations} \\ -1 & \text{odd permutations} \\ 0 & \text{otherwise.} \end{cases}$$

$$\epsilon'^{\alpha\beta\gamma\delta} = \Lambda^\alpha{}_{\alpha'} \Lambda^\beta{}_{\beta'} \Lambda^\gamma{}_{\gamma'} \Lambda^\delta{}_{\delta'} \epsilon^{\alpha'\beta'\gamma'\delta'}$$

$$= \underbrace{(\det \Lambda)}_{1} \epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta}$$

1 LT.

$$\mathcal{J}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \equiv {}^*F^{\alpha\beta}$$

Non Homogeneous eq. $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\alpha}$$

where $J^\alpha = (c\rho, \vec{J})$

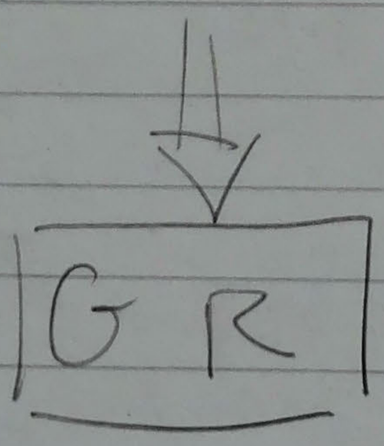
Homogeneous eq.: $\boxed{\partial_\alpha J^{\alpha\beta} = 0}$

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

Bianchi identities: $\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0.$

Lorentz eq. $\frac{dP^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta.$

How We introduce gravity ??

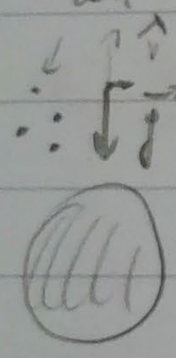


EQUIVALENCE PRINCIPLE

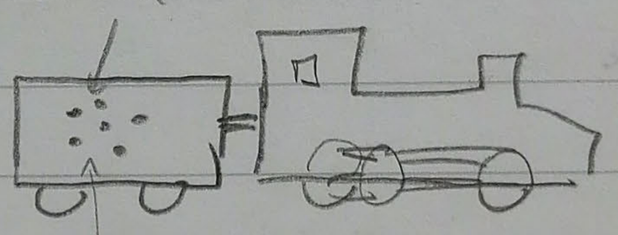
All bodies (or masses) move in a gravitational Field ($m\ddot{\vec{r}} = -G\frac{Mm}{r^2}\hat{r} \quad \forall m$) independently of their mass.

[Example. they all fall down in the gravitational field with the same acceleration.]

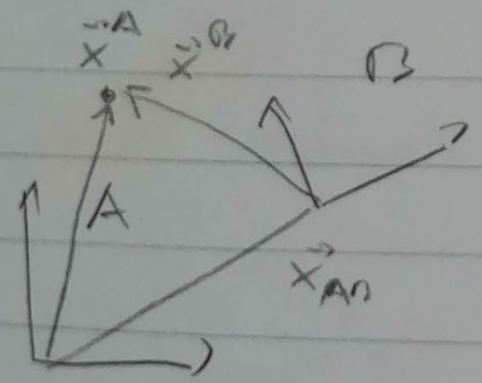
Therefore, the motion in a gravitational Field is equivalent to the motion in an accelerating non inertial reference system.



$\equiv$



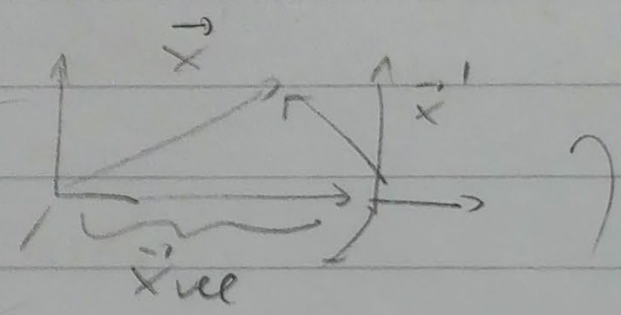
$\vec{F}_{apparent} = -m\vec{\rho} \Leftarrow \vec{a}_{rel} = \vec{\rho}$



$\vec{x} = \vec{x}' + \vec{x}_{rel}$

$\dot{\vec{x}} = \dot{\vec{x}}' + \dot{\vec{x}}_{rel}$

$\vec{a} = \vec{a}' + \vec{a}_{rel}$



$\Rightarrow \vec{a}' = -\vec{a}_{rel} = -\vec{\rho}$

↑ acceleration relative to k'.

$\vec{F}_A = \vec{F}_B + m\vec{a}_{AB}$

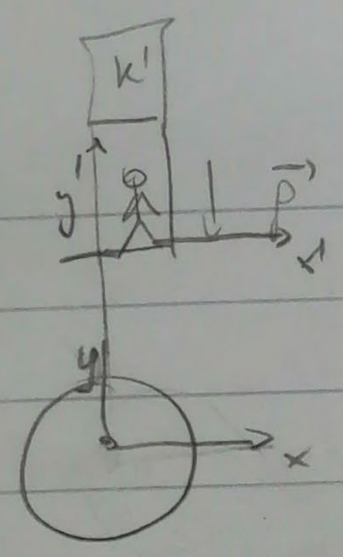
$\vec{F}_A = m\vec{a}_A$ $\quad \vec{a}_A = \frac{d^2\vec{x}_A}{dt^2}$

$\vec{F}_B = m\vec{a}_B$ $\quad \vec{a}_B = \frac{d^2\vec{x}_B}{dt^2}$

$\vec{F}_B = \vec{F}_A + \vec{F}_{apparent} = -m\vec{a}_{AB}$

apparent = fictitious

K



$\vec{a} = \vec{a}' + \vec{a}_{rel}$

$\Rightarrow \vec{a} = \vec{a}_{rel}$

$\vec{a}_{rel} = \vec{\rho} = +\rho\hat{y} = \rho\hat{y}'$

K'

$m\vec{a}' = \vec{F} + \vec{F}_{off}$

$\vec{a}' = 0$ because $y' = 0 = const$

$\vec{F}_{off} = m\vec{a}_{rel}$ $\quad \vec{F} = \vec{R} = R\hat{y}'$ $\quad m\ddot{y}' = -m\rho + R$

$\vec{F}_{off} = -\rho\hat{y}'m$ $\quad 0$ $\quad F_{offforce}$

$\rightarrow 0 = R - m\rho$

Non relativistic mechanics in a gravitational Field:

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - m\varphi$

equation: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0$

$m\dot{x} = -m\frac{\partial \varphi}{\partial x}$

$$\boxed{\dot{x} = -\frac{\partial \varphi}{\partial x}} \quad \forall m$$

Vector space E of dimension N, basis $\vec{e}_\mu$ $(\mu = 0, \dots, N-1)$.

→ Linearly independent vectors: $\sum_{n=0}^{N-1} \alpha_\mu \vec{e}_\mu = 0 \Leftrightarrow \alpha_\mu = 0 \; \forall \mu$.

→ Dot product: $\vec{e}_\mu \cdot \vec{e}_\nu = g_{\mu\nu}$ (metric tensor)

of course $g_{\mu\nu}$ is symmetric.

$\vec{v} = v_i \vec{e}_i$ $\quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

EXAMPLE ① CARTESIAN SPACE :

$$\vec{e}_i \cdot \vec{e}_j = g_{ij} = \delta_{ij}$$

EXEMPLE ② $\vec{e}_\mu \cdot \vec{e}_\nu = g_{\mu\nu} = \eta_{\mu\nu}$ Minkowski space.

Vector $V = V^\mu \vec{e}_\mu \equiv \sum_{\mu=0}^{N-1} V^\mu \vec{e}_\mu$

on contravariant.

General coordinate transformation: $x^\mu = f^\mu(x'^{\mu'})$ $\quad (x^\mu = x^\mu(x'^{\mu'}))$

$x'^{\mu'} = g^{\mu'}(x^\mu)$ $\quad (x'^{\mu'} = x'^{\mu'}(x^\mu))$

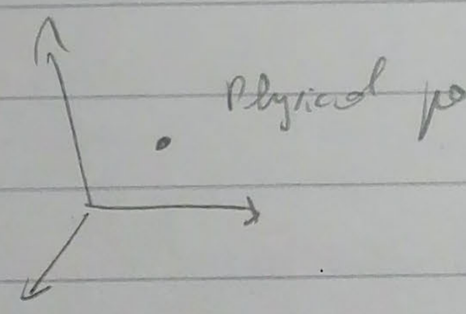
$$\Rightarrow dx^\mu = \frac{\partial f^\mu}{\partial x'^{\mu'}} dx'^{\mu'} = \frac{\partial x^\mu}{\partial x'^{\mu'}} dx'^{\mu'}$$

$\frac{\partial x^\mu}{\partial x'^{\mu'}}$ ↝ $\Lambda^\mu_{\mu'}$

$$V^\mu = \Lambda^\mu_{\mu'} V'^{\mu'}$$

↙ def of vector

How $\vec{e}_\mu$ transform?



$$V^{\mu} \vec{e}_\mu = V'^{\mu'} \vec{e}'_{\mu'}$$

$$\Lambda^{\mu}{}_{\mu'} V'^{\mu'} \vec{e}_\mu = V'^{\mu'} \vec{e}'_{\mu'} \Rightarrow \vec{e}'_{\mu'} = \Lambda^{\mu}{}_{\mu'} \vec{e}_\mu \quad (1.36)$$

Now: $dx'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} dx^{\mu} = \Lambda^{\mu'}{}_{\mu} dx^{\mu}$,

which enables us to write:

$$V'^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu}$$

and $V^{\mu} \vec{e}_\mu = V'^{\mu'} \vec{e}'_{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu} \vec{e}'_{\mu'}$

or identifying the coefficients of V^{μ}:

$$\vec{e}_\mu = \Lambda^{\mu'}{}_{\mu} \vec{e}'_{\mu'} \quad (1.37)$$

(1.36) and (1.37) $\Rightarrow$

$$\vec{e}_\mu = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu}{}_{\mu'} \vec{e}_\nu$$

$$\Rightarrow \delta^{\nu}_{\mu}$$

$$\Lambda^{\nu}{}_{\mu'} \Lambda^{\mu'}{}_{\mu} = \delta^{\nu}_{\mu} \qquad \text{If } \Lambda^{\mu}{}_{\mu'} \equiv (\Lambda)^{\mu}{}_{\mu'}$$

$$\Lambda \Lambda^{-1} = \mathbb{1} \Rightarrow \Lambda^{\mu'}{}_{\mu} \equiv (\Lambda^{-1})^{\mu'}{}_{\mu}$$

TENSOR PRODUCT

Given $V = V^\mu \vec{e}_\mu$ and $W = W^\nu \vec{e}_\nu$

$\Rightarrow T = V \otimes W$

$= V^\mu \vec{e}_\mu \otimes W^\nu \vec{e}_\nu = \underbrace{V^\mu W^\nu}_{T^{\mu\nu}} \underbrace{\vec{e}_\mu \otimes \vec{e}_\nu}_{\text{Basis in } V \otimes W}$

components

Matrix table

$$\left(T^{\mu\nu}\right) = \begin{pmatrix} V^0W^0 & V^0W^1 & \cdots \\ V^1W^0 & V^1W^1 & \ddots \end{pmatrix}$$

Transformations:

$$T^{\mu'\nu'} \equiv V^{\mu'} W^{\nu'} = \Lambda^{\mu'}_{\ \mu} V^{\mu} \Lambda^{\nu'}_{\ \nu} W^{\nu}$$

$$= \Lambda^{\mu'}_{\ \mu} \Lambda^{\nu'}_{\ \nu} T^{\mu\nu}$$

Now $\vec{e}\,'_{\mu'} = \Lambda^{\mu}_{\ \mu'} \vec{e}_\mu$

$\vec{e}\,'_{\nu'} = \Lambda^{\nu}_{\ \nu'} \vec{e}_\nu$ $\Leftarrow$ (1.36)

and $\underbrace{T^{\mu\nu}}\ \vec{e}_\mu \otimes \vec{e}_\nu = T^{\mu'\nu'} \vec{e}\,'_{\mu'} \otimes \vec{e}\,'_{\nu'}$

$= T^{\mu'\nu'} \Lambda^{\mu}_{\ \mu'} \vec{e}_\mu \otimes \Lambda^{\nu}_{\ \nu'} \vec{e}_\nu$

Comparing.

In general $\vec{e}_\mu \otimes \vec{e}_\nu$ has a fixed order and we can not commute the vectors.

General tensors

$$T = T^{\mu\nu\dots\lambda}\ \vec{e}_\mu \otimes \vec{e}_\nu \otimes \dots \otimes \vec{e}_\lambda$$

$$= T^{\mu'\nu'\dots\lambda'}\ \vec{e}'_{\mu'} \otimes \vec{e}'_{\nu'} \otimes \dots \otimes \vec{e}'_{\lambda'}$$

$$\Rightarrow T^{\mu\nu\dots\lambda} = \Lambda^{\mu}_{\mu'} \Lambda^{\nu}_{\nu'} \dots \Lambda^{\lambda}_{\lambda'}\ T^{\mu'\nu'\dots\lambda'}$$

COVARIANT VECTORS

Given the space E with metric $g_{\mu\nu}$ $\Rightarrow$ we introduce the dual space, by the relation:

$$g_{\mu\lambda} g^{\lambda\nu} = \delta^\nu_\mu .$$

$\underline{\omega^\mu}$ ∥

The dual space E^* is defined by N vectors $\vec{e}^{\,\mu}$ formed from the vectors $\vec{e}_\mu \in E$, namely

$$\vec{e}^{\,\mu} = g^{\mu\nu}\, \vec{e}_\nu .$$

(dot)

- The product $\vec{e}^{\,r}$ times $\vec{e}^{\,v}$ defines the metric of the dual space:

$$\vec{e}^{\,r} \cdot \vec{e}^{\,v} = \rho^{\mu\alpha}\,\vec{e}_\alpha \cdot \rho^{\nu\beta}\,\vec{e}_\beta =$$

$$= \rho^{\mu\alpha}\,\rho^{\nu\beta}\,\rho_{\alpha\beta} = \rho^{\mu\alpha}\,\delta^{\nu}_{\alpha} = \rho^{\mu\nu}.$$

Also $\vec{e}^{\,r} \cdot \vec{e}_v = \delta^r_v$.

$$\rho^{re}\,\vec{e}_e \cdot e_v = \rho^{re}\,\rho_{ev} = \delta^r_v$$

From the projection of V in the dual space we obtain the contravariant components V^r of the vector V,

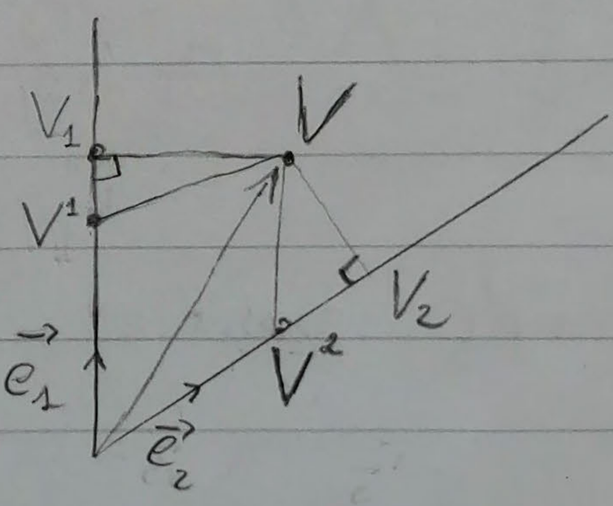
$$V \cdot \vec{e}^{\,v} = V^r\,\vec{e}_\mu \cdot \vec{e}^{\,v} = V^r\,\delta^v_\mu = V^\nu.$$

Covariant vector: $V = V_r\,\vec{e}^{\,r}$ $\left(V_r\,\vec{\omega}^{\,r}\right)$ ← Gravitation motion M.T.W

$$\vec{e}^{\,r} = \rho^{r\nu}\,\vec{e}_v \Rightarrow V = \underbrace{V_r\,\rho^{r\nu}}_{V^\nu}\,\vec{e}_v = V^v\,\vec{e}_v$$

$$\Rightarrow \boxed{V^\nu = \rho^{\nu\mu}\,V_\mu}$$

Relation between covariant and contravariant vectors.



$$V = V^r\,\vec{e}_r = V^1\,\vec{e}_1 + V^2\,\vec{e}_2$$

$$V \cdot \vec{e}_1 = V^r\,\vec{e}_r \cdot \vec{e}_1 = V^\mu\,\rho_{\mu 1} = V_1, \text{ component of } V \text{ along } \vec{e}_1$$

$$V \cdot \vec{e}_2 = V^r\,\vec{e}_r \cdot \vec{e}_2 = V^r\,\rho_{r2} = V_2.$$

Suppose for diagonal

$$\Rightarrow V_2 = \rho_{22}\,V^2 \neq V^2 \quad \left(\vec{e}^{\,1} \cdot \vec{e}_1 = 1 = \rho^{11}\,\vec{e}_1 \cdot \vec{e}_1 = 1\right) \quad \vec{e}^{\,1} \cdot \vec{e}_2 = 0$$

$$\vec{e}^{\,\alpha} \cdot \vec{e}_r = \delta^\alpha_r$$

Mathematically the covariant components of a vector behave like the derivative of a scalar function φ.

Consider two coordinate basis:

x^μ, basis $\vec{e}_\mu$,

$x'^{\mu'}$, basis $\vec{e}'_{\mu'}$,

Scalar: $\varphi(x) = \varphi'(x')$

$$\frac{\partial \varphi}{\partial x^\mu} = \frac{\partial \varphi}{\partial x'^{\mu'}} \frac{\partial x'^{\mu'}}{\partial x^\mu} = \frac{\partial \varphi}{\partial x'^{\mu'}} \Lambda^{\mu'}{}_{\mu}$$

$$\Rightarrow \left(\frac{\partial}{\partial x^\mu}\right) = \Lambda^{\mu'}{}_{\mu} \left(\frac{\partial}{\partial x'^{\mu'}}\right)$$

$$= \left(\frac{\partial x'^{\mu'}}{\partial x^\mu}\right) \frac{\partial}{\partial x'^{\mu'}}$$

$$\partial_\mu = \Lambda^{\mu'}{}_{\mu}\, \partial'_{\mu'}$$

Coordinate basis: $\vec{e}_\mu \Rightarrow \partial_\mu$

$\vec{e}^{\,\mu} \Rightarrow dx^\mu$

Im general $V_\mu = V'_{\mu'} \Lambda^{\mu'}{}_{\mu}$ $\quad \vec{e}_\mu(\vec{e}^{\,\nu}) = \delta^\nu_\mu$

$V_\mu = \Lambda^{\mu'}{}_{\mu} V'_{\mu'}$

Dirac Notation

Scalar product : $V \cdot W = \langle V | W \rangle$

$=_v \langle V| \in E^*$ (bra V) , $|W\rangle \in E$
(row) (column)

$\langle V|W\rangle = (V_0 \vec{e}^0 + V_1 \vec{e}^1 + \dots) \cdot (W^0 \vec{e}_0 + W^1 \vec{e}_1 + \dots)$

$= (V_0, V_1, \dots) \begin{pmatrix} \vec{e}^0 \cdot \vec{e}_0 & \vec{e}^0 \cdot \vec{e}_1 \\ \vdots & \end{pmatrix} \begin{pmatrix} W^0 \\ W^1 \\ \vdots \end{pmatrix}$

(δ^0_0 over $\vec{e}^0 \cdot \vec{e}_0$; δ^0_1 over $\vec{e}^0 \cdot \vec{e}_1$)

low up

$\vec{e}^{N-1} \cdot \vec{e}_0 \quad \dots \quad \delta^{N-1}_0$

Notice: $V_\mu W^\mu = (V_0, \dots V_4)(\Pi) \begin{pmatrix} W^0 \\ W^4 \end{pmatrix}$

or $= (V^0, V^4)(\delta) \begin{pmatrix} W_0 \\ W_4 \end{pmatrix}$

$\Pi = \delta^\mu_0 = \delta^1_0$

$\vec{e}^{\,\mu'} \cdot \vec{e}_{\nu'} = \delta^{\mu'}_{\nu'}$.

$\vec{e}^{\,\mu'} \cdot \Lambda^\nu_{\ \nu'} \vec{e}_\nu = \delta^{\mu'}_{\nu'}$

$\Rightarrow \vec{e}^{\,\mu'} = \Lambda^{\mu'}_{\ \mu} \vec{e}^{\,\mu}$.

You can definite the theory in terms of vectors and forms (no need for metric) or in terms of vectors and the metric (no need of forms).

Indeed: $\Lambda^{\mu'}_{\ \mu} \vec{e}^{\,\mu} \cdot \Lambda^\nu_{\ \nu'} \vec{e}_\nu = \Lambda^{\mu'}_{\ \mu} \Lambda^\nu_{\ \nu'} \delta^\mu_\nu =$

$= \Lambda^{\mu'}_{\ \mu} \Lambda^\mu_{\ \nu'} = \delta^{\mu'}_{\nu'}$

$V \cdot W = V^\mu W^\nu \delta_{\mu\nu} = V^\mu W_\mu = \Lambda^\mu_{\ \mu'} V'^{\mu'} W_\mu = \Lambda^\mu_{\ \mu'} V'^{\mu'} W'_{\mu'} \Lambda^{\mu'}_{\ \mu}$

$\Rightarrow \Lambda^\mu_{\ \mu'} \Lambda^{\mu'}_{\ e} = \delta^\mu_e$

Covariant tensor

$$T_{\mu\dots\lambda} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} \dots \Lambda^{\lambda'}{}_{\lambda} T'_{\mu'\nu'\dots\lambda'}$$

Mixed Tensor

$$T^{\mu}{}_{\dots\chi} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} \Lambda^{\lambda'}{}_{\lambda} \Lambda^{\chi'}{}_{\chi} T^{\mu'\nu'\dots}{}_{\lambda'\chi'}$$

Contractions

$$\vec{e}^{\,\mu} \cdot \vec{e}_{\nu} = g^{\mu\alpha} \vec{e}_{\alpha} \cdot \vec{e}_{\nu} = g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$$

$$V = V^{\mu} \vec{e}_{\mu} \in E$$

$$W = W_{\mu} \vec{e}^{\,\mu} \in E^{*}$$

$$V \cdot \vec{e}^{\,\nu} = V^{\mu} \vec{e}_{\mu} \cdot \vec{e}^{\,\nu} = V^{\mu} \delta_{\mu}{}^{\nu} = V^{\nu}$$

$$T^{\alpha}{}_{\beta}{}^{\gamma} = T \cdot \vec{e}^{\,\alpha} \otimes \vec{e}_{\beta} \otimes \vec{e}^{\,\gamma}$$

$$T^{\alpha}{}_{\beta}{}^{\beta} = T \cdot \vec{e}^{\,\alpha} \otimes \vec{e}_{\beta} \otimes \vec{e}^{\,\beta} = T \cdot \vec{e}^{\,\alpha} \equiv T^{\alpha}$$

Im fact $\vec{e}_{\beta} \otimes \vec{e}^{\,\beta} = |\vec{e}_{\beta}\rangle\langle\vec{e}^{\,\beta}| = \mathbb{1}$

$$T = T^{\mu}{}_{\nu}{}^{\rho} \; \vec{e}_\mu \otimes \vec{e}^{\nu} \otimes \vec{e}_\rho$$

For $\nu = \rho$: $\vec{e}^{\nu} \otimes \vec{e}_\nu = 1$

$$T \cdot \vec{e}^{\alpha} \otimes \vec{e}_\beta \otimes \vec{e}^{\gamma} = \left(T^{\mu}{}_{\nu}{}^{\rho} \; \vec{e}_\mu \otimes \vec{e}^{\nu} \otimes \vec{e}_\rho \right) \cdot \left(\vec{e}^{\alpha} \otimes \vec{e}_\beta \otimes \vec{e}^{\gamma} \right)$$

$$= T^{\mu}{}_{\nu}{}^{\rho} \; \delta^{\alpha}_{\mu} \, \delta^{\nu}_{\beta} \, \delta^{\gamma}_{\rho}$$

$$= T^{\alpha}{}_{\beta}{}^{\gamma}$$

Differentiation

The gradient is a covariant vector, $\frac{\partial}{\partial x^\alpha} = \partial_\alpha$

- Consider the tensor $T^{\beta\gamma} \Rightarrow$

$$\frac{\partial}{\partial x^\alpha}\left(T^{\beta\gamma}\right) = \Lambda^{\alpha'}{}_{\alpha} \frac{\partial}{\partial x^{\alpha'}} \left(\Lambda^{\beta}{}_{\beta'} \Lambda^{\gamma}{}_{\gamma'} T^{\beta'\gamma'} \right)$$

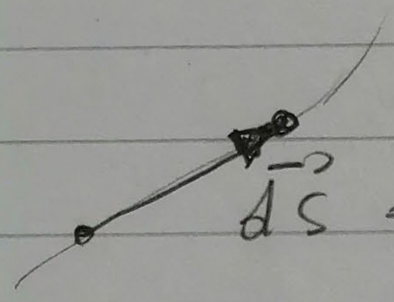
$$= \Lambda^{\alpha'}{}_{\alpha} \Lambda^{\beta}{}_{\beta'} \Lambda^{\gamma}{}_{\gamma'} \frac{\partial}{\partial x^{\alpha'}} T^{\beta'\gamma'} +$$

$$+ \Lambda^{\alpha'}{}_{\alpha} T^{\beta'\gamma'} \left(\frac{\partial x^{\beta}}{\partial x^{\beta'} \partial x^{\alpha'}} \cdot \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} + \right.$$

$$\left. + \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial^2 x^{\gamma}}{\partial x^{\gamma'} \partial x^{\alpha'}} \right)$$

$\Rightarrow \partial_\alpha T^{\beta\gamma}$ is NOT a tensor.

METRIC



$\vec{ds} = dx^\mu \vec{e}_\mu$ Oriented straight line

$ds^2 = \vec{ds} \cdot \vec{ds} = dx^\mu dx^\nu \vec{e}_\mu \cdot \vec{e}_\nu = g_{\mu\nu} dx^\mu dx^\nu$

"metric

Invariance $ds^2 = ds'^2$

$g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu'\nu'} dx'^{\mu'} dx'^{\nu'}$

$dx'^{\mu'} = \Lambda^{\mu'}_{\ \mu} dx^\mu$

$\Rightarrow g_{\mu\nu} = g'_{\mu'\nu'} \Lambda^{\mu'}_{\ \mu} \Lambda^{\nu'}_{\ \nu}$

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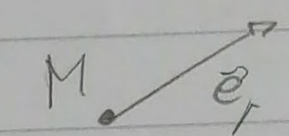
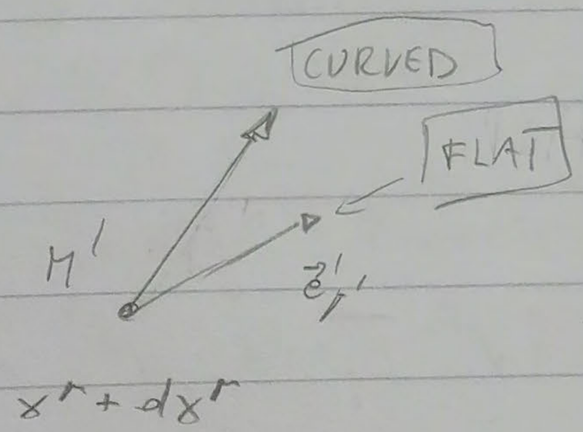
skip

Flat space and
curved space examples.





the deviation affects the direction of the vector
in a curved spacetime.

Consider: $V^\mu = V'^{\mu'} \frac{\partial x^\mu}{\partial x'^{\mu'}}$ in M, x^μ Point ↓

The value of V^μ in M' (at coordinates $x^\mu + dx^\mu$),
requires the evaluation of dV^μ

$$dV^\mu = dV'^{\mu'} \frac{\partial x^\mu}{\partial x'^{\mu'}} + V'^{\mu'} d\left(\frac{\partial x^\mu}{\partial x'^{\mu'}}\right)$$

$$= dV'^{\mu'} \frac{\partial x^\mu}{\partial x'^{\mu'}} + V'^{\mu'} \frac{\partial^2 x^\mu}{\partial x'^{\mu'} \partial x'^{\nu'}} dx'^{\nu'}$$

$\frac{\partial x^\mu}{\partial x'^{\mu'}}$: contravariant vector

$\Lambda^\mu_{\mu'} = \text{const} \Rightarrow 0$

in general does not behave like a vector.

Let us to consider:

$$V = V^\mu \vec{e}_\mu$$

$$\Rightarrow dV = dV^\mu \vec{e}_\mu + V^\mu d\vec{e}_\mu$$ ← because the direction change in curved space or in curved coordinates like polar coordinates.

$$= \frac{\partial V^\mu}{\partial x^\nu} dx^\nu \vec{e}_\mu + V^\mu \underbrace{d\vec{e}_\mu}$$

linear combination of vector basis.

$$d\vec{e}_\mu \equiv \Gamma^\alpha{}_{\mu\nu} dx^\nu \vec{e}_\alpha$$

$$\Rightarrow dV = \partial_\nu V^\mu dx^\nu \vec{e}_\mu + V^\mu \Gamma^\alpha{}_{\mu\nu} dx^\nu \vec{e}_\alpha$$

$$= \underbrace{\left(\partial_\nu V^\mu + V^\sigma \Gamma^\mu{}_{\sigma\nu}\right)}_{\nabla_\nu V^\mu \equiv \frac{DV^\mu}{dx^\nu}} dx^\nu \vec{e}_\mu$$

Covariant derivative:

$$\frac{DV^\mu}{dx^\nu} \equiv \nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu{}_{\sigma\nu} V^\sigma \equiv V^\mu{}_{;\nu}$$

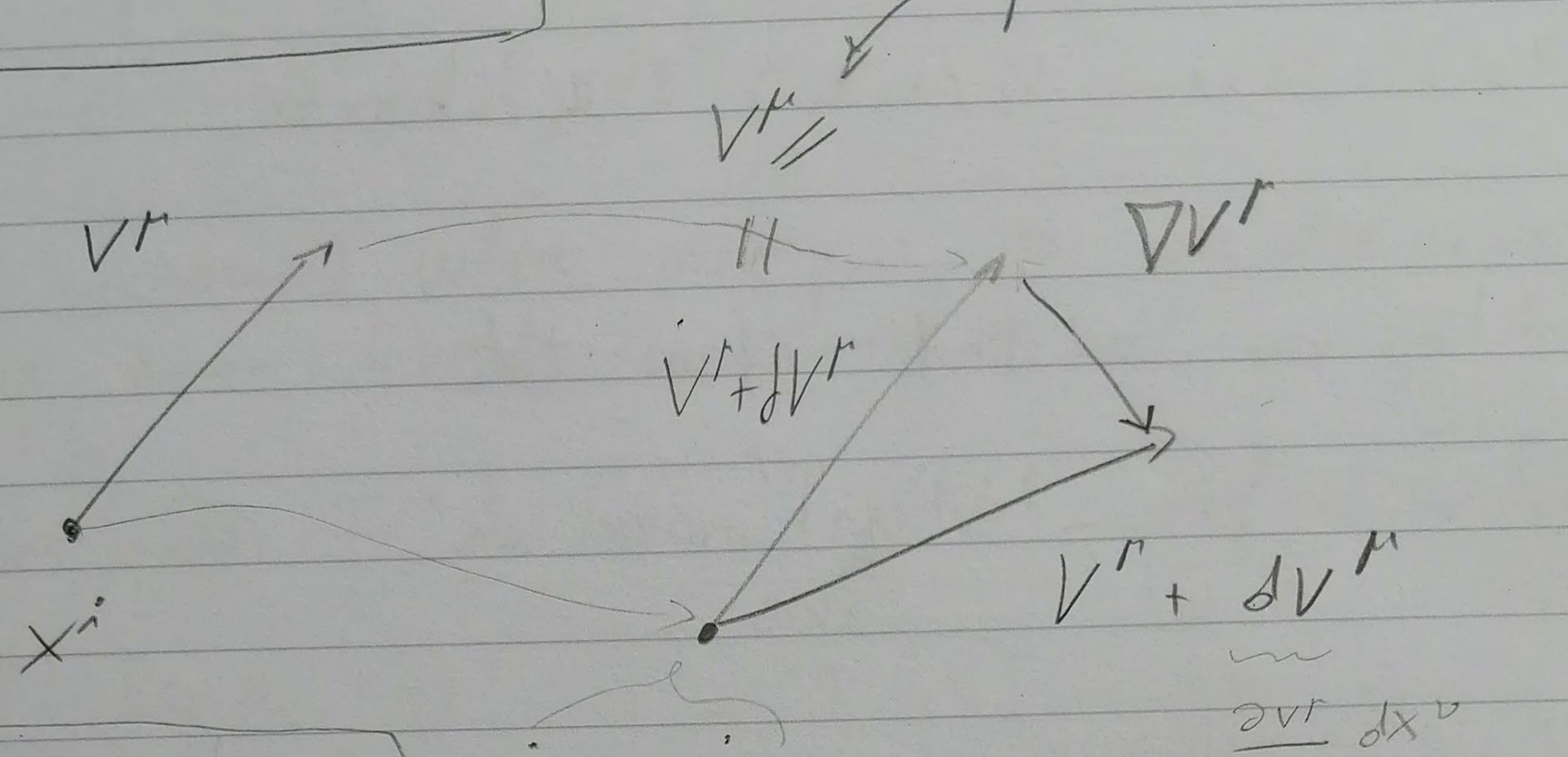
$$dV = \underbrace{\left(\nabla_\nu V^\mu\right) dx^\nu}_{(\nabla V^\mu)} \vec{e}_\mu$$

$\Gamma^\alpha{}_{\mu\nu}$:
Christ.-Hels symbols.

Measure invariant

$$\sqrt{g}\, d^4x = \sqrt{-g'(x')}\, d^4x'$$

COVARIANT DERIVATIVE, PARALLEL TRANSPORT.



V^μ : VECTOR FIELD.

$V^\mu(x^i) \equiv V^\mu$, $V^\mu(x^i + dx^i) = V^\mu + dV^\mu$

3) $\nabla V^\mu = \underbrace{\partial_\nu V^\mu\, dx^\nu}_{dV^\mu} + \underbrace{V^\sigma \Gamma^\mu_{\sigma\nu}\, dx^\nu}_{-\delta V^\mu}$

2) $V^\mu + dV^\mu - (V^\mu + \delta V^\mu) = dV^\mu - \delta V^\mu \equiv \nabla V^\mu$

1) $\nabla V^\mu + V^\mu + \delta V^\mu = dV^\mu + V^\mu \Rightarrow \nabla V^\mu = dV^\mu - \delta V^\mu$

(Note: if $V^\mu(x)$ is independent on $x \Rightarrow \frac{\partial V}{\partial x} = 0$, but $\delta V \neq 0$ anyway.)

Request properties:

$$\nabla_u \vec{v} = u^\alpha \nabla_{\vec{e}_\alpha}(v^\beta \vec{e}_\beta) = u^\alpha \underbrace{(\nabla_\alpha v^\beta)}_{\frac{Dv^\beta}{dx^\alpha}} \vec{e}_\beta$$

1) $V_{//}^{\mu}(x+\Delta x) - V^{\mu}(x) \propto \Delta x$

2) $(V_{//} + W_{//})(x+\Delta x) = V_{//}(x+\Delta x) + W_{//}(x+\Delta x)$

These conditions are satisfied if we take

$$V_{//}^{\mu}(x+\Delta x) = V^{\mu}(x) \underbrace{- V^{\lambda}(x)\Gamma^{\mu}_{\nu\lambda}(x)\cdot\Delta x^{\nu}}_{-\delta V^{\mu}} \quad (\star)$$

$$V^{\mu}(x) - (V^{r}\Gamma^{\mu}_{\nu r} + V^{\phi}\Gamma^{r}_{\nu\phi})\Delta x^{\nu}$$

$$\boxed{\mu=r}: V^{r} - (V^{r}\Gamma^{r}_{rr}\Delta r + V^{r}\Gamma^{r}_{\phi r}\Delta\phi + V^{\phi}\Gamma^{r}_{r\phi}\Delta r + V^{\phi}\Gamma^{r}_{\phi\phi}\Delta\phi)$$

Norm: $g(V,V) = V^{\mu}V^{\nu}g_{\mu\nu} = V^{r^2}g_{rr} + V^{\phi^2}g_{\phi\phi}$

$= (V^{r})^2 + r^2(V^{\phi})^2$

$\boxed{\mu=\phi}: V^{\phi}$

$V^{\phi} - V^{\lambda}\Gamma^{\phi}_{\nu\lambda}dx^{\nu}$

$= V^{\phi} - V^{\lambda}\Gamma^{\phi}_{r\lambda}dr + \dots$

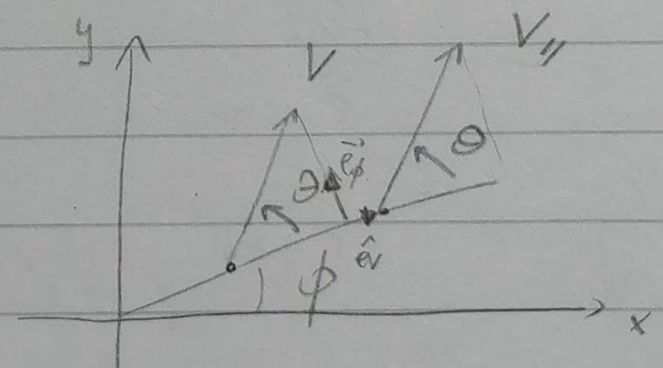
$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

Parallel transport to $(r+\Delta r, \phi)$ ← constant

$$V^{\mu}\frac{\partial}{\partial x^{\mu}} = V^{r}\partial_r + V^{\phi}\partial_{\phi}$$

$V^{r} = V\cos\theta$ ← Why? ← Verify ↓ Here: ↓

$V^{\phi} = V\underbrace{\frac{\sin\theta}{r}}$



$$V = \sqrt{g(V,V)} = \left[(V^{r})^2 + r^2(V^{\phi})^2\right]^{\frac{1}{2}}$$

$$= \left[V^2\cos^2\theta + r^2\cdot V^2\frac{\sin^2\theta}{r^2}\right]^{\frac{1}{2}} = V \quad \text{OK}$$

$$V_{//} = V_{//}^{r}\partial_r + V_{//}^{\phi}\partial_{\phi}$$

$$= V^{r}\partial_r + V\underbrace{\frac{\sin\theta}{r+\Delta r}}\partial_{\phi} = V^{r}\partial_r + \underbrace{\frac{r}{r+\Delta r}}V^{\phi}\partial_{\phi}$$

$\Rightarrow V_{//}^{r} = V^{r}$

$\Rightarrow V_{//}^{\phi} = V^{\phi} - \frac{1}{r}\Delta r V^{\phi}$ $\quad \cong V^{r}\partial_r + V^{\phi}\partial_{\phi} - \frac{\Delta r}{r}V^{\phi}\partial_{\phi}$

$\frac{r}{r(1+\frac{\Delta r}{r})} \simeq \left(1 - \frac{\Delta r}{r}\right)$ ← infinitesimal

Comparing with (*) we get $\Gamma^r_{rr} = 0$, $\Gamma^r_{r\phi} = 0$, $\Gamma^\phi_{rr} = 0$

$\Gamma^\phi_{r\phi} = \frac{1}{r}$.

$\Gamma^r_{\phi r} = 0$

In an inertial reference frame

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

It is invariant when we pass to any other inertial reference frame: Lorentz invariance.

If we pass to a non inertial reference frame

$\Rightarrow ds^2$ change.

We can consider to go in a rotating reference system:

$$x = x'\cos\Omega t - y'\sin\Omega t$$

$$y = x'\sin\Omega t + y'\cos\Omega t$$

$$z' = z$$

$$\Rightarrow ds^2 = \left(c^2 - \Omega^2(x'^2 + y'^2)\right)dt^2 - dx'^2 - dy'^2 - dz'^2$$

$$+2\Omega y'\,dx'\,dt - 2\Omega x'\,dy'\,dt$$

Therefore, we get the general quadratic form:

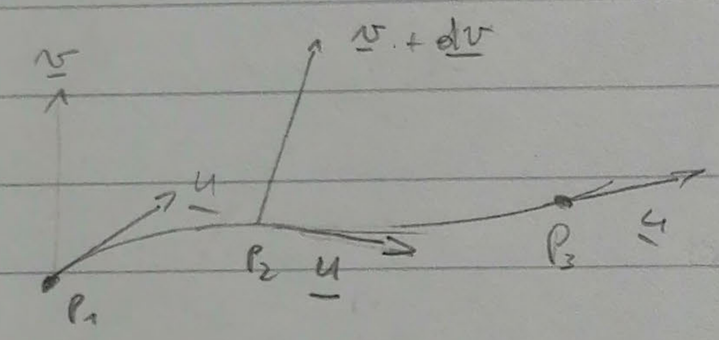
$$ds^2 = g_{ik}\,dx^i dx^k$$

$\underline{u} \parallel \underline{v}$ in P if $\nabla_{\underline{u}} \underline{v} = 0$ in P.

the vector field $\underline{v}$ is parallel transported along a curve γ if

$\underline{v}$ is parallel to the tangent vector $\underline{u}$ in any point of γ,

$u^\beta v^\alpha{}_{;\beta} = 0$ on γ.

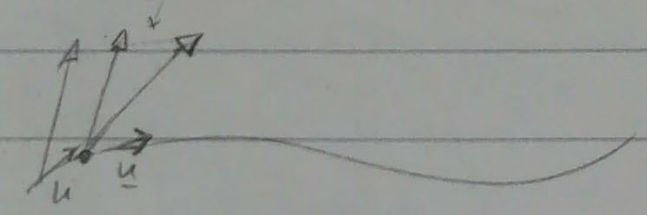


$\nabla_\beta v^\alpha$

$v^\alpha{}_{;\beta} = \partial_\beta v^\alpha + \Gamma^\alpha{}_{\gamma\beta} v^\gamma$, $\nabla_{\underline{u}} \underline{v} = u^\beta v^\alpha{}_{;\beta} \underline{e_\alpha}$

$\nabla_{\underline{e_\beta}} \underline{v}$

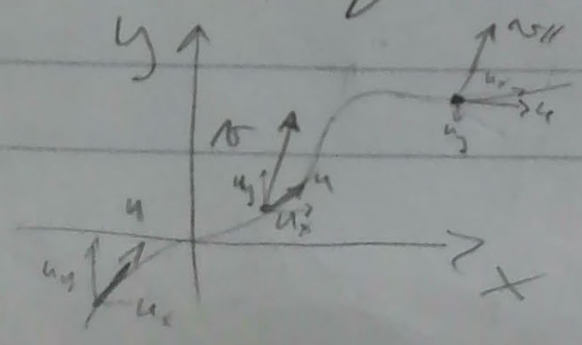
$d\underline{v} \equiv dV = (\nabla V^r) \vec{e}_r$



Geodesic $\nabla_{\underline{u}} \underline{u} = 0$.

cartesian coordinates:

In flat space and Cartesian coordinates the components of a vector $\vec{v}$ do not change under parallel transport $\Rightarrow u^i \partial_i v^i = 0$



$(\vec{u} \cdot \vec{\nabla}) \vec{v} = 0$

$u^\alpha \partial_\alpha v^\beta \rightarrow u^\alpha \nabla_\alpha v^\beta \equiv u^\alpha v^\beta{}_{;\alpha}$

- Distances and time intervals

Time interval: $ds^2 = g_{00}\,dt^2 + \dots$

For 2 events infinitesimally close, $dx^\alpha = 0 \Rightarrow$

$$ds^2 = c^2 d\tau^2 = g_{00}\,dx^0{}^2 \quad \Rightarrow \quad d\tau = \frac{1}{c}\sqrt{g_{00}}\,dx^0$$

- $\tau = \frac{1}{c}\int dx^0 \sqrt{g_{00}}$. $\qquad g_{00} > 0 \qquad (+---)$

Space interval

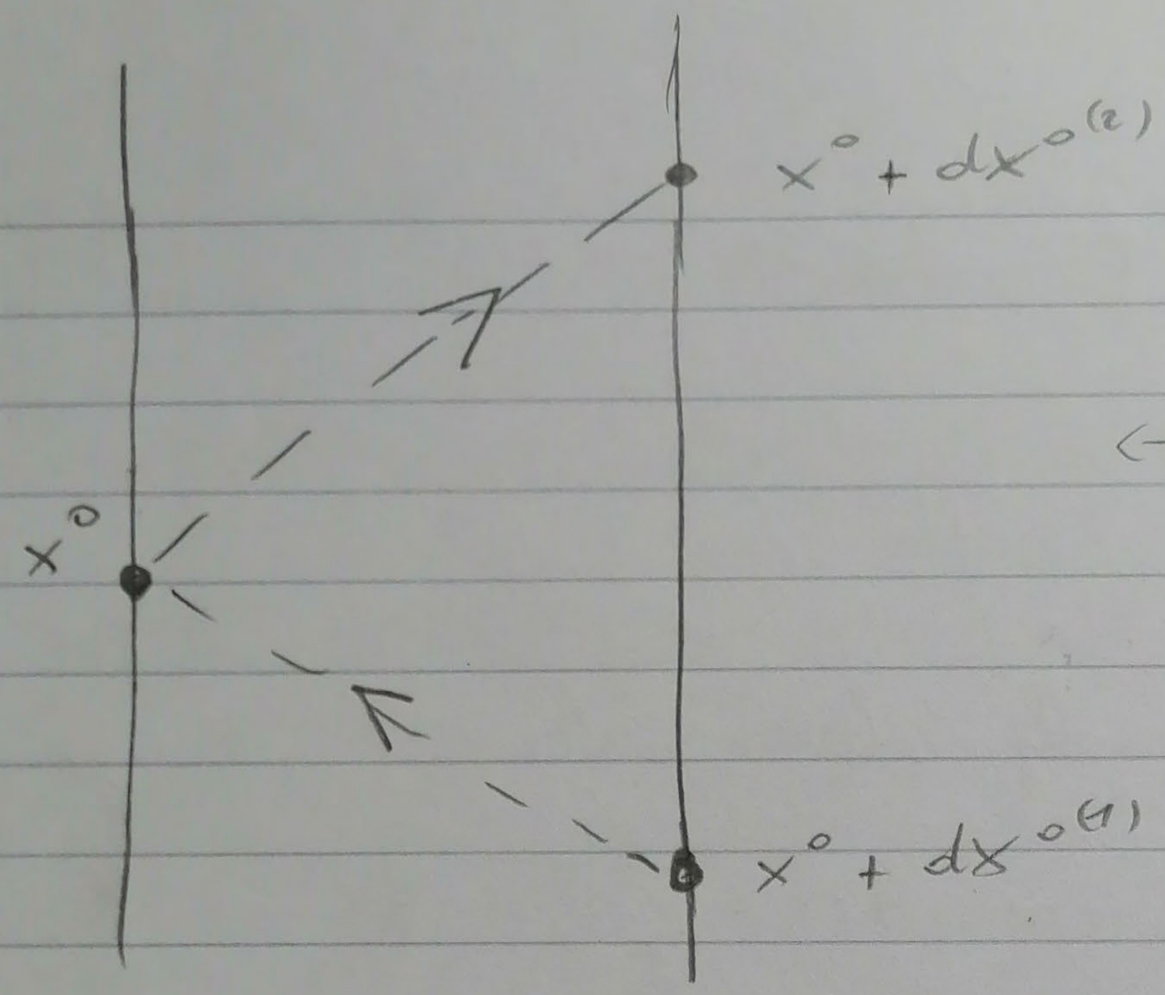
Emission of a photon from B of coordinates $x^\alpha + dx^\alpha$, towards a point A of coordinates x^α and reflected on the same path.

If $ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta + 2g_{0\alpha}\,dx^0\,dx^\alpha + g_{00}(dx^0)^2$

For the light $ds^2 = 0 \Rightarrow$

$$dx^{0(1)} = \frac{1}{g_{00}}\left(-g_{0\alpha}\,dx^\alpha - \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})\,dx^\alpha dx^\beta}\right)$$

$dx^{0(2)} =$ " + "



← Spacetime trajectories

$\Rightarrow$ The total temporal interval is : $x^\circ + dx^{\circ(2)} - (x^\circ + dx^{\circ(1)}) =$

$$= dx^{\circ(2)} - dx^{\circ(1)} = \frac{2}{g_{oo}} \sqrt{(g_{o\alpha} g_{o\beta} - g_{\alpha\beta} g_{oo}) dx^\alpha dx^\beta}$$

The real time is :

$$d\tau = \frac{\sqrt{g_{oo}}}{c} (dx^{\circ(2)} - dx^{\circ(1)})$$

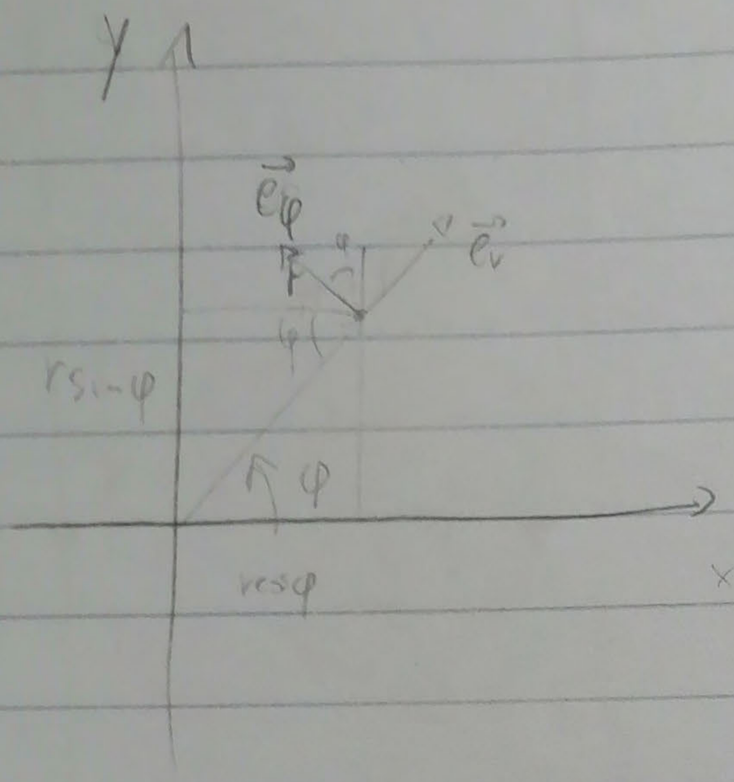
$$\Rightarrow dl^2 = \left(\frac{c}{2}\right)^2 d\tau^2 = \frac{g_{oo}}{2^2} \frac{4}{(g_{oo})^2} (g_{o\alpha} g_{o\beta} - g_{\alpha\beta} g_{oo}) dx^\alpha dx^\beta$$

$$= \left(- g_{\alpha\beta} + \frac{g_{o\alpha} g_{o\beta}}{g_{oo}} \right) dx^\alpha dx^\beta .$$

$$dl^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}}$$

tensore metrico tridimensionale.



$$\vec{e}_r = (\cos\varphi, \sin\varphi)$$

$$\vec{e}_\varphi = (-\sin\varphi, \cos\varphi)$$

$$\vec{r} = (x,y) = r(\cos\varphi, \sin\varphi) = r\hat{r} = r\vec{e}_r$$

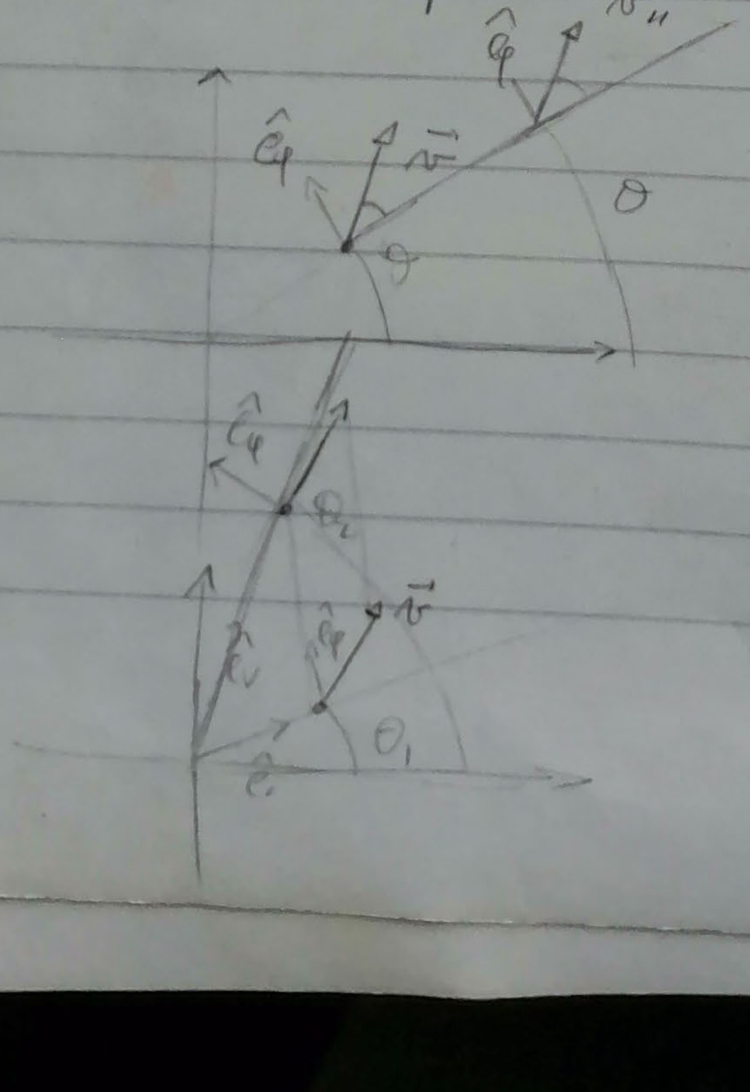
$$\dot{\vec{r}} = (\dot{x},\dot{y}) = \dot{r}(\cos\varphi, \sin\varphi) + r\dot{\varphi}(-\sin\varphi, \cos\varphi)$$

$$= \dot{r}\underbrace{\hat{r}}_{\vec{e}_r} + r\dot{\varphi}\hat{\varphi} = \dot{r}\vec{e}_r + \underbrace{r\dot{\varphi}\vec{e}_\varphi}_{r\dot{\vec{e}}_r} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r$$

$x = r\cos\varphi$
$y = r\sin\varphi$ coordinates' transformation.

$$\frac{\partial^2 x}{\partial x' \partial x'} \neq 0$$

Parallel transport: $\rightarrow \vec{v}_{//}$



$\leftarrow$ some θ if transport is only radial.

Other Christoffel symbols:

$$\delta A^i = - \Gamma^i_{\kappa\ell} A^\kappa \, dx^\ell$$

$$\Gamma_{i,\kappa\ell} = g_{im} \Gamma^m_{\kappa\ell}$$

$$\Gamma^i_{\kappa\ell} = g^{im} \Gamma_{m,\kappa\ell}$$

A scalar does not change under parallel transport, then

given two vectors A_i and B^i,

$$\delta(A_i B^i) = 0 \Rightarrow (\delta A_i) B^i + A_i (\delta B^i) = 0$$

$$B^i \delta A_i = - A_i \delta B^i = - (- \Gamma^k_{i\ell} B^i dx^\ell) A_k$$

$$\underline{B^i} \, \delta A_i = \Gamma^k_{i\ell} A_k \underline{B^i} \, dx^\ell$$

But B^i is arbitrary $\Rightarrow$ $\delta A_i = \Gamma^\kappa_{i\ell} A_\kappa dx^\ell$

Therefore, for a contravariant vector:

$$DA^i = \left(\frac{\partial A^i}{\partial x^\ell} + \Gamma^i_{\kappa\ell} A^\kappa \right) dx^\ell \equiv A^i_{;\ell} \, dx^\ell,$$

and for a covariant vector:

$$DA_i = \left(\frac{\partial A_i}{\partial x^\ell} - \Gamma^\kappa_{i\ell} A_\kappa \right) dx^\ell \equiv A_{i;\ell} \, dx^\ell.$$

Covariant derivative of a tensor,

for example $A^{ik} = A^i B^k$.

The parallel transport reads $\delta(A^i B^k) = A^i(\delta B^k) + (\delta A^i) B^k =$

$$= -\underbrace{A^i}\,\Gamma^k_{\ell m}\,\underbrace{B^\ell}\,dx^m - \underbrace{B^k}\,\Gamma^i_{\ell m}\,\underbrace{A^\ell}\,dx^m .$$

By linearity:

$$\delta A^{ik} = -A^{i\ell}_{\ m}\,\Gamma^k_{m\ell}\,dx^m - A^{mk}_{\ \ \ell}\,\Gamma^i_{m\ell}\,dx^m ,$$

$$\delta A^{ik} = -\left(A^{im}\,\Gamma^k_{m\ell} + \underbrace{A^{mk}\,\Gamma^i_{m\ell}}\right)dx^\ell ,$$

$$DA^{ik} = dA^{ik} - \delta A^{ik} =$$

$$= \underbrace{\left(\frac{\partial A^{ik}}{\partial x^\ell} + \Gamma^i_{m\ell}\,A^{mk} + \Gamma^k_{m\ell}\,A^{im}\right)}\,dx^\ell$$

$$A^{ik}{}_{;\ell}$$

Similarly:

$$A_{ik;\ell} = \frac{\partial A_{ik}}{\partial x^\ell} - \Gamma^m_{i\ell}\,A_{mk} - \Gamma^m_{k\ell}\,A_{im} .$$

(covariant)

- The covariant derivative satisfies the Leibnitz rule.

Es. $(A_i B_k)_{;e} = A_{i;e} B_k + A_i B_{k;e}$

the Christoffel symbols are symmetric.

$A_{i;k}$ is a tensor, but also $A_{i;k} - A_{k;i}$ is a tensor. suppose $A_i = \frac{\partial \varphi}{\partial x^i}$

$$\Rightarrow \frac{\partial A_i}{\partial x^k} = \frac{\partial^2 \varphi}{\partial x^i \partial x^k} = \frac{\partial A_k}{\partial x^i}$$

$$A_{k;i} = \frac{\partial A_k}{\partial x^i} - \Gamma^l_{ik} A_l$$

$$= \frac{\partial^2 \varphi}{\partial x^i \partial x^k} - \Gamma^l_{ik} \frac{\partial \varphi}{\partial x^l}$$

$$A_{k;i} - A_{i;k} = \left(\Gamma^l_{ik} - \Gamma^l_{ki}\right) \frac{\partial \varphi}{\partial x^l}$$

In Galilean coordinates $A_{k;i} \to A_{k,i} \Rightarrow (A_{k;i} - A_{i;k})\big|_P = A_{k,i} - A_{i,k} = 0$

($A_{k,i}$ ← normal derivative.)

$\Rightarrow$ in galilean coordinates $\Rightarrow \Gamma^l_{ik} = \Gamma^l_{ki}$,

but $A_{k;i} - A_{i;k}$ is a tensor then it is zero in any coordinate system $\Rightarrow \Gamma$ is symmetric in any coordinate system.

How Γ transform?

$$\Gamma'^{i}_{\kappa\ell} = \underbrace{\Gamma'^{m}_{np}\,\frac{\partial x'^{i}}{\partial x'^{m}}\,\frac{\partial x'^{n}}{\partial x^{\kappa}}\,\frac{\partial x'^{p}}{\partial x^{\ell}}}_{①} + \underbrace{\frac{\partial^2 x'^{m}}{\partial x^{\kappa}\partial x^{\ell}}\,\frac{\partial x^{i}}{\partial x'^{m}}}_{②} \qquad (*)$$

Is a tensor only under linear transformations.

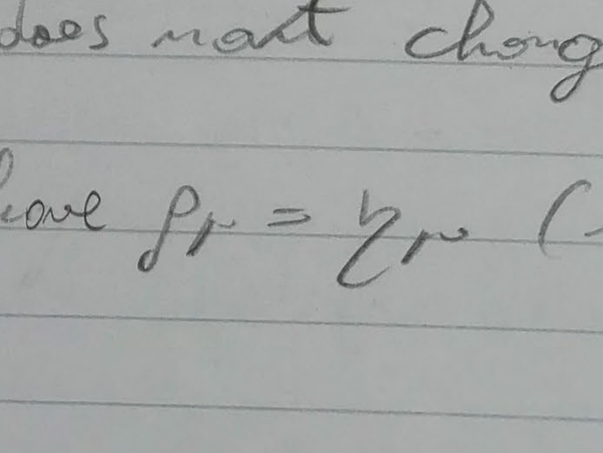
Suppose we consider the following coordinate transf.

$$x'^{i} = \overset{i}{x} + \frac{1}{2}\left(\Gamma^{i}_{\kappa\ell}\right)_0 x^{\kappa} x^{\ell} \qquad \text{(we start with } (\Gamma)_0 \neq 0\text{)}$$

We evaluate the second term ② at the origin:

$$\Rightarrow \left(\frac{\partial^2 x'^{m}}{\partial x^{\kappa}\partial x^{\ell}}\,\frac{\partial x^{i}}{\partial x'^{m}}\right)\Bigg|_0 = \left(\Gamma^{i}_{\kappa\ell}\right)_0$$

origin: $x = 0$ ⇒ $x' = 0$



coordinate origin

We evaluate ① $\Gamma'^{m}_{np}\big|_0 \left(\delta^{i}_{m}\,\delta^{n}_{\kappa}\,\delta^{p}_{\ell} + \dots\right)$

Γ ⇒ $\overset{0}{\Gamma}$ ⇒ $\overset{0}{0}$

$$\Rightarrow \Gamma'^{m}_{np}\Big|_0 \equiv 0$$

From (*) $(\Gamma)_0 = (\Gamma')_0 + (\Gamma)_0 \Rightarrow \underline{(\Gamma')_0 = 0}$

$$\left(\frac{\partial x'^{i}}{\partial x^{\kappa}}\right)\Bigg|_0 = \delta^{i}_{\kappa} \Rightarrow$$ the value of any tensor locally does not change.

In particular we can consistently have $g_{\mu\nu} = \eta_{\mu\nu}$ (locally)

and $\Gamma = 0$ (locally).

[Galilean coordinates: $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$
$g_{ij} = 0$ $i \neq j$]

Tensors Metrics and Christoffel symbols.

We know that $DA_i = g_{ik} DA^k$,

but $A_i = g_{ik} A^k \Rightarrow \underline{DA_i} = D(g_{ik} A^k)$

$DA_i = g_{ik}(DA^k)$

$= (Dg_{ik}) A^k + g_{ik} (DA^k)$

$D(g_{ik} A^k) = g_{ik}(DA^k)$

$(Dg_{ik}) A^k + g_{ik} (\cancel{DA^k}) = g_{ik} (\cancel{DA^k}) \Rightarrow ?$

$\Rightarrow (Dg_{ik}) A^k = 0 \quad \forall A^k \Rightarrow D g_{ik} = 0$

$\Rightarrow \boxed{g_{ik;\ell} = 0}$

$$g_{ik;\ell} = \frac{\partial g_{ik}}{\partial x^\ell} - g_{mk} \Gamma^m_{i\ell} - g_{im} \Gamma^m_{k\ell}$$

$$= \frac{\partial g_{ik}}{\partial x^\ell} - \Gamma_{k,i\ell} - \Gamma_{i,k\ell} = 0$$

After permuting i, k, ℓ we get 3 identities and taking the sum we end up with:

$$\Gamma^i_{k\ell} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^\ell} + \frac{\partial g_{m\ell}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m} \right).$$

Relation between g_{ik} and $\Gamma^i_{m\ell}$.

Geodesic Motion

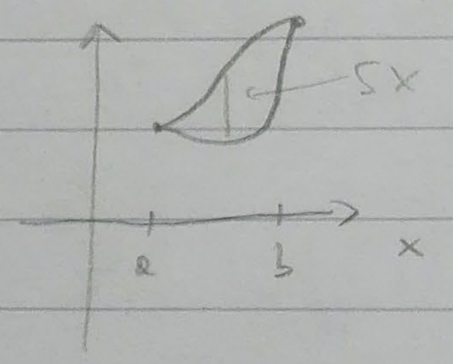
In flat space:

$$\$ = -mc\int ds \qquad \delta\$ = 0 \Rightarrow \frac{du^i}{ds} = 0 \;,\; u^i = \frac{dx^i}{ds}$$

$$\sqrt{\eta_{ij}\,dx^i dx^j} = \sqrt{\eta_{ij}\,\dot{x}^i\dot{x}^j}\,d\lambda$$

$$\frac{d^2x^i}{ds^2} = 0$$

$$\delta\$ = -mc\int \frac{1}{\not{d}s}\cdot \not{d}\,\eta_{ij}\,dx^i\,\delta(dx^j)$$



$$= -mc\int \eta_{ij}\frac{dx^i}{ds}\,\delta(dx^j)$$

$$= -mc\int \eta_{ij}\,u^i\,\underbrace{\delta(dx^j)}_{d(\delta x^j)} = -mc\,\eta_{ij}u^i(\delta x^j)\Big|_a^b + mc\int \eta_{ij}\frac{du^i}{ds}\delta x^j\,ds$$

$$\text{“}0\text{”} \qquad 0$$

$$\forall\,\delta x^j$$

$$\Rightarrow \frac{du^i}{ds} = 0$$

Covariant generalisation:

$$du^i = 0 \longrightarrow Du^i = 0 \qquad du^i + \Gamma^i_{\ell m}\,u^\ell\,dx^m = 0$$

$$\frac{du^i}{ds} + \Gamma^i_{\ell m}\,u^\ell u^m = 0$$

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\ell m}\frac{dx^\ell}{ds}\frac{dx^m}{ds} = 0$$

or: $\$ = -mc\int\sqrt{g_{\mu\nu}(x)\,dx^\mu\,dx^\nu}$

CURVATURE TENSOR

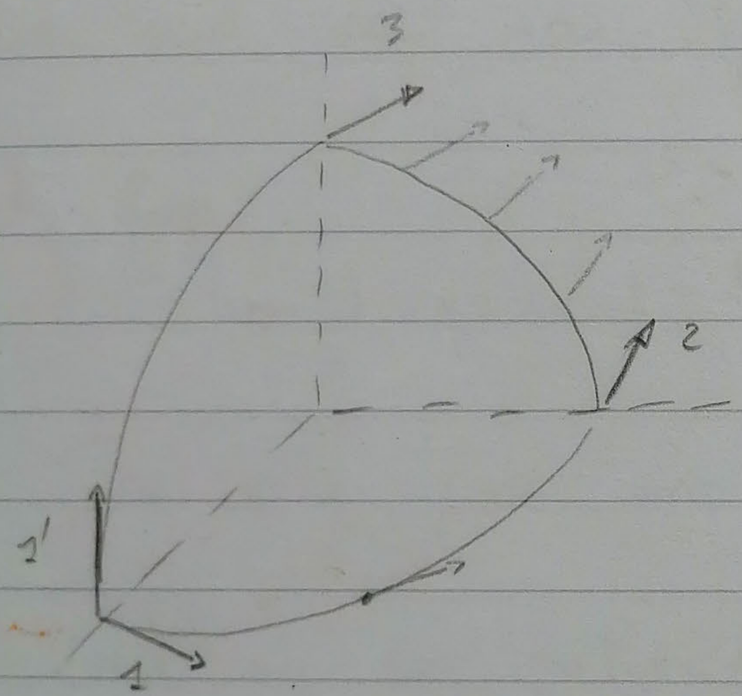
We consider the parallel transport along an infinitesimal close curve.:

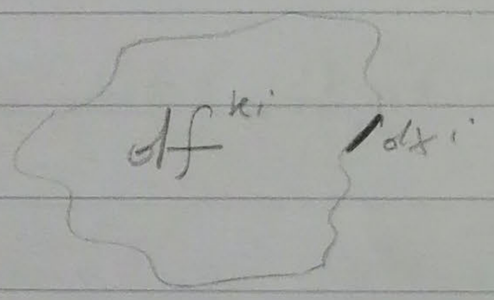
The variation is : $\Delta A_{(k)} = \oint \delta A_k$

$$= \oint \underbrace{\Gamma^i_{ke}\, A_i}_{\parallel V^i_{ke}}\, dx^e$$

$$= \oint V_{ke}\, dx^e$$

$$= \frac{1}{2}\int df^{em}\left(\frac{\partial V_{kem}}{\partial x^e} - \frac{\partial V_{kee}}{\partial x^m}\right)$$





oriented surface

Stokes theorem : $\oint dx^i(\ldots) = \int df^{ki}\, \frac{\partial}{\partial x^k}(\ldots)$

Es. For a Vector : $\oint_\ell A_i\, dx^i = \int df^{ki}\, \frac{\partial A_i}{\partial x^k} = \frac{1}{2}\int df^{ki}\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)$.

V_{kem} V_{kee}

$$\Rightarrow \Delta A_k = \frac{1}{2}\int df^{em}\left(\frac{\partial \Gamma^i_{km} A_i}{\partial x^e} - \frac{\partial \Gamma^i_{ke} A_i}{\partial x^m}\right)$$

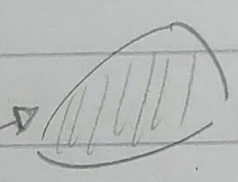
$$\Delta A_k \approx \frac{1}{2}\Delta f^{em}\left[\frac{\partial \Gamma^i_{km}}{\partial x^e} A_i - \frac{\partial \Gamma^i_{ke}}{\partial x^m} A_i + \right.$$

$$\left. + \Gamma^i_{km}\frac{\partial A_i}{\partial x^e} - \Gamma^i_{ke}\frac{\partial A_i}{\partial x^m}\right]$$

$\Gamma \perp DA$, $DA \cdot \Delta f \sim O(\delta^4)$.

$$DA_i = \left(\frac{\partial A_i}{\partial x^e} - \Gamma^m{}_{ie} A_m\right) dx^e$$

$$\frac{\partial A_i}{\partial x^e} \approx \Gamma^m{}_{ie} A_m + \mathcal{O}(\Delta^2).$$



$$\Rightarrow \Delta A_k = \frac{1}{2} R^i{}_{kem} A_i \Delta f^{em}$$

Where $R^i{}_{kem} = \frac{\partial \Gamma^i{}_{km}}{\partial x^e} - \frac{\partial \Gamma^i{}_{ke}}{\partial x^m} + \Gamma^i{}_{ne} \Gamma^n{}_{km} - \Gamma^i{}_{nm} \Gamma^n{}_{ke}$

Riemann Tensor

A scalar is invariant under parallel transport $\Rightarrow$

$$\Delta(A^k B_k) = 0$$

$$= A^k(\Delta B_k) + B_k(\Delta A^k)$$

$$= \frac{1}{2} A^k B_i R^i{}_{kem} \Delta f^{em} + B_k(\Delta A^k)$$

contravariant

$$= B_k\left(\Delta A^k + \frac{1}{2} A^i R^k{}_{iem} \Delta f^{em}\right) = 0$$

$$\Rightarrow \Delta A^k = -\frac{1}{2} R^k{}_{iem} A^i \Delta f^{em}.$$

Commutator of covariant derivatives

$$A_{i;k;e} - A_{i;e;k} = A_m R^m{}_{ike}.$$

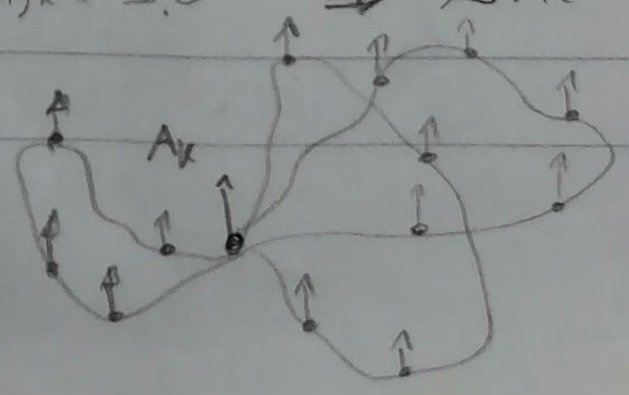
Theorem: $R^i{}_{kem} = 0 \iff$ the spacetime is flat

$\Leftarrow$: If the space is flat $\Rightarrow$ we can find a coordinate system such that $\Gamma^i{}_{em} = 0$ ($\Rightarrow \partial\Gamma = 0$) everywhere $\forall \Rightarrow R^i{}_{kem}\big|_{\text{Galilean coordinates}} = 0$, but $R^i{}_{kem}$ is a tensor $\Rightarrow R^{i'}{}_{kem} = 0$ in any coordinate system. In a flat space the parallel transport is unique and does not depend on the path along a closed loop.

$\Rightarrow$: In any space we can consider a Galilean coordinates system in an infinitesimal region around a point.

If $R^i{}_{kem} = 0 \Rightarrow$ the parallel transport is uniquely (does not depend on the path) coordinate defined and we can transport the Galilean system in all other regions to finally have a Galilean coordinates system everywhere.

$R_{ijke} = 0 \Rightarrow \Delta A_k = 0$



Properties

$$R_{ikem} = g_{im} R^{m}{}_{kem}$$

$$R_{ikem} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^e} + \frac{\partial^2 g_{ke}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{ie}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^e}\right)$$

$$+ g_{mp}\left(\Gamma^{m}_{ke}\Gamma^{p}_{im} - \Gamma^{m}_{km}\Gamma^{p}_{ie}\right)$$

$$R_{ikem} = -R_{kiem} = -R_{ikme}$$

$$R_{(ik)em} = R_{em(ik)}$$

Cyclic sum : $R_{ikem} + R_{imke} + R_{iemk} = 0$.

Bianchi identities :

$$R^{m}_{ike;m} + R^{m}_{imk;e} + R^{m}_{iem;k} = 0 .$$

This is easy to prove in a coordinate system locally geodetic where $\Gamma^{i}_{ke}\Big|_0 = 0$. R^{i}_{kem} is a tensor $\Rightarrow$ the identity is true in any coordinates system.

- Ricci tensor: $R_{ik} = g^{\ell m} R_{\ell imk} = R^{\ell}{}_{i\ell k}$

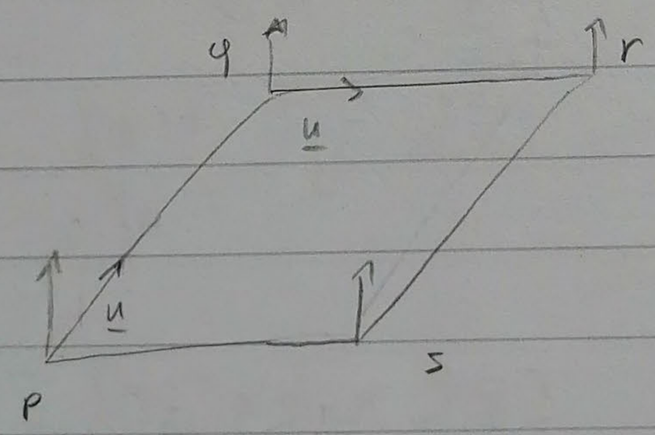
$$R_{ik} = \frac{\partial \Gamma^{\ell}_{ik}}{\partial x^{\ell}} - \frac{\partial \Gamma^{\ell}_{i\ell}}{\partial x^{k}} + \Gamma^{\ell}_{ik}\Gamma^{m}_{\ell m} - \Gamma^{m}_{i\ell}\Gamma^{\ell}_{km}.$$

$$R_{ik} = R_{ki}$$

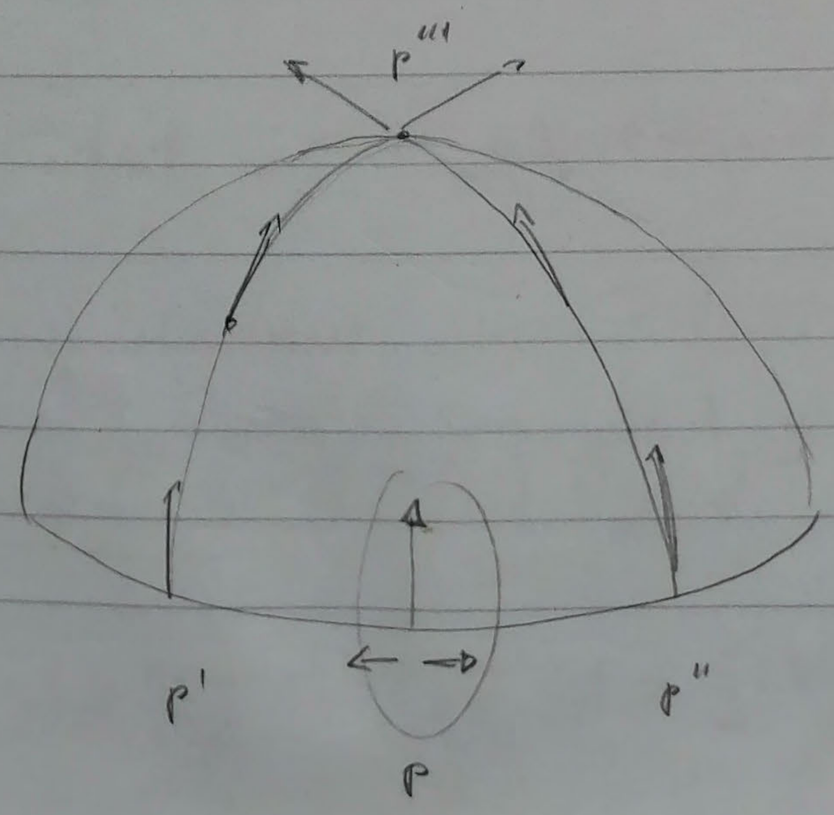
- Ricci scalar: $R = g^{ik} R_{ik}$.

PARALLEL TRANSPORT

FLAT SPACE



CURVED SPACE



$x(\lambda)$

$$\frac{dT}{d\lambda} = 0$$ constant vector in flat space.

$$\frac{dT}{d\lambda} = \frac{dx^{r}}{d\lambda}\frac{\partial T}{\partial x^{r}} = 0$$

$\frac{dx}{d\lambda} = u^{r}$

↓ generalization

$$\frac{DT}{d\lambda} = 0$$

$$\frac{dx^{r}}{d\lambda} \nabla_{r} T$$

u^{r}

Infinitesimal coordinates transformation.

$$x'^{i} = x^{i} + \xi^{i},$$

$$\rho'^{ik}(x'^{\ell}) = \rho^{\ell m}(x^{\ell}) \frac{\partial x'^{i}}{\partial x^{\ell}} \cdot \frac{\partial x'^{k}}{\partial x^{m}} =$$

$$\rho'^{ik}(x^{\ell}) + \xi^{\ell} \frac{\partial \rho'^{ik}}{\partial x^{\ell}} + o(\xi^{2}) = \rho^{\ell m}(x^{\ell}) \left(\delta^{i}_{\ell} + \frac{\partial \xi^{i}}{\partial x^{\ell}}\right)\left(\delta^{k}_{m} + \frac{\partial \xi^{k}}{\partial x^{m}}\right)$$

$$\simeq \rho^{ik}(x^{\ell}) + \rho^{k\ell}(x^{\ell}) \frac{\partial \xi^{i}}{\partial x^{\ell}} + \rho^{im}(x^{\ell}) \frac{\partial \xi^{k}}{\partial x^{m}}$$

$$\rho'^{ik}(x^{\ell}) = \rho^{ik}(x^{\ell}) - \xi^{\ell} \frac{\partial \rho^{ik}}{\partial x^{\ell}} + \rho^{k\ell}(x^{\ell}) \frac{\partial \xi^{i}}{\partial x^{\ell}} + \rho^{i\ell}(x^{\ell}) \frac{\partial \xi^{k}}{\partial x^{m}}.$$

some "x".

or:

$$\rho'^{ik} = \rho^{ik} + \delta\rho^{ik}, \quad \delta\rho^{ik} = \xi^{i;k} + \xi^{k;i}. \text{ Homework.}$$

$$\rho'_{ik} = \rho_{ik} + \delta\rho_{ik}, \quad \delta\rho_{ik} = -\xi_{i;k} - \xi_{k;i}.$$

Notice

Killing eq. $\quad \xi_{i;k} + \xi_{k;i} = 0 \Rightarrow \rho'_{ik}(x) = \rho_{ik}(x)$

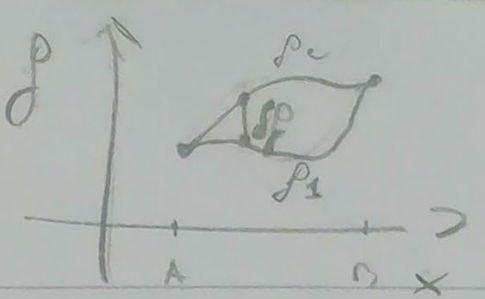
ISOMETRY

$$R^{2}(d\theta'^{2} + \sin^{2}\theta' \, d\varphi'^{2}) = ds^{2}_{sh} \qquad \rho'_{ob}(x') = \begin{pmatrix} 1 & \\ & \sin^{2}\theta' \end{pmatrix}$$

$$\theta' \to \theta \quad \rho'_{ob}(x) = \begin{pmatrix} 1 & \\ & \sin^{2}\theta \end{pmatrix} = \rho_{ob}(x)$$

$$\delta(\partial f) = f_2'(x) - f_1'(x) = \frac{\partial}{\partial x}(f_2 - f_1) = \partial(\delta f)$$

$$\mathcal{L}_m(f_{1\sim}, \partial_e f_{1\sim})$$



$$\delta f = f_2(x) - f_1(x)$$

$$\frac{\partial}{\partial x} f_2(x) - \frac{\partial f_1(x)}{\partial x} = \frac{\partial}{\partial x}(f_2 - f_1)$$

$$\delta\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\delta f$$

$$\$_{matter} = \frac{1}{c}\int d^4x \sqrt{-g}\, \mathcal{L}_m$$

$$\delta\$ = \frac{1}{c}\int\left\{ \frac{\partial(\sqrt{-g}\,\mathcal{L}_m)}{\partial g^{ik}}\delta g^{ik} + \frac{\partial(\sqrt{-g}\,\mathcal{L}_m)}{\partial\left(\frac{\partial g^{ik}}{\partial x^e}\right)}\delta\left(\frac{\partial g^{ik}}{\partial x^e}\right)\right\} d\Omega$$

$$= \frac{1}{c}\int\left\{ \frac{\partial(\sqrt{-g}\,\mathcal{L}_m)}{\partial g^{ik}} - \frac{\partial}{\partial x^e}\frac{\partial(\sqrt{-g}\,\mathcal{L}_m)}{\partial\left(\frac{\partial g^{ik}}{\partial x^e}\right)}\right\}\delta g^{ik}\, d\Omega$$

d^4x

$$\text{Def:}\quad T_{ik} = \frac{2}{\sqrt{-g}}\left\{\frac{\partial(\sqrt{-g}\,\mathcal{L}_m)}{\partial g^{ik}} - \frac{\partial}{\partial x^e}\frac{\partial(\sqrt{-g}\,\mathcal{L}_m)}{\partial\left(\frac{\partial g^{ik}}{\partial x^e}\right)}\right\}$$

$$\Rightarrow \delta\$ = \frac{1}{2c}\int T_{ik}\,\delta g^{ik}\sqrt{-g}\, d\Omega$$

$$= -\frac{1}{2c}\int T^{ik}\,\delta g_{ik}\sqrt{-g}\, d\Omega$$

$$g_{ik}\,g^{ke} = \delta_i^e$$

$$(\delta g_{ik})\,g^{ke} + g_{ik}(\delta g^{ke}) = 0$$

Replacing δg_{ik} for small coordinate transformation in $\delta\$$ we get : (symmetric tensor $T_{ik} = T_{ki}$)

$$\delta\$ = \frac{1}{2c}\int T_{ik}\left(\xi^{i;k} + \xi^{k;i}\right)\sqrt{-g}\, d\Omega$$

$$= \frac{1}{c}\int T_{ik}\,\xi^{i;k}\,\sqrt{-g}\, d\Omega$$

$$\delta S = \frac{1}{c}\int \left(\left(T^{\kappa}{}_{i}\,\xi^{i}\right)_{;\kappa}\sqrt{-g}\,d\Omega - T^{\kappa}{}_{i;\kappa}\,\xi^{i}\sqrt{-g}\,d\Omega \right)$$

$g_{;i} = 0 \Leftrightarrow \nabla_i g_{\kappa i} = 0$

$$= \frac{1}{c}\int \frac{\partial}{\partial x^{\kappa}}\left(\sqrt{-g}\,T^{\kappa}{}_{i}\,\xi^{i}\right) d\Omega - \int T^{\kappa}{}_{i;\kappa}\,\xi^{i}\sqrt{-g}\,d\Omega$$

$\downarrow$ $\qquad\qquad$ $\downarrow$

0 $\qquad\qquad$ d^4x

$\delta S = 0$ because the action is invariant under coordinate transformation $\Rightarrow \forall \xi^i : \boxed{T^{\kappa}{}_{i;\kappa} = 0}$

$\uparrow$

$\partial_\kappa T^{\kappa}{}_{i} = 0$ in Galilean coordinates.

general infinitesimal coordinate transformation.

- Einstein Equation

$$S_{EH} = -2\kappa_4^{-2} \int d^4x \sqrt{-g}\, R$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \Rightarrow \delta S_{EH} = -2\kappa_4^{-2} \int d^4x \left[(\delta\sqrt{-g})\, R + \sqrt{-g}\, \delta R \right]$$

- We evaluate $\delta(\sqrt{-g})$:

Jacobi's Formula: $\boxed{\frac{\partial (\det A)}{\partial A_{ij}} = (\det A)\,(A^{-1})_{ji}}$

$$\Rightarrow dg = g\, g^{ik}\, dg_{ik} = -g\, g_{ik}\, dg^{ik}.$$

(g_{ik}^{-1})

$$g_{ik} g^{kj} = \delta_i^j \Rightarrow dg_{ik}\, g^{kj} = -g_{ik}\, dg^{kj}$$

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-)\,\delta g = \frac{-1}{2\sqrt{-g}}(-g\, g_{ik}\, \delta g^{ik})$$

$$\frac{-g}{\sqrt{-g}} \cdot \frac{\sqrt{-g}}{\sqrt{-g}} = \sqrt{-g}$$

$$= -\frac{1}{2}\sqrt{-g}\, g_{ik}\, \delta g^{ik}.$$

$$\delta R = \delta(g^{ik} R_{ik}) = \delta g^{ik} R_{ik} + g^{ik} \delta R_{ik}$$

- $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$

$R_{\sigma\nu} = \partial_\mu \Gamma^{\mu}_{\nu\sigma} - \partial_\nu \Gamma^{\mu}_{\mu\sigma} + \dots$

$$\delta R^{\rho}_{\sigma\mu\nu} = \partial_\mu \delta\Gamma^{\rho}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\rho}_{\mu\sigma} +$$

$$+ \delta\Gamma^{\rho}_{\mu\lambda}\cdot\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\cdot\delta\Gamma^{\lambda}_{\nu\sigma}$$

$$- \delta\Gamma^{\rho}_{\nu\lambda}\cdot\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\nu\lambda}\cdot\delta\Gamma^{\lambda}_{\mu\sigma}$$

$\delta\Gamma^{\rho}_{\nu\sigma}$ is the difference of two connection then it is a tensor (see 2 pages later).

We observe that the variation of the Riemann tensor can be simplified to:

$$\delta R^{\rho}_{\sigma\mu\nu} = \nabla_\mu(\delta\Gamma^{\rho}_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^{\rho}_{\mu\sigma})$$

$$\delta R_{\mu\nu} = \delta R^{\rho}_{\mu\rho\nu} = \nabla_\rho(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\rho\mu})$$

Therefore: $\delta R = R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\left[\nabla_\rho(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\rho\mu})\right]$

$$= R_{\mu\nu}\,\delta g^{\mu\nu} + \nabla_\rho\left(g^{\mu\nu}\,\delta\Gamma^{\rho}_{\nu\mu}\right) +$$

$$- \nabla_\nu\left(g^{\mu\nu}\,\delta\Gamma^{\rho}_{\rho\mu}\right)$$

$$= R_{\mu\nu}\,\delta g^{\mu\nu} + \nabla_\sigma\underbrace{\left(g^{\mu\nu}\,\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\,\delta\Gamma^{\rho}_{\rho\mu}\right)}_{\text{Vector}}$$

Note that:

$$\int d^4x\,\sqrt{-g}\,\nabla_\mu V^\mu = \int d^4x\,\nabla_\mu\left(\sqrt{-g}\,V^\mu\right) \quad (\text{because } \nabla_\mu g_{\nu\rho} = 0)$$

↑ compatibility

Now: $\Gamma^i_{ki} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^m} \right)$

$= \frac{1}{2} g^{im} \frac{\partial g_{mi}}{\partial x^k}$,

but $dg = g\, g^{ik}\, dg_{ik}$

$\Rightarrow \Gamma^i_{ki} = \frac{1}{2} \frac{1}{g} \cdot g \cdot g^{im} \frac{\partial g_{im}}{\partial x^k}$

$= \frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^k} = \frac{\partial (\ln \sqrt{-g})}{\partial x^k}$. $= \frac{1}{2} \left(\frac{1}{-g} \right) \cdot \frac{\partial (-g)}{\partial x^k}$

we evaluate

$A^i{}_{;i} \equiv \nabla_i A^i = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{ei} A^e$

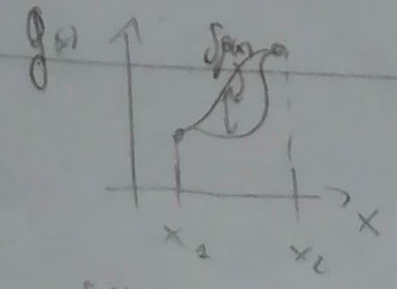
$= \frac{\partial A^i}{\partial x^i} + \frac{\partial (\ln \sqrt{-g})}{\partial x^i_e} \cdot A^e$.

$= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g}\, A^i \right)$

$\Rightarrow \int d^4x\, \nabla_\mu (\sqrt{-g}\, V^\mu) = \int d^4x\, \sqrt{-g}\, \nabla_\mu V^\mu$

$= \int d^4x\, \sqrt{-g}\, \frac{1}{\sqrt{-g}}\, \partial_\mu (\sqrt{-g}\, V^\mu)$

$= \int d^4x\, \partial_\mu (V^\mu \sqrt{-g})$



Footnote $\int \Gamma$ is a tensor. Coordinate changing $x \to y$.

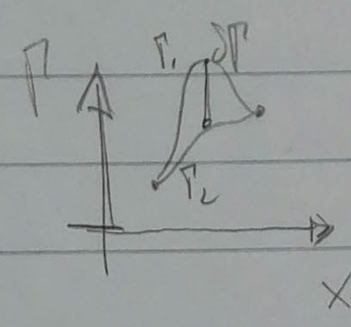
$$\frac{\partial y^\tau}{\partial x^\lambda} \Gamma^{(x)\lambda}_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu} \frac{\partial y^\sigma}{\partial x^\nu} \Gamma^{(y)\tau}_{\rho\sigma} + \frac{\partial^2 y^\tau}{\partial x^\mu \partial x^\nu}$$

$$\Delta\Gamma^\lambda_{\mu\nu}(x) = \Gamma^{(1)\lambda}_{\mu\nu}(x) - \Gamma^{(2)\lambda}_{\mu\nu}(x)$$

↳ transform as:

$$\frac{\partial y^\tau}{\partial x^\lambda}\left(\Delta\Gamma^{(x)\lambda}_{\mu\nu}\right) = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}\left(\Delta\Gamma^{(y)\tau}_{\rho\sigma}\right)$$

The second term cancels because it is independent on Γ, but only depends on the coordinate transformation.



- Full variation

$$\delta S_{EH} = -2\kappa_4^{-2} \int d^4x \left[-\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}\, R \;+ \right.$$

$$\left. + \sqrt{-g}(\delta g^{\mu\nu}) R_{\mu\nu} \;+ \sqrt{-g} g^{\mu\nu}\, \delta R_{\mu\nu} \right]$$

$$= -2\kappa_4^{-2} \int_{\partial M(4)} d^4x \left\{ \sqrt{-g} \left[\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} \right] + \right.$$

$$\left. + \sqrt{-g}\, \nabla_\sigma V^\sigma \right\}$$

$\Downarrow$

0 because $\delta g|_{\partial M(4)} = 0$

$$\Rightarrow -\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \qquad \text{Vacuum EOM.}$$

- Including matter

$$S' = -2\kappa_4^{-2} \int d^4x \sqrt{-g}\, R + \int d^4x \sqrt{-g}\, \mathcal{L}_m$$

$$\delta S' = -2\kappa_4^{-2} \int d^4x \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu} + \frac{1}{2} \int d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, \mathcal{L}_m)}{\delta g^{\mu\nu}}$$

$$\Rightarrow \quad -2\kappa_4^{-2} G_{\mu\nu} = -\frac{1}{2} T_{\mu\nu}$$

$$G_{\mu\nu} = \frac{\kappa_4^2}{4} T_{\mu\nu} \qquad \kappa_4 = 32\pi G_N \quad (c=1)$$

$$\boxed{G_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$ Einstein EOM.

Expression for a point particle again

$x^0 = ct$

$g_{00} \frac{dx^0}{dt} \frac{dx^0}{dt} = c^2$

If flat spacetime: $S = -mc \int ds = -mc \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$

$\simeq -mc \int \sqrt{c^2 - \frac{dx}{dt} \frac{dx}{dt}}\, dt$

$= -mc^2 \int \left(1 - \frac{1}{2} \frac{v^2}{c^2} + \dots\right) dt$

$= -mc \int \left(c - \frac{1}{2} \frac{v^2}{c} + \dots\right) dt$

In presence of gravity:

$S = -mc \int \left(c - \frac{v^2}{2c} + \frac{\varphi}{c}\right) dt$ (A)

(φ: gravitational potential; $\rightsquigarrow \frac{1}{2} m \dot{x}^2 - m\varphi$)

but $S = -mc \int \underbrace{\sqrt{g_{\mu\nu}(x)\, dx^\mu dx^\nu}}_{ds}$ (B) $\Rightarrow$ coupling (A) and (B):

$ds \simeq \left(c - \frac{v^2}{2c} + \frac{\varphi}{c}\right) dt$

$\Rightarrow ds^2 \simeq \left(c^2 + 2\varphi - 2 \cdot \frac{v^2}{2c} \cdot c + O\left(\frac{1}{c^2}\right)\right) dt^2$

$= \left(c^2 + 2\varphi\right) dt^2 - \left(\frac{d\vec{r}}{dt}\right)^2 dt^2$

$= \left(c^2 + 2\varphi\right) dt^2 - d\vec{r}^2 = \left(1 + \frac{2\varphi}{c^2}\right) dx^{0\,2} - d\vec{r}^2$

$\Rightarrow \boxed{g_{00} = 1 + \frac{2\varphi}{c^2}}$

$g_{\mu\nu} = c^2 \left(1 + \frac{2\varphi}{c^2}\right) dt^2 - d\vec{x}^2$

$= (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$

$\Rightarrow h_{\mu\nu} = \begin{pmatrix} \frac{2\varphi}{c^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Let's consider $T^{\kappa}_{\ i} = \mu c^2 u_i u^{\kappa}$

↑ energy density or mass density

system of particles with $|\vec{u}| \ll c$ and interacting only gravitationally.

$u^i = (1, 0)$, $\vec{u} = 0$ (because $|\vec{u}| \ll c$) $\Rightarrow T^0_{\ 0} \neq 0$

$T^0_{\ 0} = \rho c^2$

$u^\alpha u_\alpha = 1$

$u^0 u_0 + u^i u_i = 1$

$\Downarrow$ $\quad\quad\quad

(B) $\Delta\varphi = 4\pi G \rho$ (μ) LAPLACE EQUATION

$\Rightarrow$ (A) (↓) (B) $\Rightarrow$ $R^0_0 = \frac{4\pi G}{c^2}\rho_{=\mu}$

$$\varphi = -G_N \int \mu \frac{dV}{R}$$

Therefore, the constant in $G_N = \frac{8\pi G_N}{c^4}$ was the correct one.

Energy tensor of a macroscopic body

$$T^i_k = \begin{pmatrix} \epsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$ in Galilean coordinates.

In a general coordinates system:

$$T^i_k = (p+\epsilon)\, u^i u_k - p\,\delta^i_k$$ | perfect fluid

→ for dust $T^i_k = \epsilon u_i u^k$ because $p = 0$

→ for radiation $T^i_i = 0 \Rightarrow p = \frac{1}{3}\epsilon \Rightarrow T^{ik} = \frac{4}{3}\epsilon u^i u^k - \frac{1}{3}\epsilon g^{ik}$

$p = \epsilon = \rho \leftarrow$ Def.

$$T^1_1 = (p+\epsilon)\, u^1 u_1 - p\,\delta^1_1$$

$$T^0_0 = (p+\epsilon)\, u^0 u_0 - p$$

$$u^0 u_0 + \underbrace{u^2 u_2}_{0} = 1 \Rightarrow u^0 u_0 = 1.$$

$$g^{ij} T_{ij} = \frac{4}{3}\epsilon \underbrace{u^i u_i}_{1} - \frac{1}{3}\epsilon \cdot 4 = 0$$

Schwarzschild metric

coordinates (r, θ, φ, t) $(1, 2, 3, 4)$

$g_{\mu\nu}$ has 10 independent components, but we assume spherical symmetry.

1) A spherically symmetric metric $\Rightarrow$ invariance under rotations.

2) A static spacetime metric $\Rightarrow$ $g_{\mu\nu}(\cancel{x})$ is independent on the coordinate t, $\frac{\partial g_{\mu\nu}}{\partial t} = 0$. Therefore, $ds^2(t) = ds^2(-t)$.

3) A vacuum solution is one such that $T_{ab} = 0$,

$\Rightarrow R_{ab} = 0$.

4) Signature $(+\,+\,+\,-)$

First <u>simplification</u>. Under the coordinate transformation

$(r, \theta, \varphi, t) \to (r, \theta, \varphi, -t)$ we know:

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = -g_{\mu\nu} \quad (\text{for } \mu \neq \nu),$$

$\mu = 1, 2, 3$; $\frac{\partial x^\alpha}{\partial x'^\mu}$ $\mu \neq \nu$ $(-\delta^\beta_\nu)$; $g_{\mu\nu}$ $\mu \neq \nu$ $(1, 2, 3)$

$r = r'$, $\theta = \theta'$, $\varphi = \varphi'$, $t = -t'$.

$t = -t'$

The metric does not depend on t, then for $t \to -t$ can not change,

$\Rightarrow g_{\mu 4} = 0$.

but $g'_{\mu 4} = g_{\mu 4}$ because the metric does not depend on t

$\Rightarrow g_{\mu 4} = 0 \quad (\mu \neq 4)$.

Similarly, under $(r, \theta, \varphi, t) \to (r, \theta, -\varphi, t)$

or

$\to (r, -\theta, \varphi, t)$

the metric does not change, then

$g_{\mu 3} = 0, \mu \neq 3,$

$g_{\mu 2} = 0, \mu \neq 2.$

Finally $g_{\mu\nu} = 0$ for $\mu \neq \nu$, and

$$ds^2 = g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\varphi^2 + g_{44} dt^2.$$

We define $g_{11} = A(r)$, $g_{44} = B(r)$.

On the hypersurface of constant t and r the metric is

$$dl^2 = r_0^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$\Rightarrow$ comparing $\rho_{22} = r^2$ and $\rho_{33} = r^2 \sin^2\theta$.

Thus the metric reads,

$$ds^2 = A(r)\,dr^2 + B(r)\,dt^2 + r^2(\sin^2\theta\,d\varphi^2 + d\theta^2).$$

We now calculate $\Gamma^\mu_{\nu\kappa}$ from $\rho_{\mu\nu}$.

$$\Gamma^1_{ik} = \begin{pmatrix} \frac{A'}{2A} & & & 0 \\ & -\frac{r}{A} & -r\frac{\sin^2\theta}{A} & \\ & 0 & & -\frac{B'}{2A} \end{pmatrix}$$

$$\Gamma^2_{ik} = \begin{pmatrix} 0 & \frac{1}{r} & & 0 \\ \frac{1}{r} & 0 & & \\ & & -\sin\theta\cos\theta & \\ 0 & & & 0 \end{pmatrix}$$

$$\Gamma^3_{ik} = \begin{pmatrix} 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & \cot\theta & \\ \frac{1}{r} & \cot\theta & 0 & 0 \\ 0 & & 0 & \end{pmatrix}$$

$$\Gamma^4_{ik} = \begin{pmatrix} 0 & 0 & 0 & \frac{B'}{2B} \\ 0 & & & \\ 0 & & 0 & \\ \frac{B'}{2B} & & & \end{pmatrix}$$

In vacuum $R_{\alpha\beta} = 0$,

then

$$\Gamma^{\varepsilon}_{\beta\nu,\varepsilon} - \Gamma^{\varepsilon}_{\varepsilon\nu,\beta} + \Gamma^{\varepsilon}_{\varepsilon\lambda}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\varepsilon}_{\beta\lambda}\Gamma^{\lambda}_{\varepsilon\nu} = 0$$

Only 4 eq. are non trivial:

1) $4A'B^2 - 2rB''AB + rA'B'B + rB'^2A = 0$

2) $rA'B + 2A^2B - 2AB - rB'A = 0$

3) $-2rB''AB + rA'B'B + rB'^2A - 4B'AB = 0$

4) $(\times \sin^2\theta)$.

① − ② $\Rightarrow$ $A'B + AB' = 0$ for $A, B \neq 0$

$\frac{d(AB)}{dr} = 0$

$\Rightarrow$ $AB = \text{const} = K$ $\quad A'B = -AB'$

We replace $AB = K$ in ②

$(-A'B) = AB'$, $K = AB$

$r(A'B - AB') + 2AB(A-1) = 0$

$r(2A'B) + 2K(A-1) = 0$

$2A'B = \frac{2K}{A}$

$rA' + A(A-1) = 0$

$r\frac{dA}{dr} = -A(A-1)$, $\frac{dA}{A(A-1)} = -\frac{dr}{r} = -\ln r$

$-\ln\left(\frac{A}{1-A}\right) = -\ln r \Rightarrow \frac{A}{1-A} = r$

C

$$A = \frac{r}{r+c} = \frac{1}{1+\frac{c}{r}}.$$

solution :

$$A = \left(1 + \frac{c}{r}\right)^{-1} \qquad A' = -\left(1+\frac{c}{r}\right)^{-2} \cdot \left(-\frac{c}{r^2}\right)$$

$$\left(\frac{c}{r^2}\right)\left(1+\frac{c}{r}\right)^{-2} = \left(1+\frac{c}{r}\right)^{-1}\left(1 - \frac{1}{1+\frac{c}{r}}\right)$$

$$\frac{c}{r}\frac{1}{\left(1+\frac{c}{r}\right)^2} = \frac{1}{\left(1+\frac{c}{r}\right)} \qquad \frac{\cancel{1+\frac{c}{r}} - \cancel{1}}{\left(1+\frac{c}{r}\right)}$$

OK

$$\Rightarrow ds^2 = \left(1+\frac{c}{r}\right)^{-1} dr + r^2 d\Omega^{(2)} + k\left(1+\frac{c}{r}\right) dt^2$$

B
||

$$g_{44} \to -\left(1+\frac{2\varphi}{c^2}\right) = -\left(1 - \frac{2GM}{rc^2}\right) = k\left(1+\frac{c}{r}\right)$$

$$\Rightarrow k = -1 \quad \text{and} \quad c = -\frac{2MG}{c^2}$$

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \frac{dr^2}{1 - \frac{2MG}{c^2 r}} + r^2 d\Omega^{(2)}.$$

- Large distance limit

$$+g_{00} = 1 - \frac{2M}{r} \simeq 1 + 2V(\vec{x}) \qquad V(\vec{x}) = -\frac{M}{r}$$

Singularities

The metric is singular in $r=0$ and $r=2M$.

- Let us start considering the singularity in $r=2M$.

Does the spacetime metric exist for $r \leq 2M$?

Let us consider the new coordinates:

$$c\cdot\tau = \pm ct \pm \int \frac{f(r)\,dr}{1-\frac{r_g}{r}} \quad , \quad R = ct + \int \frac{dr}{\left(1-\frac{r_g}{r}\right) f(r)}$$

- $\Rightarrow ds^2 = \frac{1-\frac{r_g}{r}}{1-f^2}\left(c^2 d\tau^2 - f^2 dR^2\right) - r^2 d\Omega^{(2)}$ (Homework)

We can take $f(r) = \sqrt{\frac{r_g}{r}} \Rightarrow R - c\tau = \int \frac{(1-f^2)\,dr}{\left(1-\frac{r_g}{r}\right)\cdot f} =$

$$= \int \sqrt{\frac{r}{r_g}}\,dr = \frac{2}{3}\frac{r^{3/2}}{r_g^{1/2}}$$

or $r = \left[\frac{3}{2}(R-c\tau)\right]^{2/3} r_g^{1/3}$, $\Rightarrow$

- $ds^2 = c^2 d\tau^2 - \frac{dR^2}{\left[\frac{3}{2}(R-c\tau)\right]^{2/3} r_g^{2/3}} - \left[\frac{3}{2}(R-c\tau)\right]^{4/3} r_g^{2/3}\, d\Omega^{(2)}$

In the new coordinates $v = v_p$ corresponds to

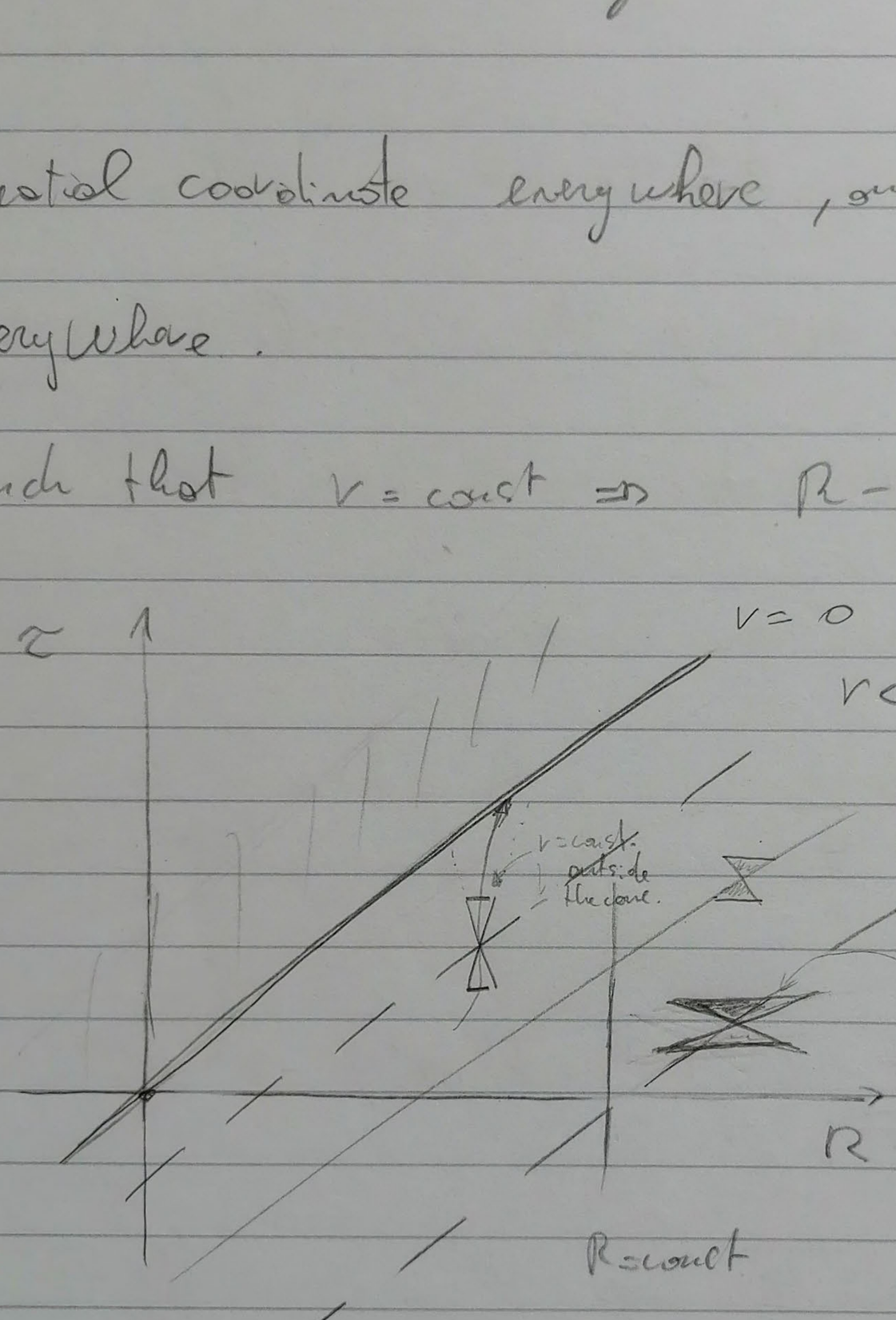
$\frac{1}{2}(R - cz) = v_p$, therefore, the metric is regular in

the new coordinate for $v = v_p$.

R is a spatial coordinate everywhere, and z is a time

coordinate everywhere.

Particles such that $v = const \Rightarrow R - cz = const$



Light propagation $ds^2 = 0$ $(\theta = \varphi = const)$

$$\Rightarrow \frac{dz}{dR} = \pm\left[\frac{3}{2v_p}(R - cz)\right]^{-\frac{1}{3}} = \pm\sqrt{\frac{v}{v_p}}$$

$$\begin{cases} < 1 & \text{for } v > v_p \\ = 1 & \text{for } v = v_p \\ > 1 & \text{for } v < v_p \end{cases}$$

Point particle

$$ds^2 = \sqrt{dx^2}$$

$$x' = f(x)$$

$$S = -m\int ds = -m\int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{dx'}\frac{dx^\nu}{dx'}}\,dx' = \int \mathcal{L}\,dx'$$

$$\frac{dx^\mu}{dx'} = \left(\frac{dx'}{dx^\mu}\right)^{-1}$$

$$\Rightarrow \text{I. choice} \quad x' = s \quad \text{or} \quad dx' = ds \quad \Rightarrow \quad \frac{dx}{ds} = 1$$

$$\mathcal{L} = \frac{1}{2}\frac{ds^2}{dx'^2} = \frac{1}{2}\left(1 - \frac{r_g}{r}\right)\left(\frac{dt}{dx}\right)^2 - \frac{1}{2}\left(\frac{dr}{dx}\right)^2 \frac{1}{1-\frac{r_g}{r}} - \frac{1}{2}r^2\left[\left(\frac{d\theta}{dx}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{dx}\right)^2\right]$$

$$= \frac{1}{2}\left(1-\frac{r_g}{r}\right)\dot{t}^2 - \frac{1}{2}\dot{r}^2\frac{1}{1-\frac{r_g}{r}} - \frac{1}{2}r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)$$

$\theta = \varphi = 0$, radial motion:

$$\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{1}{2}\left(1-\frac{r_g}{r}\right)\dot{t} = E \quad \Rightarrow \quad \dot{t} = \frac{E^{\text{full}}}{g_{tt}} = \frac{E^{\text{full}}}{1-\frac{r_g}{r}}$$

pour cond

$$\mathcal{L} = \frac{1}{2} \quad \Rightarrow \quad g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 = 1$$

$$\left(1-\frac{r_g}{r}\right)\frac{E^2}{\left(1-\frac{r_g}{r}\right)^2} - \frac{\dot{r}^2}{1-\frac{r_g}{r}} = 1$$

$$\dot{r}^2 + \left(1-\frac{r_g}{r}\right) = E^2$$

$$r \to \infty \Rightarrow \dot{r} = 0 \quad \Rightarrow \quad E = 1 \quad \Rightarrow \quad \dot{r} = \sqrt{\frac{r_g}{r}}$$

$$\sqrt{\frac{r_g}{r}} = \frac{dr}{dx^0}$$

$$\frac{dr}{d\tilde{\tau}} = -\sqrt{\frac{r_g}{r}}$$

$$\sqrt{\frac{r}{r_g}}\, dr = -d\tilde{\tau}$$

$$\frac{1}{r_g^{\frac{1}{2}}} \cdot \frac{r^{+\frac{1}{2}+1}}{+\frac{1}{2}+1} = -\tilde{\tau} + \text{cond}r$$

$$\frac{2}{3}\frac{r^{3/2}}{r_g^{\frac{1}{2}}} = -\tilde{\tau} + \text{const}$$

for $\tilde{\tau} = 0$, $r = r_0 \Rightarrow$ const $= \frac{2}{3}\frac{r_0^{3/2}}{r_g^{\frac{1}{2}}}$

$$\tilde{\tau} = \frac{2}{3}\left(\frac{r_0^{3/2}}{r_g^{\frac{1}{2}}} - \frac{r^{3/2}}{r_g^{\frac{1}{2}}}\right)$$

for $r = 0$ and $r_0 = r_g$

$r = 0$ for $\tilde{\tau} < \infty$.

$$\tilde{\tau} = \frac{2}{3} r_g \frac{1}{c} = \frac{4}{3} M = \frac{2}{3}\frac{1}{c}\frac{2MGw}{c^2}$$

$$r_g = \frac{2MGw}{c^2}$$

$M_\odot \Rightarrow r_{g\odot} = 3\,Km$

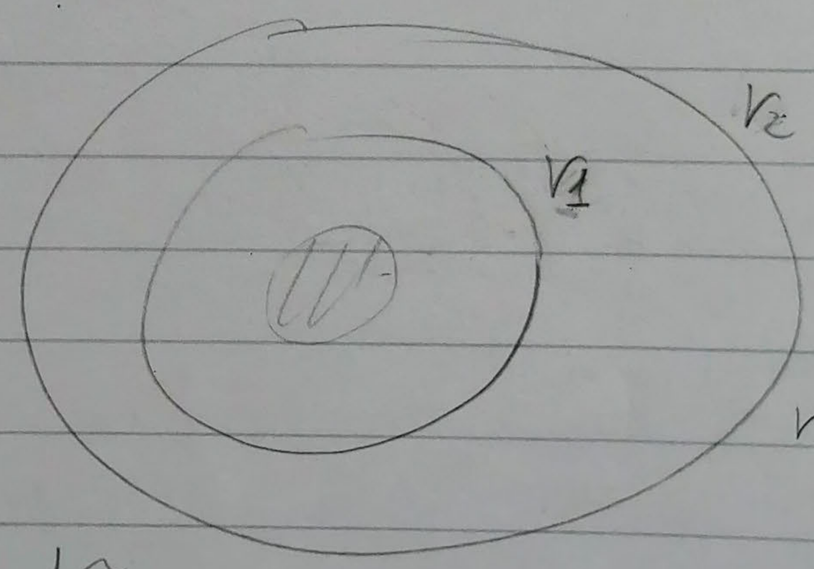
—

ds^2 is invariant for $t \to t + \text{cost} \Rightarrow$

$$d\tau_2^2 = \left(1 - \frac{r_g}{r_2}\right) dt$$

$$d\tau_1^2 = \left(1 - \frac{r_g}{r_1}\right) dt$$

some "dt"



$$d\tilde{\tau}_2 = \sqrt{\frac{\left(1 - \frac{r_g}{r_2}\right)}{\left(1 - \frac{r_g}{r_1}\right)}}\, d\tilde{\tau}_1, \quad d\tilde{\tau}_2 > d\tilde{\tau}_1$$

$d\tilde{\tau}_1\big|_{r=r_g} < \infty \Rightarrow d\tilde{\tau}_\infty\big|_{r=r_g} = \infty$

$r_2 \to \infty \Rightarrow d\tilde{\tau}_2 = \frac{d\tilde{\tau}_1}{\sqrt{1 - \frac{r_g}{r_1}}} \Rightarrow d\tilde{\tau}_2\big| \longrightarrow \infty$ (as $r_1 \to r_g$)

$$\frac{\omega_2}{\omega_1} = \sqrt{\frac{1-\frac{v_p}{r_1}}{1-\frac{v_p}{r_2}}} \longrightarrow \sqrt{1-\frac{v_p}{r_1}} \longrightarrow 0$$

$r_2 \to \infty$ for $r_1 \to v_p$

$\omega_2 < \omega_1$

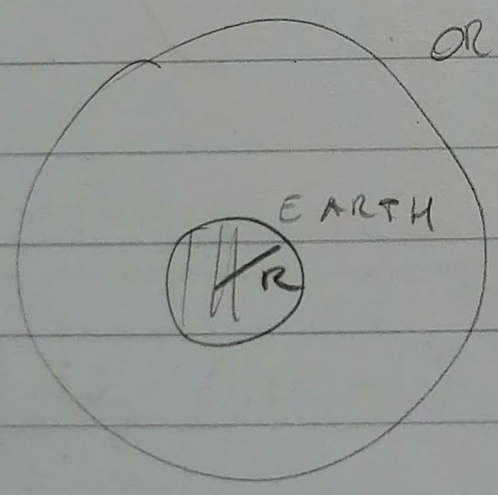
For the observer at infinity the body is completely frozen.

Conformal BH:

$$\sqrt{\frac{1+\frac{L^4}{r_1^4}}{1+\frac{L^4}{r_2^4}}} \qquad \sqrt{1-\frac{r_p}{r_1}} \qquad \frac{1}{\sqrt{1-\frac{r_p}{r_2}}} \qquad r_1 \to r_p \Longrightarrow \left(1+\frac{L^4}{r_p^4}\right)\cdot\infty$$

$$\| \quad \frac{s_2}{s_1}$$

$r_1 = r_p + \epsilon$



$$d\tau_\epsilon = \sqrt{1-\frac{2M}{R}} \qquad d\tau_{orbit} = \sqrt{1-\frac{2M}{R+h}}$$

$$d\tau_\epsilon = d\tau_0 \sqrt{\frac{1-\frac{2M}{R}}{1-\frac{2M}{R+h}}} \Rightarrow d\tau_\epsilon < d\tau_{orbit}.$$

$$\frac{2}{k}\cdot\frac{\lambda}{c} = \frac{2}{c} = \frac{2}{300.000}\,s. = 10^{-6}\,sec \qquad \tau_p \frac{M^3}{M_p^3}$$

Kruskal - Szekeres coordinates

schwarzschild coordinates $(t, r, \theta, \varphi) \to (T, X, \theta, \varphi)$

$$T = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \quad \left.\right\} \; r > 2GM$$

$$X = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) \quad \left.\right)$$

$$T = \left(1 - \frac{r}{2GM}\right)^{\frac{1}{2}} \dots \quad \left.\right\} \; 0 < r \leq 2GM.$$

$$X = \left(1 - \frac{r}{2GM}\right)^{\frac{1}{2}} \dots \quad \left.\right)$$

$$\Rightarrow T^2 - X^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$$

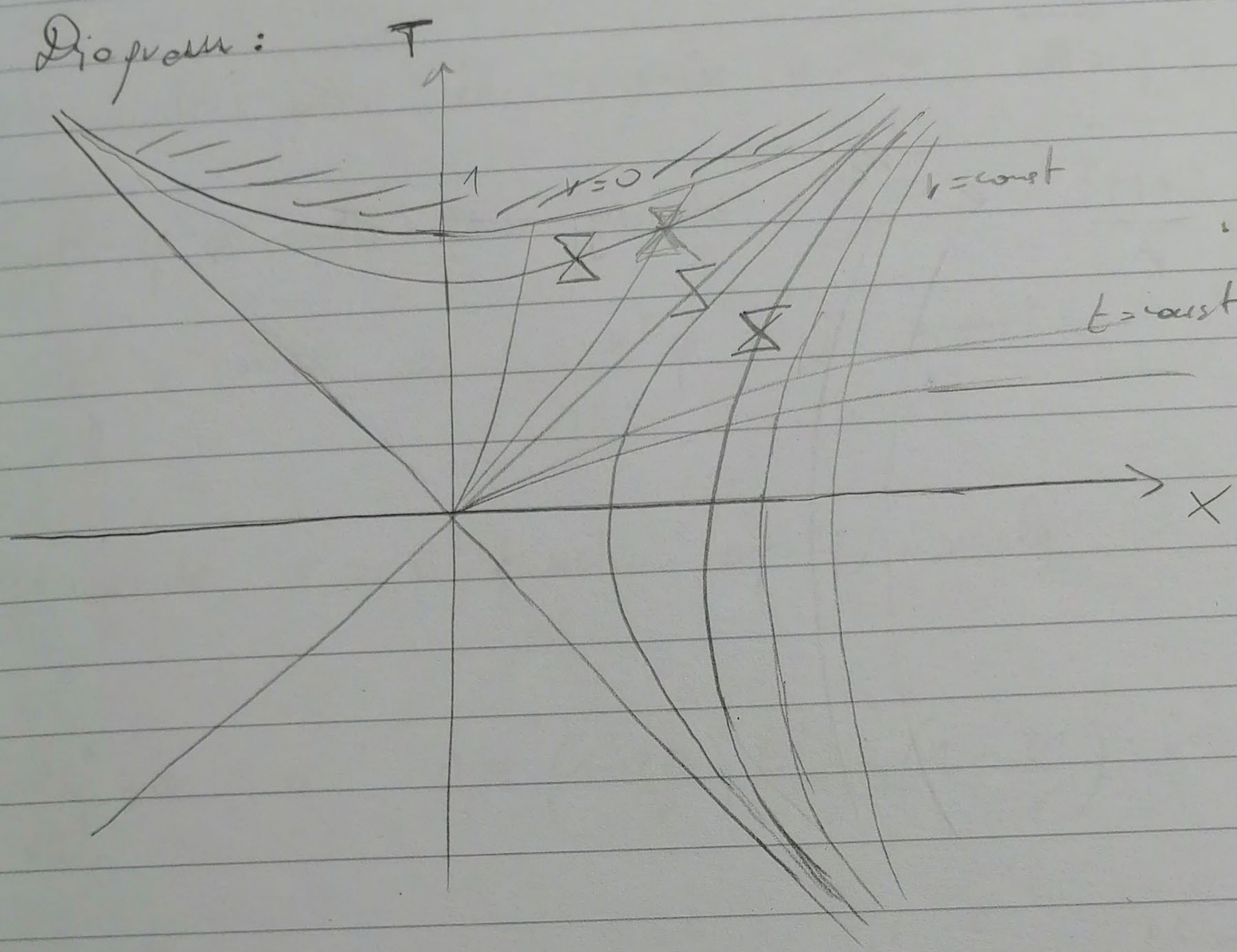
In these coordinates:

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM} \left(-dT^2 + dX^2\right) + r^2 d\Omega^{(2)}$$

Location of the horizon ($r = 2GM$): $T = \pm X$

" " " singularity ($r = 0$): $T^2 - X^2 = 1$

Diagram:



Singularity: $r=0$, $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}$

Let's change coordinates to (x, y, θ, φ)

defined by ($G=1$)

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = xy$$

$$e^{t/2M} = \frac{x}{y}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} = \frac{dr}{2M\left(1 - \frac{2M}{r}\right)}$$

$$\frac{dx}{x} - \frac{dy}{y} = \frac{dt}{2M}$$

$$\Rightarrow ds^2 = \frac{32M^3}{r} e^{-r/2M} dx\,dy + r^2 d\Omega^2$$

We consider also the angular motion, $\vartheta = \frac{\pi}{2}$, $\dot{\vartheta} = 0$,

but $\varphi \neq$ const.

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \text{const.} \Rightarrow \quad r^2 \dot{\varphi} = J = \text{const}$$

We consider r as a function of φ, namely

$$\frac{ds^2}{d\tau^2} = 1 \Rightarrow \quad 1 = \left(1 - \frac{2M}{r}\right)^{-1} E^2 - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - J^2/r^2$$

$$r' \equiv \frac{\dot{r}}{\dot{\varphi}} = \frac{dr}{d\varphi} \quad , \quad \dot{r}^2 = r'^2 \dot{\varphi}^2 = r'^2 \frac{J^2}{r^4}$$

$$\Rightarrow \quad 1 - \frac{2M}{r} = E^2 - J^2 \frac{r'^2}{r^4} - J^2 \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} .$$

Now we introduce the Keplerian coordinate u,

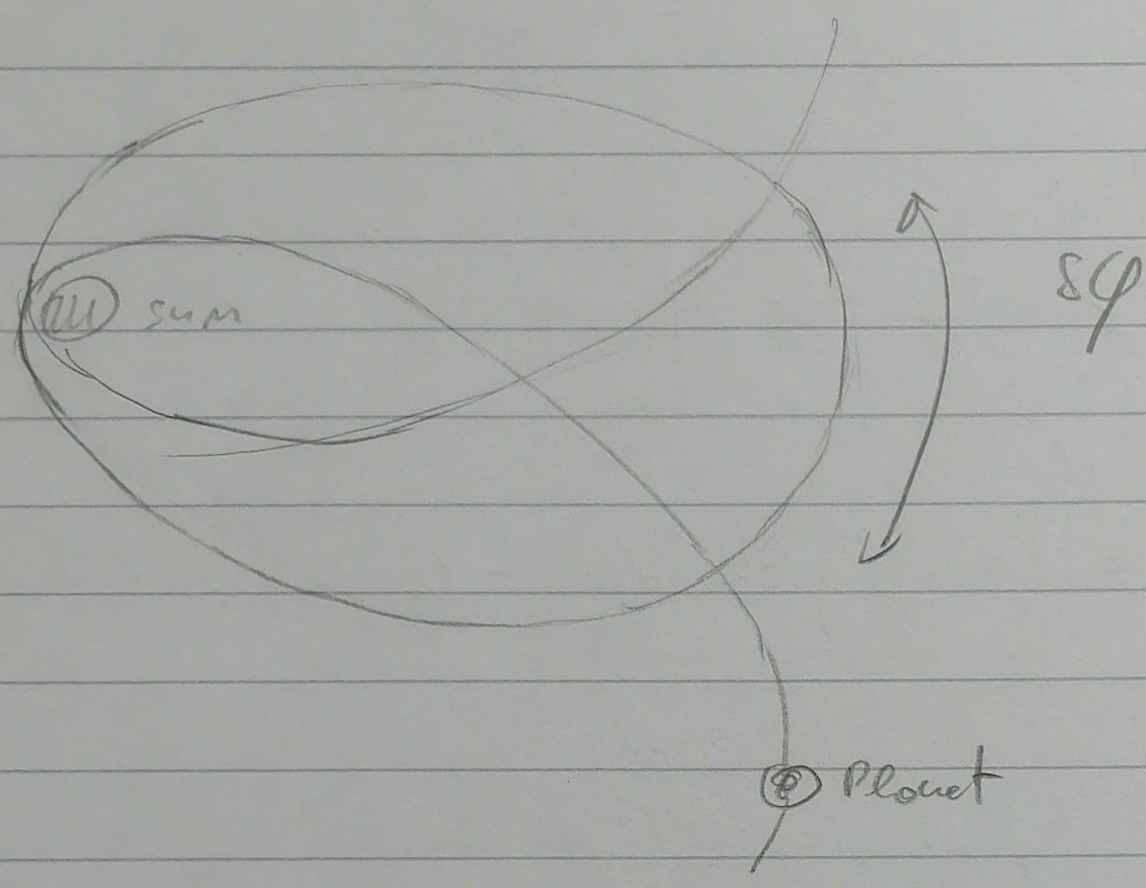
$$r = \frac{1}{u} \quad , \quad r' = -u'/u^2$$

$$1 - 2Mu = E^2 - J^2 u'^2 - J^2 u^2 (1 - 2Mu)$$

$$\Rightarrow \quad \frac{du}{d\varphi} = \sqrt{(2Mu - 1)\left(u^2 + \frac{1}{J^2}\right) + \frac{E^2}{J^2}} .$$

Formally

$$\varphi - \varphi_0 = \int_{u_0}^{u} du \left(\frac{\epsilon^2 - 1}{J^2} + \frac{2Mu}{J^2} - u^2 + 2Mu^3 \right)^{-\frac{1}{2}}$$



We take the derivative respect to φ:

$$\frac{2M}{J^2} u' - 2u'u'' - 2uu' + 6Mu^2u' = 0,$$

for $u' \neq 0 \Rightarrow$ $u'' + u = \frac{M}{J^2} + \underbrace{3Mu^2}$

relativistic correction (small)

$\Rightarrow$ $u'' + u = A + \epsilon u^2$.

The unperturbed period is 2π, but since we have

now u^2 term $\Rightarrow \quad T = 2\pi(1+\alpha\epsilon)$.

We look for a perturbative solution:

$$u = A + B\cos[(1-\alpha\epsilon)\varphi] + \epsilon u_1(\varphi) + O(\epsilon^2)$$

$$u'' = -B(1-2\alpha\epsilon)\cos[(1-\alpha\epsilon)\varphi] + \epsilon u_1''(\varphi) + O(\epsilon^2)$$

$\Rightarrow$ for u_1 :

$\cos(\varphi)$: 2π

$\cos(\alpha\varphi)$: $\frac{2\pi}{\alpha} = \frac{2\pi}{1-2\epsilon} \simeq 2\pi(1+2\epsilon)$

$$u_1'' + u_1 = (-2\alpha B + 2AB)\cos\varphi + B^2\cos^2\varphi + A^2,$$

(Forced pendulum).

We take $\alpha = A$ to override the resonance (u_1 must be small $\forall\varphi$ and stay small)

$$\Rightarrow \quad u_1'' + u_1 = \frac{1}{2}B^2(\cos^2\varphi + 1) + A^2$$

$$u_1 = \frac{1}{2}B^2\left(1 - \frac{1}{2^2-1}\cos 2\varphi\right) + A^2. \qquad (u(0) = u(2\pi))$$

$$u_1 = \frac{1}{2}B^2\left(1 - \frac{1}{3}\cos 2\varphi\right) + A^2 \qquad u(0) = -\frac{1}{6}B^2 + A^2 = u(2\pi).$$

For an orbit periodic in φ we should take,

$$T = 2\pi(1+A\epsilon) \quad \Rightarrow \quad \delta\varphi = 2\pi A\epsilon = 2\pi\frac{3M^2}{J^2}.$$

$u(0) = A + B + \epsilon u_1(0)$

$u(2\pi) = A + B\cos[(1-\alpha\epsilon)2\pi] + \epsilon u_1(2\pi)$

$u(2\pi) - u(0) = 0 \Rightarrow \cos[(1-\alpha\epsilon)2\pi] = 1$

$(1-\alpha\epsilon)2\pi = 2n\pi \quad \Rightarrow T = \frac{2\pi}{1-2\epsilon} \simeq 2\pi(1+2\epsilon)$

Cosmology

Cosmological principle : the universe is homogeneous and isotropic.

⇒ We choose a cosmic time coordinate "t" such that

$$ds^2 = dt^2 - \rho_{ij}(x,t)\, dx^i dx^j$$

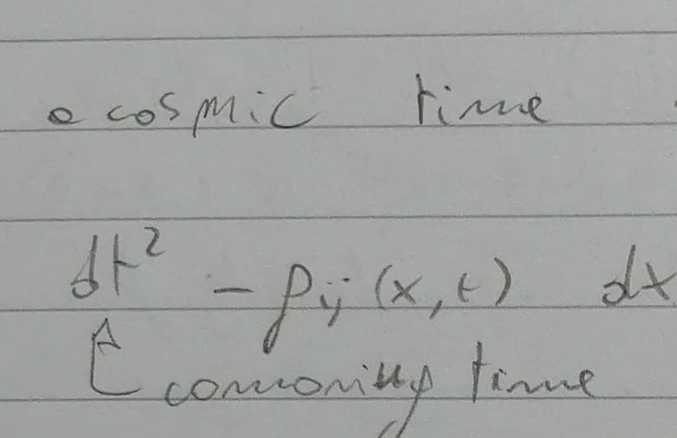
↑ comoving time

spatial section : $\frac{x_1^2}{k} + x_2^2 + x_3^2 + \frac{x_4^2}{k} = \frac{R(t)^2}{k}$

where $k = 0, \pm 1$.

close universe

$k = 1$



$c < 2\pi a$

$2\pi a = 2\pi R\theta$

$a = R\theta$

$c = 2\pi r$

$c = 2\pi R \sin\vartheta$

$< 2\pi R\theta$

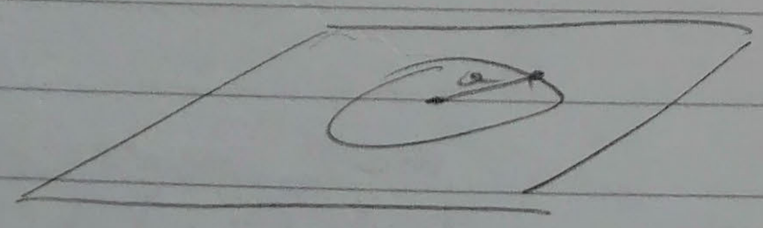
$r = R\sin\vartheta$

$b = R\cos\theta$

$\sin\vartheta < \theta$

flat universe

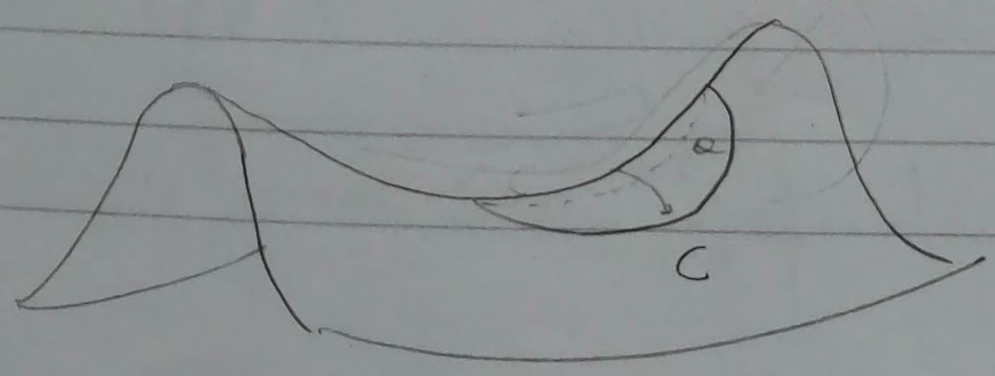
$k = 0$



$c = 2\pi a$

open universe

$k = -1$



$c > 2\pi a$

Curvature $K = \frac{k}{R^2(t)}$

$$d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{dx_4^2}{k}$$

In polar coordinates : $\longleftarrow$
$$\begin{cases} x_1 = R\,r\,\sin\theta\cos\phi, \\ x_2 = R\,r\,\sin\theta\sin\phi, \\ x_3 = R\,r\,\cos\theta \\ x_4 = R(1-kr^2)^{\frac{1}{2}} \end{cases}$$

$$ds^2 = c^2dt^2 - d\sigma^2$$

$$= c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

Energy momentum tensor for matter of density ρ :

$$T_{\mu\nu} = \rho c^2 u_\mu u_\nu .$$

For $g_{\mu\nu} = \eta_{\mu\nu}$:

$$u^\mu = \frac{dx^\mu}{d\tau}, \quad \vec{u} = \frac{d\vec{x}}{d\tau} = \gamma\vec{v}, \quad u_0 = \frac{dt}{d\tau} = \gamma$$

For matter at rest : $T_{00} = \rho c^2$, $T_{0i} = T_{ij} = 0$.

For a perfet fluid $\begin{cases} (i)\ \text{no heat conduction} \\ (ii)\ \text{no viscosity} \Rightarrow T_{ij} = 0 \end{cases} \rightarrow T_{0i} = 0$ that is at reast, but has Random non relativistic internal motion, we have

instead :

$T_{00} = \sum_\alpha m_\alpha c^2 = \rho c^2$ $\quad m_\alpha$: mass in unit of volume

$T_{ii} = \frac{1}{3}\sum_\alpha m_\alpha v_\alpha^2 = P \qquad i = 1, 2, 3.$

In any Lorentz frame :

$$T_{\mu\nu} = (p + \rho c^2)\, u_\mu u_\nu - p\, g_{\mu\nu}.$$

For radiation $E = c|\vec{p}|$, $T_{00} = \sum E = \rho c^2$

$$T_{ii} = \frac{1}{3} \sum \frac{|\vec{p}|^2 c^2}{E} = \frac{1}{3} \sum E = \frac{1}{3} \rho c^2$$

$$T^r{}_r = (p + \rho)\, \underbrace{u^r u_r}_{1} - g p = \rho - 3p$$

For radiation $T^r{}_r = 0 \Rightarrow p = \frac{1}{3}\rho$ $(c = 1)$.

Let's solve the Einstein eq.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{\Lambda}{c^2} g_{\mu\nu} = -\frac{8\pi G_N}{c^4} T_{\mu\nu}$$

1) $\frac{\ddot{R}}{R} = -\frac{4\pi G_N}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}$ $\quad (r,0) = (0,0)$

2) $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_N \rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3}$ $\quad (r,0) = (i,j)$ ← Only one eq. because of isotropy.

1) $dE = TdS - pdV$

$\Lambda = 0 \Rightarrow \quad \frac{d}{dt}\left(\rho R^3 c^2\right) = -p \frac{d}{dt}(R^3)$ ⇐ 2) $\quad \overset{dr}{T^i{}_{0;i}} = 0$

$\Rightarrow ds = 0$

For $p = 0$, $k = 0$: $\rho R^3 = \text{const} \Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G \rho}{3} \sim \frac{1}{R^3}$

$R \sim t^{2/3}$!

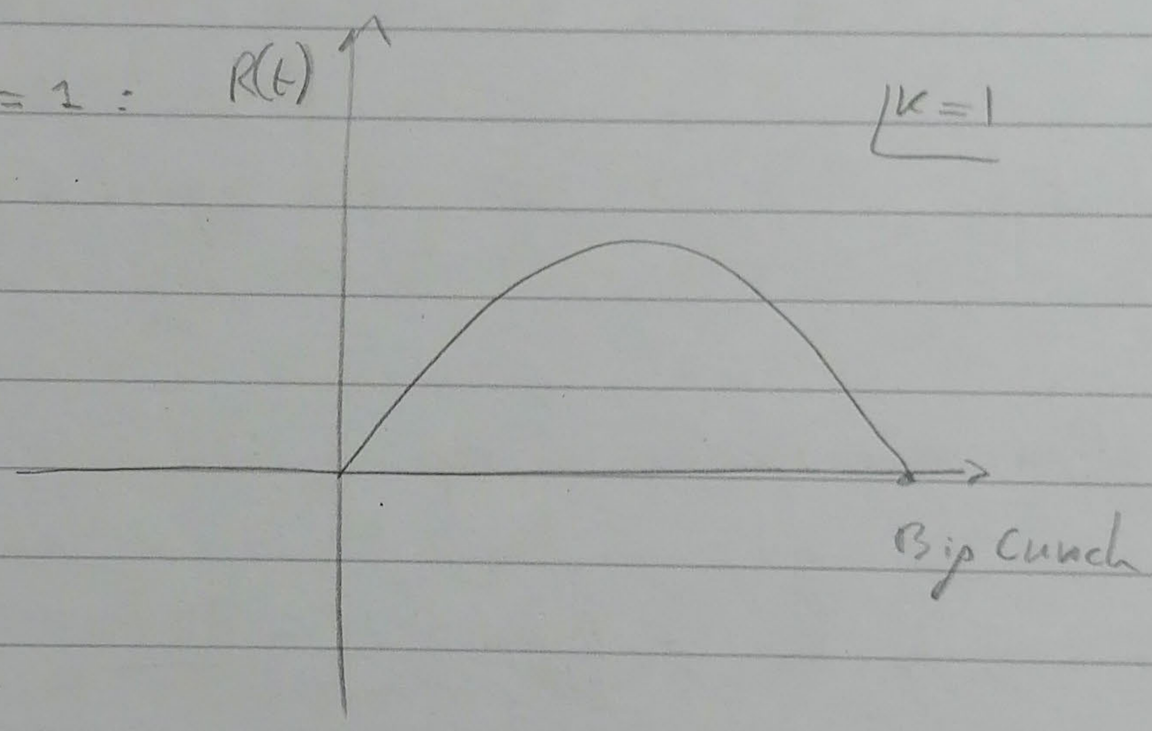
Thermal equilibrium ⇒ max entropy

Radiation, $P = \frac{1}{3}\rho c^2 \Rightarrow \frac{d}{dt}(\rho R^4) = 0$

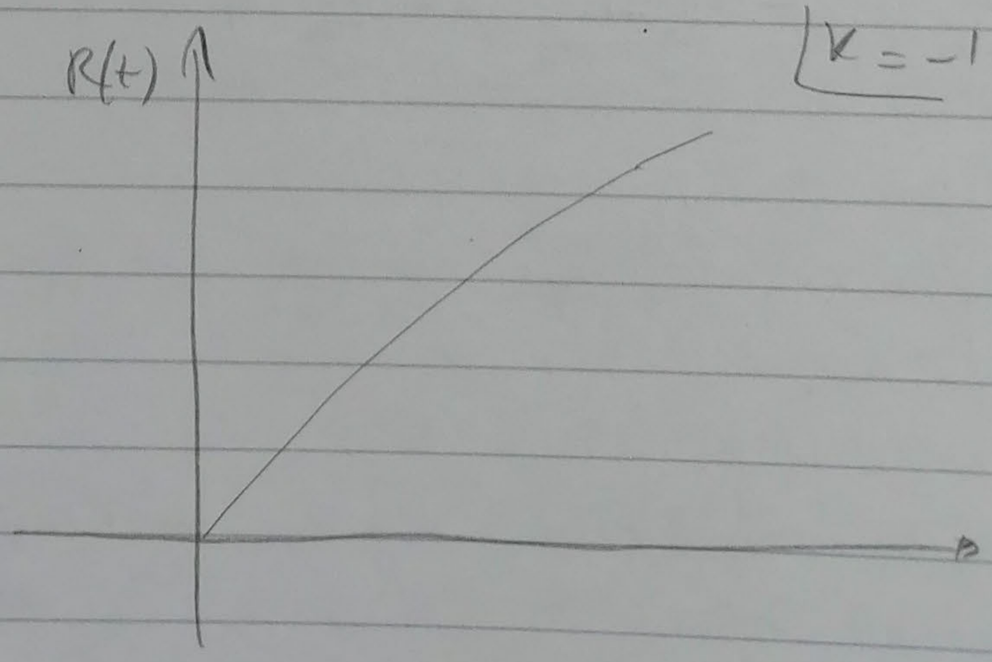
$$\Rightarrow H = \left(\frac{\dot{R}}{R}\right)^2 \sim \frac{1}{R^4} \Rightarrow R \sim t^{\frac{1}{2}}$$

Radiation dominates at the Big-Bang.

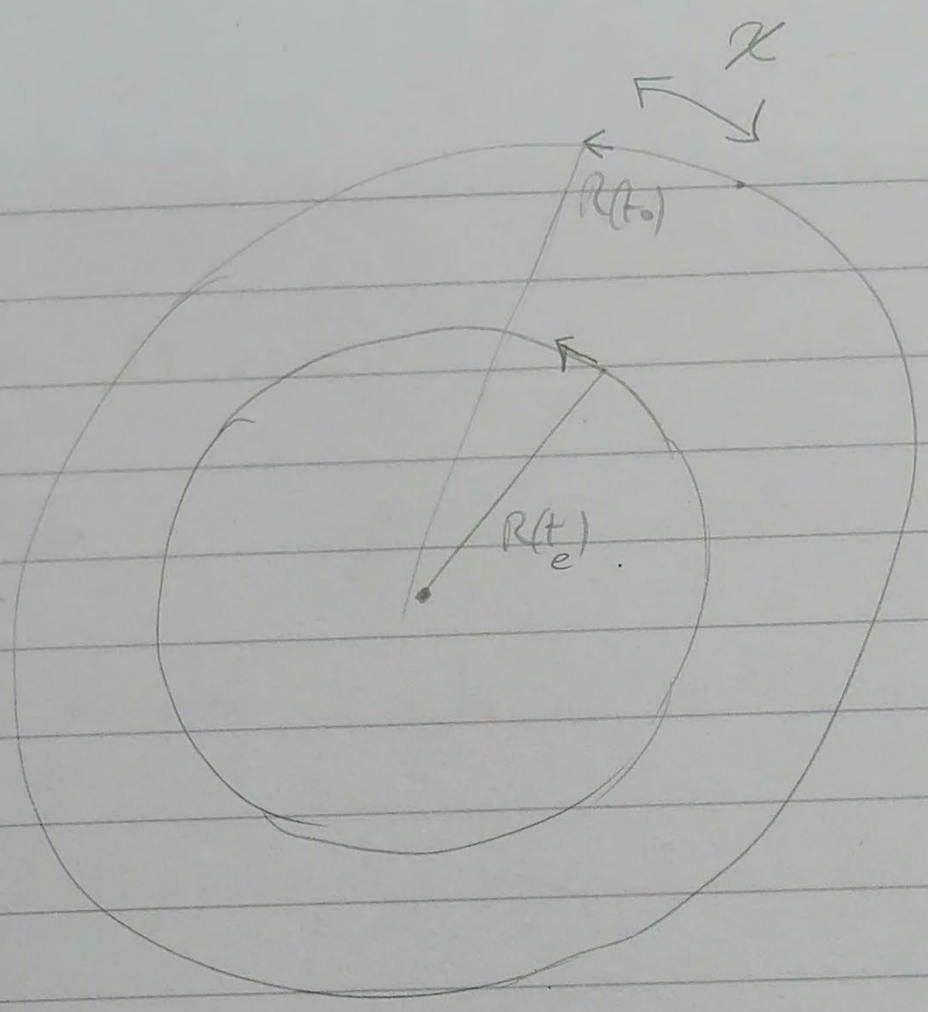
Dust again: $k = 1$:



$k = -1$



$$ds^2 = R^2(t)\left[d\chi^2 + \sinh^2\chi\, d\Omega^2\right]$$



$$z = \frac{\lambda_o}{\lambda_e} - 1$$

$\lambda_o = R(t_o)\,\Delta\chi$ ↖ comoving coordinates

$\lambda_e = R(t_e)\,\Delta\chi$

Redshift: $\quad z+1 = \dfrac{R(t_o)}{R(t_e)} \Rightarrow z = \dfrac{R(t_o)}{R(t_e)} - 1$

$H = \dfrac{\dot R}{R} \quad \Rightarrow z \cong \dfrac{R(t_o)}{R(t_o) + (t_e - t_o)\,\dot R(t_o)} - 1$

$$= \frac{R(t_o)}{R(t_o) + (t_e - t_o)\,H(t_o)\cdot R(t_o)} - 1$$

$$\cong (t_o - t_e)\,H(t_o)$$

$$\cong \frac{D}{c}\,H(t_o) \qquad \text{or} \qquad c\cdot z = D\,H(t_o) \equiv v_r$$

Not directly observed.

Recession velocity

$$\boxed{v_r = H_o\, D}$$ Hubble's law.

$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$.

Linearisation : $g_{ik} = \eta_{ik} + h_{ik}$

Jauge invariance : $h'_{ik} = h_{ik} - \partial_k \xi_i - \partial_i \xi_k$, $h' = h - 2\partial^i \xi_i$

Jauge fixing : $\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$

$\rightarrow \partial^\alpha \bar{h}_{\alpha\beta} = 0$ & conditions

$\Rightarrow \Box \bar{h}_{\alpha\beta} = -16\pi G_N T_{\alpha\beta}$

$\bar{h}'_{\alpha\beta} = h'_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h'$ $\qquad h' = h - \frac{1}{2}\cdot 4h = -h$

$= h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha - \frac{1}{2}\eta_{\alpha\beta}(h - 2\partial^i \xi_i)$

$= \bar{h}_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha + \eta_{\alpha\beta}\partial^i \xi_i$

$\partial^\alpha \bar{h}'_{\alpha\beta} = \partial^\alpha \bar{h}_{\alpha\beta} - \Box \xi_\beta - \partial_\beta \partial^\alpha \xi_\alpha + \partial_\beta \partial^i \xi_i$

$0 \qquad\qquad 0$

$\Box \xi_\beta = 0 \Rightarrow$ other 4 conditions $\Rightarrow$ 2 polarisations.

$$\bar{h}_{ij}(x) = 4G \int d^3\vec{x}' \; \frac{T_{ij}(\vec{x}', t - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}$$

→ Radial geodesics in Sch. and Kerr.

→ $\nabla^r G_{r\cdot} = ?$

$\nabla^r T_{r\cdot} = ?$

→ What is the meaning of Einstein Eq.

$$\mathcal{L} = -m\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -m\sqrt{\rho_{00}(r)\dot{t}^2 + \rho_{11}(r)\dot{r}^2 + \rho_{33}(\sin^2\theta\dot{\varphi}^2 + \dot{\theta}^2)}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = \frac{-m\,1}{2\sqrt{\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\cdot\rho_{00}\,2\dot{t} = \frac{m^2}{\mathcal{L}}\rho_{00}\dot{t} = \text{const} \equiv E$$

$$\dot{t} = \frac{E}{m^2\rho_{00}}\frac{\mathcal{L}}{} = -\frac{E}{m}\frac{1}{\rho_{00}}$$

$$\dot{x}^2 = 1 \quad \Rightarrow \quad \mathcal{L} = -m$$

$$\rho_{00}\dot{t}^2 + \rho_{11}\dot{r}^2 = 1$$

$$\rho_{00}\frac{E^2}{m^2}\frac{1}{\rho_{00}^2} + \rho_{11}\dot{r}^2 = 1$$

$$\rho_{00}\rho_{11}\dot{r}^2 - \rho_{00} = -\frac{E^2}{m^2}$$

b) If $g_{00} = \frac{-1}{\rho_{11}} \Rightarrow -\dot{r}^2 - \rho_{00} = -\frac{E^2}{m^2}$

$$\dot{r}^2 + \rho_{00} = \frac{E^2}{m^2}$$

c), d)

If $\rho_{00} = 1 - \frac{2m}{r} \Rightarrow \dot{r}^2 + \left(1 - \frac{2m}{r}\right) = \frac{E^2}{m^2}$

If $\dot{r}^2 \to$ for $r \to \infty \Rightarrow E = m$

$\Rightarrow \dot{r}^2 = \frac{2m}{r}$ $\qquad \frac{dr}{d\tau} = -\sqrt{\frac{2m}{r}} \Rightarrow r^{\frac{1}{2}}dr = -\sqrt{2m}\,d\tau$

$\frac{r^{3/2}}{3/2} = -\sqrt{2m}\cdot\tau + C$

$0 = 2$ for $r_0 = r$ $\quad \frac{2}{3}r_0^{3/2} = C$

$\dot{r}^2 + f(r) = 1 \qquad f(r) = 1 - \frac{2m(r)}{r} \Rightarrow \dot{r}^2 = \frac{2m(r)}{r}$

$$\frac{2}{3}\left(r^{3/2} - r_0^{3/2}\right) = -\sqrt{2MG}\,\tau$$

$$\tau = \frac{2}{3\sqrt{2MG}}\left(r_0^{3/2} - r^{3/2}\right)$$

e) $\dot{r}^2 + f(r) = 1$

$$\dot{r}^2 = \frac{2MGr^2}{r^3 + \alpha(GM)^3}$$

$$\dot{r} = -\frac{\sqrt{2MG}\,r}{\sqrt{r^3 + \alpha(GM)^3}}$$

for $r \to 0 \Rightarrow \dot{r} \simeq -\frac{\sqrt{2MG}\,r}{\sqrt{\alpha(GM)^3}}$

$$\frac{dr}{r} \simeq -\sqrt{\frac{2MG}{\alpha(GM)^3}}\,d\tau$$

$$\ln r \simeq -\sqrt{\frac{2MG}{\alpha(GM)^3}}\,\tau + \text{Const}$$

$$\int \frac{d\lambda}{\lambda^{1+\frac{1}{2}}} = -\frac{2}{\lambda^{\frac{1}{2}}} + C$$

for $\tau = 0 \Rightarrow r = r_0 \Rightarrow C = \ln r_0$

$$\tau = -\sqrt{\frac{\alpha(GM)^3}{2MG}}\left(\ln r - \ln r_0\right)$$

$$= -4 \ln\left(\frac{r}{r_0}\right)$$

$$= \sqrt{\frac{\alpha(GM)^3}{2MG}}\,\ln\frac{r_0}{r}$$

(2) a) $ds^2 = dt^2 - a^2(t)\, d\vec{x}^2$

$$= a^2(t)\left[\frac{dt^2}{a^2(t)} - d\vec{x}^2\right]$$

$$\Rightarrow \frac{dt}{a(t)} = d\tau \quad \Rightarrow \tau = \int \frac{dt}{a(t)} \quad \Rightarrow \tau = \tau(t)$$

and

$t = t(\tau)$.

$$\Rightarrow ds^2 = a^2(\tau)\,(d\tau^2 - d\vec{x}^2)$$

Minkowski

b) $d\vec{x}^2 = d\tau^2 \Rightarrow \sum_{i=1}^{3} x_i^2 = \tau^2$ ← light.

b) $\mathcal{L} = -m\sqrt{\dot{t}^2 - a^2 \dot{x}_i^2}$

$\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} = 1$

$\Rightarrow \mathcal{L} = -m$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -m\,\frac{1}{2\sqrt{\dot{t}^2 - a^2\dot{x}_i^2}}\, 2\dot{t} = +\frac{m^2}{\mathcal{L}}\,\dot{t} \qquad \frac{\partial \mathcal{L}}{\partial t} \neq 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = -m\,\frac{1}{2\sqrt{\;}}\,(-a^2)\,2\dot{x}_i = +\frac{m^2 a^2 \dot{x}_i}{\mathcal{L}} \qquad \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

$$-m a^2 \dot{x}_i = \text{constant} \qquad \dot{x}_i = \frac{c_i}{a^2}, \quad x_i = \int$$

Ex. 5

$$D\left(1+\frac{D}{n^2}\right)h = 0$$

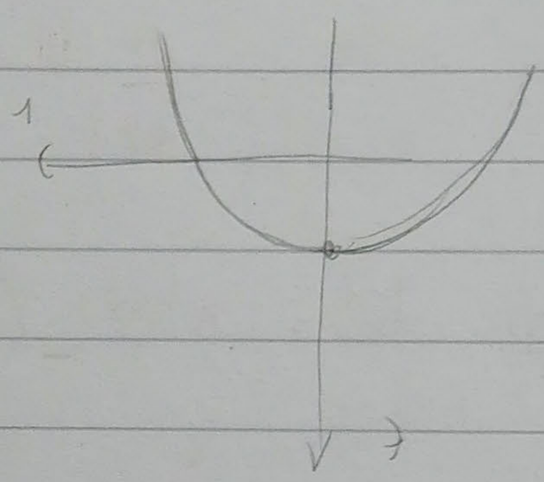
$$h_{\mu\nu} = \epsilon_{\alpha\beta} e^{ik_1 x} + \tilde{\epsilon}_{\alpha\beta} e^{ik_2 x}$$

$$k_1^2 = 0$$

$$\frac{-k_2^2}{n^2} + 1 = 0 \qquad k_2^2 = n^2$$

$$k_0 = \pm\sqrt{\vec{k}^2 + n^2}$$

$$\omega$$



c) $\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}$

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}}$$

$$\frac{da}{a} = \sqrt{\frac{\Lambda}{3}}\, dt$$

$$\ln a = \sqrt{\frac{\Lambda}{3}}\, t + \text{const}$$

$$a = a_0 e^{\sqrt{\frac{\Lambda}{3}}\, t} \qquad (\ln a_0 = \text{const})$$

$$(x_1^0 - x_2^0)^{m^2} V = 7$$

$$7^{m^2} = x_1^0 - x_2^0$$

$$\ln 7^{m^2} = \ln(x_1^0 - x_2^0)$$

$$\frac{1}{m^2} \approx \frac{\ln 7^0}{\ln 1^0}$$

$$\left(1 - \frac{1}{m^2}\right) = \frac{\ln 7^0}{\ln 1^0}$$

$$0 > \frac{\ln 7^0}{\ln 1^0}$$

$$\left|\frac{1}{m^2} - 1\right| = \left|\frac{\ln 7^0}{\ln 1^0}\right|$$

$$\sim\left(\frac{1}{m^2} - 1\right) = {}_2\left(\frac{7^0}{1^0}\right)$$

③

$\theta = \frac{\pi}{2}$: EQUATOR.

(+ − − −)

$$ds^2 = \underbrace{\left(1 - \frac{r_s r}{\rho^2}\right)}_{\rho_{00}} dt^2 - \underbrace{\frac{\rho^2}{\Delta}}_{\rho_{11}} dr^2 + \underbrace{\frac{2 r_s r a}{\rho^2}}_{2\rho_{t\phi}} \sin^2\theta \, dt\, d\phi - \left(r^2 + a^2 + \frac{r_s r a^2 \sin^2\theta}{\rho^2}\right) \underbrace{1}_{\rho_{\phi\phi}} \underbrace{\sin^2\theta \, d\varphi^2}_{1}$$

$$\frac{ds^2}{d\tau^2} = 1 \qquad \rho^2 = r^2 + a^2\cos^2\theta = r^2$$

$$\Delta = r^2 - r_s r + a^2$$

$$\mathcal{L} = -m\sqrt{\rho_{00}\dot{t}^2 + \rho_{11}\dot{r}^2 + 2\rho_{t\phi}\dot{\phi}\dot{t} + \rho_{\phi\phi}\dot{\varphi}^2} = -m\sqrt{\dot{x}^2}$$

$m \to f\phi$ (−) se η_{00} (−+++) (−)

$m \to f\phi$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -m\,\frac{1}{m \to f\phi\, 2\sqrt{\;}}\,(2\dot{t}\,\rho_{00} + 2\rho_{t\phi}\dot{\phi}) \qquad \rho_{\phi\phi} = -\left(r^2 + a^2 + \frac{r_s r a^2}{\rho^2}\right)$$

(−)

$$= \frac{m^2}{\mathcal{L}}(\dot{t}\,\rho_{00} + \rho_{0\phi}\dot{\phi}) = -E \quad \text{const.}$$

(−) $m^2 \to f^2$ (↑ en entrant le repoussoir)

$m^2 \to f^2 f^2$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = +\frac{m^2}{\mathcal{L}}(\rho_{0\phi}\dot{t} + \rho_{\phi\phi}\dot{\varphi}) = \ell \quad \text{const.}$$

(−)

$\dot{x}^2 = -1$

$$\frac{ds^2}{d\tau^2} = 1 \;(-1) \Rightarrow \rho_{00}\dot{t}^2 + \rho_{11}\dot{r}^2 + \rho_{\phi\phi}\dot{\phi}^2 + 2\rho_{t\phi}\dot{\phi}\dot{t} = 1 \;(-)$$

$$\Rightarrow \mathcal{L} = -m$$

$$-\phi$$

$$\ell = 0 \Rightarrow \dot{\varphi} = -\frac{\rho_{0\phi}}{\rho_{\phi\phi}}\dot{t}$$

$$\Rightarrow -E = \frac{m^2}{\mathcal{L}}\left(\dot{t}\,\rho_{00} - (\rho_{0\phi})^2\frac{1}{\rho_{\phi\phi}}\dot{t}\right) \Rightarrow E = +m\dot{t}\left(\rho_{00} - \frac{\rho_{0\phi}^2}{\rho_{\phi\phi}}\right)$$

(−) $m \to f\phi$

$$\Rightarrow \dot{t} = +\frac{E}{m}\left(\frac{\rho_{\phi\phi}}{\rho_{00}\rho_{\phi\phi} - \rho_{0\phi}^2}\right)$$

(−) $m \to \phi f$

$g_{\mu\nu} - \rho_{\mu\nu}$

$$\dot{x}^2 = -1 \gtrless$$

$$g_{00}\dot{t}^2 + g_{11}\dot{r}^2 + g_{\phi\phi}\frac{(g_{0\phi})^2}{(g_{\phi\phi})^2}\dot{t}^2 - 2\frac{g_{\phi\phi}}{g_{\phi\phi}}\dot{t}^2 = 1 \quad \uparrow (-)$$

$$g_{00}\dot{t}^2 - \frac{g_{0\phi}^2}{g_{\phi\phi}}\dot{t}^2 + g_{11}\dot{r}^2 = 1 \quad \uparrow (-)$$

$$\dot{t}^2\left(g_{00}g_{\phi\phi} - g_{0\phi}^2\right)\frac{1}{g_{\phi\phi}} + \dot{r}^2 g_{11} = 1 \quad \uparrow (-)$$

$$\frac{E^2}{m^2}\frac{g_{\phi\phi}}{(g_{00}g_{\phi\phi} - g_{0\phi}^2)} \cancel{(g_{00}g_{\phi\phi} - g_{0\phi}^2)}\frac{1}{g_{\phi\phi}} + \dot{r}^2 g_{11} = 1 \quad \downarrow (-)$$

$E = m \qquad \frac{E^2}{m^2} \to \frac{E^2}{f^2\phi^2}$ $\qquad (-1$ per segnatura $(-+++))$

$$g_{11}\dot{r}^2 + \frac{g_{\phi\phi}}{g_{00}g_{\phi\phi} - g_{0\phi}^2} = 1 \quad \uparrow \frac{E^2}{f^2\phi^2}\frac{E^2}{f^2\phi^2}\frac{1}{fK_4^2 S^{-1}}\left(\frac{S}{S^2}\right) = 1 \quad (-)$$

$$g_{11}(g_{00}g_{\phi\phi} - g_{0\phi}^2)\dot{r}^2 + g_{\phi\phi} = (g_{00}g_{\phi\phi} - g_{0\phi}^2) \quad \uparrow \frac{E^2}{f^2\phi^2} = \frac{1}{S^{-1}} = S$$

$$g_{00} = 1 - \frac{r_s}{r} \qquad g_{11} = -\frac{r^2}{r^2 - r_s r + a^2}$$

$$g_{0\phi} = \frac{r_s}{r}\cdot a \qquad g_{\phi\phi} = -\left(r^2 + a^2 + \frac{r_s a^2}{r}\right)$$

$$S\dot{r}^2 - \frac{(r^3 + a^2 r + a^2 r_s)}{r^3} = -\frac{r^2 - r_s r + a^2}{r^2}$$

$$S\dot{r}^2 = -\frac{r^2 - r_s r + a^2}{r^2} + \frac{r^3 + a^2 r + a^2 r_s}{r^3} \xrightarrow{a=0} -\left(1 - \frac{r_s}{r}\right) + 1 = \frac{r_s}{r}$$

$$\hat{g}_{00}\hat{g}_{11}\dot{r}^2 + \hat{g}_{11} = -\frac{1}{S} \quad (-)$$

$$g_{00} = \frac{1}{\hat{g}_{00}S} \qquad g_{11} = -\frac{1}{\hat{g}_{11}S}$$

$$\hat{g}_{11}\dot{r}^2 + \frac{\hat{g}_{\phi\phi}}{S(\hat{g}_{00}\hat{g}_{\phi\phi} - \hat{g}_{0\phi}^2)} = 1 \quad (-)$$

$$g_{0\phi} = 0 \Rightarrow S\hat{g}_{11}\dot{r}^2 + \frac{1}{S\hat{g}_{00}} = -1 \quad (-)$$

$$S^2\hat{g}_{00}\hat{g}_{11}\dot{r}^2 + 1 = S\hat{g}_{00}$$

$$S^2\hat{g}_{11}\dot{r}^2 + \frac{\hat{g}_{\phi\phi}}{\hat{g}_{00}\hat{g}_{\phi\phi} - \hat{g}_{0\phi}^2} = S$$

$$-S^2\dot{r}^2 + S\left(-1 + \frac{2M}{r}\right) = -1$$

$\leftarrow (+---)$

for the Cartan-Gauss cond.

$$S\,\frac{r^2}{a^2}\dot r^2 - e^2 S^{-1}\,\frac{a^2 r_s}{a^2 r} = -1$$

$$\frac{r^2}{a^2}\dot r^2 - e^2\frac{1}{S^2}\frac{r_s}{r} = -\frac{1}{S^6}$$

$$\frac{r^2}{a^2}\dot r^2 \simeq \frac{e^2}{\left(1+\frac{L^2}{r^2}\right)^8}\frac{r_s}{r} - \frac{1}{\left(1+\frac{L^2}{r^2}\right)^6}$$

$$\frac{r^2}{a^2}\dot r^2 \simeq \frac{r_s}{r} - \frac{r^{16}}{L^{16}}e^2 - \frac{r^8}{L^8}$$

$$\dot r^2 \simeq -\frac{a^2 r^6}{L^8}$$

$r \to 0$ $\qquad g_{00} \to -\frac{r_s}{r}$

$g_{11} \to -\frac{r^2}{a^2}$ $\qquad g_{0\phi} \to \frac{r_s a}{r}$

$g_{\phi\phi} \to -\frac{r_s a^2}{r}$

$$-\frac{r^2}{a^2}\left(-\frac{r_s}{r}\cdot\frac{-r_s a^2}{r^2} - \left(-\frac{r_s}{r}a\right)\right)\dot r^2 - \frac{r_s a^2}{r} = -\left(-\frac{r_s}{r}a\right)^2$$

$$-\frac{r^2}{a^2}\left(-\frac{r_s}{r}\right)\frac{r_s^2}{r^2}a^2\dot r^2 - \frac{r_s a^2}{r} = -\frac{r_s^2}{r^2}a^2$$

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S $\qquad S^{-1}$

$r \to 0$

$$g_{11}\dot r^2 + \frac{r_s}{r} \simeq 1$$

S $\qquad (-1)$ $\qquad S^{-1}$

$$\left(-\frac{r^2}{r^2 - r_s r + a^2}\right)\dot r^2 + \frac{r_s}{r} \simeq 1$$

S $\qquad S^{-1}$

$$-\frac{\dot r^2}{1 - \frac{r_s}{r} + \frac{a^2}{r^2}} + \frac{r_s}{r} \simeq 0$$

S $\qquad (-)$ $\qquad S^{-1}$

$$\frac{\dot r^2}{1 - \frac{r_s}{r} + \frac{a^2}{r^2}} = \frac{r_s}{r}$$

$S^2 \to$ $\boxed{\dot r^2 \sim \frac{r_s a^2}{r^3}}$

$$g_{11}\dot r^2 \simeq 1$$

$$-\frac{r^2}{a^2}\dot r^2 = 1 \qquad \dot r^2 = -\frac{a^2}{r^2}$$

$$(S)\ \dot r = -\sqrt{\frac{r_s}{r^3}}\,a$$

$$(S)\ r^{3/2}\frac{dr}{d\tau} = -\sqrt{r_s}\,a$$

$$\left(\left(1+\frac{L^2}{r^2}\right)^4\right)\ r^{3/2}\frac{dr}{d\tau} = -\sqrt{r_s}\,a \approx \frac{L^8}{r^8}\cdot r^{3/2}dr = -\sqrt{r_s}\,a\,d\tau$$

$$\frac{3}{2} - 7 = \frac{3+14}{2} = -\frac{11}{2}$$

$$S = \left(1 + \frac{L^2}{r^2}\right)^4$$

$$\frac{L^8}{r^8}r^{3/2}\frac{dr}{d\tau} = -\sqrt{r_s}\cdot a \sim \tau$$

$$L^8\frac{r^{\frac{3}{2}-8+1}}{\frac{3}{2}-8+1} = -\sqrt{r_s}\,a\,\tau$$

$$\frac{r^{-11/2}}{-\frac{11}{2}} = -\frac{\sqrt{r_s}\,a}{L^8}\,\tau$$

$$r^{-11/2} = \frac{2}{11}\frac{\sqrt{r_s}\,a}{L^8}\,\tau$$