Parziale 21/12/23

martedì 19 dicembre 2023   15:37

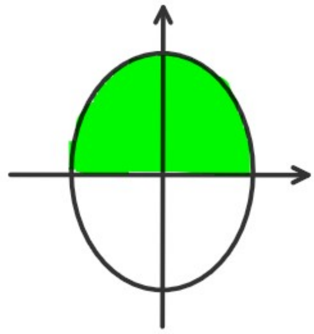
1)

a)

$$\iint_D \frac{9x^2y}{x^2+y^2}\,dx\,dy \qquad D=\left\{\begin{array}{l} x^2+y^2\le 1, \\ y\ge 0 \end{array}\right\}$$

uso le coordinate polari:

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\,\text{sen}\,\theta \end{cases} \qquad \begin{array}{l}\theta\in[0,\pi] \\ \rho\in[0,1]\end{array}$$



$$\Longrightarrow$$

$$\iint_D \frac{9x^2y}{x^2+y^2}\,dx\,dy =$$

$$=\int_0^1\left(\int_0^\pi \frac{9\rho^3\cos^2\theta\,\text{sen}\,\theta}{\rho^2}\,\overbrace{\rho}^{\text{Jacobiano}}\,d\theta\right)d\rho =$$

$$=\int_0^1 3\rho^2\left[-\cos^3\theta\right]_0^\pi\,d\rho =$$

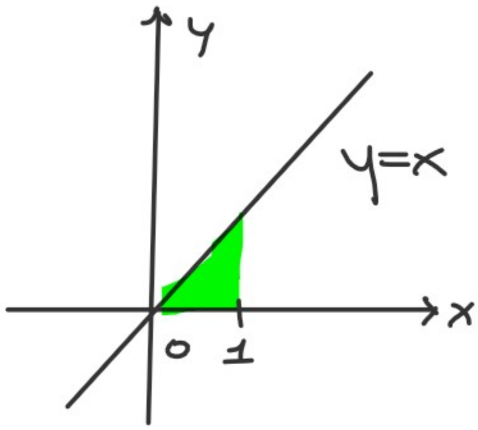
$$=\int_0^1 3\rho^2\left(-(-1)+1\right)d\rho =$$

$$=2\int_0^1 3\rho^2\,d\rho = 2\left[\rho^3\right]_0^1 = 2(1-0)=2.$$

b) $\iiint_T z\,dx\,dy\,dz$

$$T=\left\{0\le x\le 1,\ 0\le y\le x\ ,\ \frac{x+y}{2}\le z\le\sqrt{x+y}\right\}$$

$$\int_0^1 \left(\int_0^x \left(\int_{\frac{x+y}{2}}^{\sqrt{x+y}} z\,dz\right)dy\right)dx$$



$$= \int_0^1 \left(\int_0^x \left(\left[\frac{z^2}{2}\right]_{\frac{x+y}{2}}^{\sqrt{x+y}}\right)dy\right)dx$$

$$= \int_0^1 \left(\int_0^x \left(\frac{x+y}{2} - \frac{(x^2+y^2+2xy)}{8}\right)dy\right)dx$$

$$= \int_0^1 \left[\frac{xy}{2}+\frac{y^2}{2}-\frac{x^2y}{8}-\frac{y^3}{24}+\frac{xy^2}{8}\right]_0^x dx =$$

$$= \int_0^1 \left(\frac{x^2}{2}+\frac{x^2}{2}-\cancel{\frac{x^3}{8}}-\frac{x^3}{24}+\cancel{\frac{x^3}{8}}\right)dx =$$

$$= \int_0^1 \left(x^2-\frac{x^3}{24}\right)dx = \left[\frac{x^3}{3}-\frac{x^4}{4\cdot 24}\right]_0^1 = \frac{1}{3}-\frac{1}{4\cdot 24}.$$

2) $\int_\gamma e^{2x}\,ds$ $\qquad \gamma(x)=(x,e^x)$, $x\in[1,2]$

$= \int_1^2 e^{2x}\cdot\sqrt{1+e^{2x}}\,dx \qquad \gamma'(x)=(1,e^x)$

$\|\gamma'(x)\|=\sqrt{1+e^{2x}}$

$$= \left[\frac{(1+e^{2x})^{3/2}}{3/2\cdot 2}\right]_1^2 = \frac{(1+e^4)^{3/2}}{3}-\frac{(1+e^2)^{3/2}}{3}.$$

3) $F(x,y)=\left(y^2e^{xy},\ e^{xy}+xye^{xy}\right)$

$$\frac{\partial F_1}{\partial y}=2ye^{xy}+y^2e^{xy}\cdot x$$

$$\frac{\partial F_2}{\partial x} = y e^{xy} + y e^{xy} + xy e^{xy} y = 2y e^{xy} + xy^2 e^{xy}$$

irrotazionale su $\mathbb{R}^2$ s.c. $\Rightarrow$ conservativo.

potenziale: $g_x = F_1$, $g_y = F_2$.

$$\Rightarrow g = \int F_1 \, dx = \int y^2 e^{xy} \, dx = y e^{xy} + \phi(y)$$

$$g_y = e^{xy} + y e^{xy} \cdot x + \phi'(y)$$

$$g_y = F_2 \Rightarrow e^{xy} + xy e^{xy} + \phi'(y) = e^{xy} + xy e^{xy}$$

$$\Rightarrow \phi'(y) = 0 \Rightarrow \phi = c$$

$$\Rightarrow g(x,y) = y e^{xy} + c .$$

γ: segmento che congiunge $(0,1)$ e $(2,1)$.

$$L = \int_\gamma F \cdot T_\gamma \, ds = g(2,1) - g(0,1) = 1 \cdot e^{2 \cdot 1} - 1 \cdot e^{0 \cdot 1}$$
$$= e^2 - 1 .$$

4) $f(x,y,z) = 2xy$.

$$\Sigma = \left\{ \left(x, y, -\frac{x^2}{2} - \frac{y^2}{2}\right), (x,y) \in [0,1] \times [0,1] \right\}$$

$$\| \varphi_x \wedge \varphi_y \| = \sqrt{1 + (-x)^2 + (-y)^2} = \sqrt{1 + x^2 + y^2}$$

$$\int_\Sigma f \, d\sigma = \int_0^1 \left(\int_0^1 2xy \sqrt{1 + x^2 + y^2} \, dy \right) dx =$$

$$= \int_0^1 x \cdot \left[\frac{(1+x^2+y^2)^{3/2}}{3/2} \right]_0^1 dx =$$

$$= \int_0^1 \frac{2}{3} x \left((2+x^2)^{3/2} - (1+x^2)^{3/2} \right) dx =$$

$$= \frac{1}{3} \left[\frac{(2+x^2)^{5/2}}{5/2} - \frac{(1+x^2)^{5/2}}{5/2} \right]_0^1 =$$

$$= \frac{1}{3} \cdot \frac{2}{5} \left(3^{5/2} - 2^{5/2} - 2^{5/2} + 1 \right)$$

5) $\begin{cases} y' = (1+y)\,\mathrm{sen}\,x \\ y(2\pi) = 1 \end{cases}$

$$\frac{y'}{1+y'} = \mathrm{sen}\,x \qquad \int \frac{dy}{1+y} = \int \mathrm{sen}\,x \, dx \Rightarrow$$

$$\ln(1+y) = -\cos x + c \Rightarrow$$

$$1+y = e^{-\cos x + c} \Rightarrow \quad y = e^{-\cos x + c} - 1 .$$

$$y(2\pi) = e^{-\cos(2\pi) + c} - 1 = e^{-1+c} - 1 .$$

impongo $e^{-1+c} - 1 = 1 \Rightarrow \quad e^{c-1} = 2$

$$c - 1 = \ln 2 \Rightarrow c = \ln(2) + 1 .$$

soluzione: $y(x) = e^{-\cos x + \ln(2) + 1} - 1$

$$\left(y(x) = 2e \cdot e^{-\cos x} - 1 \right)$$

$$\left(y(x) = 2e \cdot e^{-\cos x} - 1\right)$$

6) $y'' - y' - 2y = 0$

$\lambda^2 - \lambda - 2 = 0 \qquad \Delta = 1 + 8 = 9$

$\lambda_{1,2} = \dfrac{1 \pm \sqrt{9}}{2} = \begin{cases} -1 \\ 2 \end{cases}$

$\Rightarrow y(x) = k_1 e^{-x} + k_2 e^{2x}.$

$y'' - y' - 2y = 2x + 1 :$

$y_s = ax + b.$

$y_s' = a$

$y_s'' = 0.$

$y_s'' - y_s' - 2y_s = 0 - a - 2(ax + b) =$

$= -2ax - a - 2b.$

impongo $-2ax - a - 2b = 2x + 1$

$\Rightarrow \begin{cases} -2a = 2 \\ -a - 2b = 1 \end{cases} \Rightarrow \begin{cases} a = -1 \\ 1 - 2b = 1 \end{cases}$

$a = -1, \quad -2b = 0 \Rightarrow b = 0$

$\Rightarrow y_s(x) = -x.$

$y(x) = k_1 e^{-x} + k_2 e^{2x} - x$

$$\Rightarrow y(x) = k_1 e^{-x} + k_2 e^{2x} - x .$$

riserve:

limite:

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$$

$x=0$ $\quad \lim_{y\to 0} \frac{0}{\sqrt{y^2}} = 0$

$y=0$ $\quad \lim_{x\to 0} \frac{x}{\sqrt{x^2}} = \lim_{x\to 0} \frac{x}{|x|}$ non esiste!

$\left(\frac{x}{|x|} = \pm 1\right)$

derivate composte:

$g(v,r) = (e^{v+r}, 2v+r)$. $\qquad F(v,r) = f(g(v,r))$

$$F_v = f_x \cdot x_v + f_y \cdot y_v = e^{v+r} f_x + 2 f_y .$$

$$F_r = f_x \cdot x_r + f_y \cdot y_r = e^{v+r} \cdot f_x + f_y .$$

$$F_{vv} = (e^{v+r} f_x)_v + (2 f_y)_v =$$

$$= e^{v+r} \cdot f_x + e^{v+r}(f_{xx} \cdot e^{v+r} + 2 f_{xy}) + 2(f_{yx} e^{v+r} + 2 f_{yy})$$

$$= e^{v+r} f_x + e^{2(v+r)} f_{xx} + 2e^{v+r} f_{yy} + 2e^{v+r} f_{yx} + 4f_{yy}$$

$$= e^{v+r}\left(f_x + e^{v+r} f_{xx} + 2f_{yy} + 2f_{yx}\right) + 4f_{yy}.$$

$$F_{wr} = e^{v+r}\left(f_x + e^{v+r} f_{xx} + 2f_{yy} + 2f_{yx}\right) +$$

$$+ e^{v+r}\left(f_{xx} e^{v+r} + f_{xy} + e^{v+r} f_{xx} + e^{v+r}(f_{xxx} e^{v+r} + f_{xxy}) + 2f_{yyx} e^{v+r} + 2f_{yyy} + 2f_{yx} e^{v+r} + 2f_{yxy}\right) +$$

$$+ 4\left(f_{yyx} e^{v+r} + f_{yyy}\right).$$

MAX / MIN vincolo: $x^2 + y^2 \leq 4$

$f(x,y) = 4y + 2xy$

$f_x = 2y \qquad f_y = 4 + 2x$

$f_{xx} = 0 \quad f_{xy} = 2 \quad f_{yy} = 0$

$$\begin{cases} 2y = 0 \\ 4 + 2x = 0 \end{cases} \qquad \begin{cases} y = 0 \\ x = -2 \end{cases}$$

$$Hf(-2,0) = \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} < 0 \quad \text{sella.}$$

$g(x,y) = x^2 + y^2 \leq 4. \qquad \nabla f = \lambda \nabla g,\ g = 0.$

$$\begin{cases} 2y = 2\lambda x \\ 4 + 2x = 2\lambda y \end{cases} \qquad \begin{cases} \lambda = \frac{y}{x} \\ \ldots \end{cases} \qquad x \neq 0$$

$$\begin{cases} 2y = 2\lambda x \\ 4+2x = 2\lambda y \\ x^2+y^2=4 \end{cases} \qquad \begin{cases} \lambda = \frac{y}{x} & x\neq 0 \\ \lambda = \frac{2+x}{y} & y \neq 0 \\ x^2+y^2=4 \end{cases}$$

$$\frac{y}{x} = \frac{2+x}{y} \Rightarrow y^2 = 2x+x^2$$

$$\overset{\text{III}}{\Rightarrow} x^2+(2x+x^2)=4 \qquad 2x^2+2x-4=0$$

$$x^2+x-2=0 \qquad \Delta = 1+4\cdot 2 = 9$$

$$x_{1,2} = \frac{-1\pm 3}{2} = \begin{cases} -2 \\ 1 \end{cases}$$

$$y^2 = 2x+x^2 \Rightarrow y = \pm\sqrt{2x+x^2}$$

$$x=-2 \Rightarrow y = \pm\sqrt{-4+4} = 0 \qquad \lambda = \frac{0}{x} = 0 \qquad (-2,0,0)$$

$$x = 1 \Rightarrow y = \pm\sqrt{2+1} = \pm\sqrt{3} \qquad \lambda = \frac{y}{x} = \pm\sqrt{3}$$

$$(1,-\sqrt{3},-\sqrt{3}),\ (1,\sqrt{3},\sqrt{3})$$

$$f(-2,0) = 4\cdot 0 + 2\cdot(-2)\cdot 0 = 0$$

$$f(1,-\sqrt{3}) = 4(-\sqrt{3}) + 2\cdot 1\cdot(-\sqrt{3}) = -4\sqrt{3} - 2\sqrt{3} = -6\sqrt{3} \quad \text{MIN}$$

$$f(1,\sqrt{3}) = 4\sqrt{3} + 2\cdot 1\cdot\sqrt{3} = 6\sqrt{3} \quad \text{MAX}$$