Some Extremal Principles of Mechanics

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Linear elasticity

In Cartesian coordinates, the equations of motion are given by

$$\sigma_{ij,i} + f_j = \rho u_{j,tt} \qquad i, j = 1, 2, 3, \tag{1}$$

where σ_{ij} are the components of the stress tensor, u_i are the components of the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$, f_i are the components of the volume force vector $\mathbf{f} = (f_1, f_2, f_3)$, and ρ is the material density. The stress components σ_{ij} describe the force interactions between portions of an elastic body. The following notation will be used for partial differentiation with respect to the spatial coordinates x_i and time t:

$$(\cdot)_{,i} = \frac{\partial(\cdot)}{\partial x_i}, \qquad (\cdot)_{,t} = \frac{\partial(\cdot)}{\partial t}.$$

In this chapter we modify our notation for partial derivatives in order to use Einstein's convention for repeated subscripts i, j, k, l, m, n.

Motion equations and strains

Equation (1) may be written as

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho \frac{\partial^2 u_j}{\partial t^2}, \qquad j = 1, 2, 3.$$

In elasticity it is shown that the matrix $[\sigma_{ij}]$ is symmetric so that $\sigma_{ij} = \sigma_{ji}$. The Cartesian components ε_{sk} of the strain tensor are given by

$$\varepsilon_{sk} = \frac{1}{2}(u_{s,k} + u_{k,s}). \tag{2}$$

By definition, the matrix $[\varepsilon_{sk}]$ is also symmetric.

Hooke's law

The stress and strain tensors are related by the generalized version of Hooke's law. For an isotropic body this takes the form

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{3}$$

where μ and λ are *Lamé's constants*. Relation (3) is an extension of Hooke's law for the rod: $\sigma = E\varepsilon$. Young's modulus E may be obtained from Lamé's constants as

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

Hooke's law

A more general form of Hooke's law for an anisotropic body is

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn}$$

where the *elastic moduli* C_{ijmn} (the components of a tensor of elastic moduli) satisfy

$$C_{ijmn} = C_{jimn} = C_{ijnm} = C_{mnij}.$$

As a result, the set C_{ijmn} consists of no more than 21 independent constants. For an isotropic body, the number of independent elastic constants is two; they can be chosen as the Lamé constants μ and λ so that

$$C_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}),$$

where δ_{ij} is Kronecker's symbol.

Motion equations

Substitution of (3) and (2) into (1) yields the equations of motion in terms of the displacements:

$$(\lambda + \mu) u_{i,ki} + \mu u_{k,ii} + \rho f_k = \rho u_{k,tt} \qquad i, k = 1, 2, 3. \tag{4}$$

For equilibrium problems, the equations of motion reduce to

$$\sigma_{ij,i} + f_j = 0, \qquad j = 1, 2, 3 \tag{5}$$

or, in terms of displacements,

$$(\lambda + \mu) u_{i,ki} + \mu u_{k,ii} + f_k = 0, \qquad k = 1, 2, 3. \tag{6}$$

Boundary conditions

Two types of boundary conditions occur in the formulation of boundary value problems in elasticity. Suppose the boundary $S = \partial V$ of a body consists of two nonoverlapping portions S_1 and S_2 so that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. If the displacement vector is given on S_1, we have a boundary condition of the form

$$u_i\Big|_{S_1} = u_i^0, \qquad i = 1, 2, 3, \tag{7}$$

where u_i^0 is a given function. If external forces (p_1, p_2, p_3) act over S_2, the condition is

$$n_i\sigma_{ij}\Big|_{S_2} = p_j, \qquad j = 1, 2, 3, \tag{8}$$

where the p_j are given functions and n_i are the components of the outward unit normal to S.

The dynamic problems of elasticity also require initial conditions of the form

$$u_i\Big|_{t=0} = \hat{u}_i, \quad u_{i,t}\Big|_{t=0} = \hat{v}_i, \qquad i = 1, 2, 3.$$

Strain energy density

In elasticity, the strain energy function W is introduced as a quadratic function of the ε_{mn}:

$$W(\varepsilon_{mn}) = \frac{1}{2}\varepsilon_{ij}C_{ijmn}\varepsilon_{mn}.$$

For an isotropic material this reduces to

$$W(\varepsilon_{mn}) = \frac{1}{2}\lambda\varepsilon_{ii}^2 + \mu\varepsilon_{ij}\varepsilon_{ij}. \tag{9}$$

From thermodynamic considerations it follows that W is positive definite:

$$W(\varepsilon_{mn}) > 0 \quad \text{whenever } \varepsilon_{mn} \neq 0. \tag{10}$$

This implies the following inequalities for the elastic moduli:

$$3\lambda + 2\mu > 0, \qquad \mu > 0. \tag{11}$$

It can be shown that W is the potential for stresses:

$$\sigma_{ij} = W_{,\varepsilon_{ij}}.$$

Lagrange's variational principle

The existence of W allows us to formulate *Lagrange's variational principle for elasticity*:

Theorem

A stationary point $\mathbf{u} = (u_1, u_2, u_3)$ *of the total potential energy functional*

$$\mathcal{E}(\mathbf{u}) = \iiint_V W(\varepsilon_{mn})\, dV - \iiint_V f_i u_i\, dV - \iint_{S_2} p_i u_i\, dS$$

on the set of admissible displacements subject to (7) *satisfies the equilibrium equations* (5) *in the volume* V *and the boundary condition* (8). *The converse also holds. This stationary point is the minimum of* $\mathcal{E}$.

Proof.

Using the formula

$$\delta\mathcal{E} = \frac{d}{d\tau}\mathcal{E}(\mathbf{u} + \tau\boldsymbol{\varphi})\Big|_{\tau=0}, \qquad \boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3),$$

let us find the first variation of $\mathcal{E}$:

$$\begin{aligned}\delta\mathcal{E} &= \iiint_V \frac{1}{2} W_{,\varepsilon_{ij}}(\varphi_{i,j} + \varphi_{j,i})\, dV - \iiint_V f_i\varphi_i\, dV - \iint_{S_2} p_i\varphi_i\, dS \\ &= \iiint_V W_{,\varepsilon_{ij}}\varphi_{j,i}\, dV - \iiint_V f_i\varphi_i\, dV - \iint_{S_2} p_i\varphi_i\, dS \\ &= \iiint_V \sigma_{ij}\varphi_{j,i}\, dV - \iiint_V f_i\varphi_i\, dV - \iint_{S_2} p_i\varphi_i\, dS.\end{aligned}$$

□

Proof.

We show that if $\delta\mathcal{E} = 0$ for all admissible φ_i, then (5) and (8) hold. The Gauss–Ostrogradski formula gives

$$\begin{aligned} 0 = \delta\mathcal{E} &= \iiint_V \sigma_{ij}\varphi_{j,i}\,dV - \iiint_V f_i\varphi_i\,dV - \iint_{S_2} p_i\varphi_i\,dS \\ &= -\iiint_V \left(\sigma_{ij,i} + f_j\right)\varphi_j\,dV + \iint_{S_1} n_k\sigma_{kj}\varphi_j\,dS \\ &\quad + \iint_{S_2} (n_k\sigma_{kj} - p_j)\varphi_j\,dS. \end{aligned}$$

Recall that the φ_i satisfy the homogeneous version of (7), i.e., $\varphi_i|_{S_1} = 0$.
□

Proof.

From the arbitrariness of φ_i, a two-step derivation (first for the volume integrals where we take $\varphi = 0$ on S_2, then for the surface integrals) yields the required equations

$$\sigma_{ij,i} + f_j = 0 \text{ in } V, \qquad n_k \sigma_{kj}\big|_{S_2} = p_j.$$

□

Proof.

Conversely, on a solution $\mathbf{u}$ of the equilibrium problem we have $\delta\mathcal{E} = 0$ for any admissible φ_i that vanishes on S_1. Indeed, multiply the ith equation of (5) by φ_i, add the results, then integrate over V. Using a similar formula obtained by multiplying the ith equation of (8) by φ_i, summing, and integrating over S_2, we get

$$\begin{aligned}
0 &= \iiint_V (\sigma_{ki,k} + f_i)\, \varphi_i \, dV - \iint_{S_2} (n_k \sigma_{ki} - p_i) \varphi_i \, dS \\
&= -\iiint_V \sigma_{kj} \varphi_{j,k} \, dV + \iiint_V f_i \varphi_i \, dV \\
&\quad + \iint_S n_k \sigma_{ki} \varphi_i \, dS - \iint_{S_2} (n_k \sigma_{ki} - p_i) \varphi_i \, dS \\
&= -\iiint_V \sigma_{kj} \varphi_{j,k} \, dV + \iiint_V f_i \varphi_i \, dV + \iint_{S_2} p_i \varphi_i \, dS \\
&= -\delta\mathcal{E}.
\end{aligned}$$

Proof.

Hence a stationary point of $\mathcal{E}$ is a solution to the equilibrium problem for the elastic body, and vice versa.

Finally we show that $\mathcal{E}$ attains its minimum at the stationary point. The proof uses the fact that W is a positive definite quadratic form in the strain components. Let $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ be another admissible vector function satisfying (7) and consider the difference

$$\Delta\mathcal{E} = \mathcal{E}(\tilde{\mathbf{u}}) - \mathcal{E}(\mathbf{u}).$$

□

Proof.

We get

$$\begin{aligned}\Delta\mathcal{E} &= \iiint_V W(\tilde{\varepsilon}_{mn})\,dV - \iiint_V f_i\tilde{u}_i\,dV - \iint_{S_2} p_i\tilde{u}_i\,dS\\ &\quad - \iiint_V W(\varepsilon_{mn})\,dV + \iiint_V f_i u_i\,dV + \iint_{S_2} p_i u_i\,dS\\ &= \iiint_V [W(\tilde{\varepsilon}_{mn}) - W(\varepsilon_{mn})]\,dV\\ &\quad - \iiint_V f_i(\tilde{u}_i - u_i)\,dV - \iint_{S_2} p_i(\tilde{u}_i - u_i)\,dS.\end{aligned}$$

□

Proof.

Let $\varphi_i = \tilde{u}_i - u_i$. Because $\tilde{u}_i$ and u_i coincide on S_1, we have $\varphi_i|_{S_1} = 0$. Next,

$$\begin{aligned}
2\left[W(\tilde{\varepsilon}_{mn}) - W(\varepsilon_{mn})\right] &= \lambda\tilde{\varepsilon}_{ii}^2 + 2\mu\tilde{\varepsilon}_{ij}\tilde{\varepsilon}_{ij} - \lambda\varepsilon_{ii}^2 - 2\mu\varepsilon_{ij}\varepsilon_{ij} \\
&= \lambda\tilde{\tilde{\varepsilon}}_{ii}^2 + 2\mu\tilde{\tilde{\varepsilon}}_{ij}\tilde{\tilde{\varepsilon}}_{ij} + 2\lambda\tilde{\tilde{\varepsilon}}_{ii}\tilde{\varepsilon}_{ii} + 4\mu\tilde{\tilde{\varepsilon}}_{ij}\tilde{\varepsilon}_{ij} \\
&= 2W(\tilde{\tilde{\varepsilon}}_{mn}) + 2\lambda\tilde{\tilde{\varepsilon}}_{ii}\tilde{\varepsilon}_{ii} + 4\mu\tilde{\tilde{\varepsilon}}_{ij}\tilde{\varepsilon}_{ij}
\end{aligned}$$

where

$$\tilde{\varepsilon}_{mn} = \frac{1}{2}(\tilde{u}_{m,n} + \tilde{u}_{n,m}), \qquad \tilde{\tilde{\varepsilon}}_{mn} = \frac{1}{2}(\varphi_{m,n} + \varphi_{n,m}).$$

□

Proof.

Therefore

$$\begin{aligned}
\Delta\mathcal{E} &= \iiint_V W(\tilde{\tilde{\varepsilon}}_{mn})\,dV + \iiint_V (\lambda\varepsilon_{ii}\tilde{\tilde{\varepsilon}}_{ii} + 2\mu\varepsilon_{ij}\tilde{\tilde{\varepsilon}}_{ij})\,dV \\
&\quad - \iiint_V f_i\varphi_i\,dV - \iint_{S_2} p_i\varphi_i\,dS \\
&= \iiint_V W(\tilde{\tilde{\varepsilon}}_{mn})\,dV + \iiint_V \sigma_{ij}\varphi_{j,i}\,dV \\
&\quad - \iiint_V f_i\varphi_i\,dV - \iint_{S_2} p_i\varphi_i\,dS \\
&= \iiint_V W(\tilde{\tilde{\varepsilon}}_{mn})\,dV + \delta\mathcal{E}.
\end{aligned}$$

$\square$

Proof.

Because $\mathbf{u} = (u_1, u_2, u_3)$ is a solution, the first variation $\delta\mathcal{E} = 0$ for any admissible φ and we have

$$\Delta\mathcal{E} = \iiint_V W(\tilde{\tilde{\varepsilon}}_{mn})\, dV. \tag{12}$$

The positive definiteness of W means that $\Delta\mathcal{E} \geq 0$ for any admissible $\tilde{u}_i$. Hence the set of u_i are a global minimizer of $\mathcal{E}$. $\square$

Virtual work principle

Theorem

Sufficiently smooth functions u_i *that vanish on* S_1 *are a solution to the boundary value problem* (5), (7), (8), *if and only if the equation*

$$\iiint_V \sigma_{ij}\varphi_{j,i}\,dV - \iiint_V f_i\varphi_i\,dV - \iint_{S_2} p_i\varphi_i\,dS = 0, \tag{13}$$

with σ_{ij} *given by* (3), *holds for any sufficiently smooth functions* φ_i *that also vanish on* ∂S_1.

The virtual work principle underlies the notion of weak solutions in elasticity. It is more general than Lagrange's principle as it can be extended to nonconservative systems for which total potential energy functionals do not exist.

Hamilton’s least action principle

Hamilton’s least action principle is the basis for variational formulations in dynamics. Let the kinetic energy density of a body be given by

$$K = \frac{1}{2}\rho(u_{1,t}^2 + u_{2,t}^2 + u_{3,t}^2).$$

In this case we say that a function is admissible if it (1) vanishes on S_1 and (2) takes the values of a solution to the dynamical problem at time instants t_1 and t_2. This means that we consider the admissible variations $\varphi_i(x_1, x_2, x_3, t)$ of the solution to the problem such that $\varphi_i\big|_{S_1} = 0$ and $\varphi_i\big|_{t=t_1} = 0 = \varphi_i\big|_{t=t_2}$.

Hamilton’s variational principle

Theorem

A solution to a boundary value problem in the dynamics of elastic solids (i.e., a solution to (1), (7), *and* (8)*) is a stationary point of the* ***action functional***

$$\mathcal{E}_A(\mathbf{u}) = \int_{t_1}^{t_2} \left(\iiint_V (K - W)\, dV + \iiint_V f_i u_i \, dV + \iint_{S_2} p_i u_i \, dS \right) dt$$

in the class of admissible functions that satisfy (7) *and take prescribed values coincident with the solution at time instants t_1 and t_2. Conversely, a stationary point of $\mathcal{E}_A$ in the class of admissible functions is a solution to the dynamical boundary value problem for an elastic body.*

Hamilton’s variational principle

Proof.

The first variation of $\mathcal{E}_A$ is

$$\delta\mathcal{E}_A = \int_{t_1}^{t_2} \left(\iiint_V (\rho u_{i,t}\varphi_{i,t} - \sigma_{ij}\varphi_{j,i} + f_i\varphi_i)\, dV + \iint_{S_2} p_i\varphi_i\, dS \right) dt.$$

Integrating by parts, we have

$$\delta\mathcal{E}_A = \int_{t_1}^{t_2} \left(\iiint_V (-\rho u_{i,tt}\varphi_i + \sigma_{ij,i}\varphi_j + f_i\varphi_i)\, dV - \iint_S n_k\sigma_{kj}\varphi_j\, dS \right.$$
$$\left. + \iint_{S_2} p_i\varphi_i\, dS \right) dt + \iiint_V \rho u_{i,t}\varphi_i\, dV \Big|_{t=t_1}^{t=t_2}.$$

□

Proof.

But $\varphi_i\big|_{S_1} = 0$ and $\varphi_i\big|_{t=t_1} = 0 = \varphi_i\big|_{t=t_2}$, so

$$\delta\mathcal{E}_A = \int_{t_1}^{t_2} \left(\iiint_V (-\rho u_{j,tt} + \sigma_{ij,i} + f_j)\varphi_j \, dV \right. \\ \left. - \iint_S (n_k\sigma_{kj} + p_j)\varphi_j \, dS \right) dt.$$

Hence if $\delta\mathcal{E}_A = 0$ for all admissible φ_i, the equations of motion (1) and the boundary conditions (8) follow.

Conversely, if $\mathbf{u}$ is a solution to the dynamic problem, the first variation of $\mathcal{E}_A$ is zero. The proof is similar to the proof of the corresponding part of Lagrange's principle. The difference lies in the sets of admissible functions and in the domain of integration, which for Hamilton's principle is $V \times [t_1, t_2]$.

□

Comments

We should note that Hamilton’s principle is not minimal; it yields stationary points of the action functional.

Other variational principles in elasticity bear names such as Castigliano, Reissner, Washizu, Tonti, and Hashin-Strikman. Some are minimal or maximal like Lagrange’s principle; others are stationary like Hamilton’s principle.

In addition to their roles in proving existence theorems, they form the basis for practical engineering approaches such as the finite element method. Moreover, extensions of variational methods turned out to be useful in the theory of more complex problems in nonlinear elasticity, plasticity, viscoelasticity, and so on.

Reissner–Mindlin plate theory

In Reissner–Mindlin plate theory, the bending of an elastic plate is described by the equations

$$M_{11,1} + M_{21,2} - Q_1 = \rho J \vartheta_{1,tt}, \tag{14}$$

$$M_{12,1} + M_{22,2} - Q_2 = \rho J \vartheta_{2,tt}, \tag{15}$$

$$Q_{1,1} + Q_{2,2} + p = \rho h w_{,tt}, \tag{16}$$

where the $M_{\alpha\beta}$ are the bending and twisting moments ($\alpha, \beta = 1, 2$), the Q_α are the transverse shear forces, the ϑ_α are the averaged rotations of fibers normal to the plate midsurface before deformation, w is the deflection, ρ is the density, J is the moment of inertia, h is the plate thickness, and p is the transverse load.

Constitutive equations

The constitutive equations — i.e., the relations between the bending and twisting moments, the transverse shear forces, and the surface strain measures — are given by

$$M_{11} = D(\vartheta_{1,1} + \nu\vartheta_{2,2}), \qquad M_{22} = D(\vartheta_{2,2} + \nu\vartheta_{1,1}), \tag{17}$$

$$M_{12} = M_{21} = \frac{D(1-\nu)}{2}(\vartheta_{1,2} + \vartheta_{2,1}), \tag{18}$$

$$Q_1 = \Gamma(w_{,1} + \vartheta_1), \qquad Q_2 = \Gamma(w_{,2} + \vartheta_2), \tag{19}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad \Gamma = k\mu h, \tag{20}$$

where E is Young's modulus, μ is the shear modulus, ν is Poisson's ratio, D is the bending stiffness, Γ is the transverse shear stiffness, and k is the shear correction factor. For k, Reissner proposed $k = 5/6$ whereas Mindlin took $k = \pi^2/12$. Other values of k also appear in the literature.

Boundary conditions

In this theory, on the boundary contour ∂S or a portion ∂S_1, kinematic boundary conditions consist of given deflections and rotations:

$$w\big|_{\partial S_1} = w^0, \qquad \vartheta_\alpha\big|_{\partial S_1} = \vartheta_\alpha^0. \tag{21}$$

Static boundary conditions are

$$n_\alpha M_{\alpha\beta}\big|_{\partial S_2} = M_\beta^0, \qquad Q_\alpha n_\alpha\big|_{\partial S_2} = Q_n^0. \tag{22}$$

In (21)–(22), the quantities w^0, θ_α^0, M_β^0, and Q_n^0 are given functions of the arc-length parameter s. The quantities n_1 and n_2 are the components of the outward unit normal to ∂S.

Equilibrium equations

In equilibrium, equations (14)–(16) reduce to

$$M_{11,1} + M_{21,2} - Q_1 = 0, \tag{23}$$

$$M_{12,1} + M_{22,2} - Q_2 = 0, \tag{24}$$

$$Q_{1,1} + Q_{2,2} + p = 0. \tag{25}$$

Solving (23) and (24) for Q_1 and Q_2, and substituting these into (25), we obtain

$$M_{11,11} + 2M_{12,12} + M_{22,22} + p = 0. \tag{26}$$

Strain energy density

The strain energy density for plate bending is

$$\begin{aligned} W(\varkappa_{\alpha\beta}, \gamma_\alpha) &= \frac{1}{2}\left[M_{\alpha\beta}\varkappa_{\alpha\beta} + Q_\alpha\gamma_\alpha\right] \\ &= \frac{D}{2}\left[\varkappa_{11}^2 + \varkappa_{22}^2 + 2\nu\varkappa_{11}\varkappa_{22} + \frac{1-\nu}{2}(\varkappa_{12}^2 + 2\varkappa_{12}\varkappa_{21} + \varkappa_{21}^2)\right] \\ &\quad + \frac{\Gamma}{2}(\gamma_1^2 + \gamma_2^2), \end{aligned}$$

where $\varkappa_{\alpha\beta}$ are the components of the bending tensor (or tensor of change of curvature), and γ_α are the shear strain components defined by

$$\varkappa_{\alpha\beta} = \vartheta_{\alpha,\beta}, \quad \gamma_\alpha = w_{,\alpha} + \vartheta_\alpha.$$

It can be directly verified that

$$M_{\alpha\beta} = \frac{\partial W}{\partial \varkappa_{\alpha\beta}}, \qquad Q_\alpha = \frac{\partial W}{\partial \gamma_\alpha}. \tag{27}$$

Kinetic energy

In the Reissner–Mindlin plate theory, the kinetic energy density is

$$K = \frac{\rho h}{2}(w_{,t})^2 + \frac{\rho J}{2}\left[(\vartheta_{2,t})^2 + (\vartheta_{2,t})^2\right].$$

Plate theory features variational principles similar to those in linear elasticity.

Lagrange's variational principle

Theorem

A solution of boundary value problem (23)–(25), (21), (22) *is a stationary point of the energy functional*

$$\mathcal{E}(w,\vartheta_1,\vartheta_2)=\iint_S W\,dS-\iint_S pw\,dS-\int_{\partial S_2}\left(Q_n^0 w+M_\beta^0\vartheta_\beta\right)ds. \tag{28}$$

Conversely, sufficiently smooth functions ϑ_α *and* w *that constitute a stationary point of* $\mathcal{E}$ *in the class of all admissible functions (i.e., satisfying the kinematic boundary conditions* (21)*), satisfy the equilibrium equations* (23)–(25) *and boundary conditions* (22)*. Moreover, at a stationary point* $\mathcal{E}$ *takes its global minimum value.*

Virtual work principle

Theorem

Sufficiently smooth functions w, ϑ_α *that vanish on* ∂S_1 *constitute a solution of the boundary value problem* (23)–(25), (21), (22) *if and only if the equation*

$$\iint_S (M_{\alpha\beta}\varphi_{\beta,\alpha} + Q_\alpha\varphi_{0,\alpha} + Q_\alpha\varphi_\alpha - p\varphi_0)\, dS - \int_{\partial S_2} (Q_n^0\varphi_0 + M_\beta^0\varphi_\beta)\, ds = 0, \tag{29}$$

with $M_{\alpha\beta}$ *and* Q_α *given by* (17)–(19), *holds for any sufficiently smooth functions* $\varphi_0, \varphi_1, \varphi_2$ *that also vanish on* ∂S_1.

Equation (29) forms the basis for various versions of the finite element method in plate theory.

Hamilton’s variational principle

Hamilton’s variational principle holds for dynamic problems in plate theory:

Theorem

A solution to the dynamical boundary value problem (14)–(16), (21), (22) *is a stationary point of the action functional*

$$\mathcal{E}_A = \int_{t_1}^{t_2} \left(\iint_S (K - W)\, dS + \iint_S pw\, dS + \int_{\partial S_2} (Q_n^0 w + M_\beta^0 \vartheta_\beta)\, ds \right) dt.$$

in the class of admissible functions (i.e., satisfying (21) *and taking prescribed values coincident with the solution at times t_1 and t_2). Conversely, a stationary point of $\mathcal{E}_A$ in the class of admissible functions is a solution of the dynamical boundary value problem for the plate.*

Kirchhoff plate theory

The classical Kirchhoff theory is easily derived from the Reissner–Mindlin theory. In the former, the rotations ϑ_α and the deflection w are related by

$$\vartheta_1 = -w_{,1}, \qquad \vartheta_2 = -w_{,2}. \tag{30}$$

So in Kirchhoff theory, bending of the plate is described by one function: the deflection $w(x, y, t)$. This allows us to return to a simpler notation for the partial derivatives of w. We shall write $w_{,1} = w_x$, $w_{,12} = w_{xy}$, etc. The constitutive equations for the moments now take the form

$$M_{11} = -D(w_{xx} + \nu w_{yy}), \qquad M_{22} = -D(w_{yy} + \nu w_{xx}), \tag{31}$$

$$M_{12} = M_{21} = -D(1 - \nu) w_{xy}. \tag{32}$$

Equilibrium equation, etc.

Equilibrium equation (26) reduces to an equation in w that we saw in Chapter 1:

$$D\Delta w + p = 0. \tag{33}$$

The strain energy functional for a Kirchhoff plate is

$$\mathcal{E}(w) = \iint_S W\, dS - \iint_S pw\, dS, \tag{34}$$

$$W = \frac{D}{2}\left[w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx}w_{yy} + 2(1-\nu)w_{xy}^2\right]. \tag{35}$$

To avoid awkward formulas we assumed here an absence of boundary loads, i.e., $Q_n^0 = M_\beta^0 = 0$. In Kirchhoff's plate theory the kinetic energy becomes

$$K = \frac{\rho h}{2}(w_t)^2.$$

Hamilton's variational principle reduces to finding stationary points of the functional

$$\mathcal{E}_A(w) = \int_{t_1}^{t_2} (K - W + pw)\, dS\, dt$$

Interaction of a plate with elastic beams

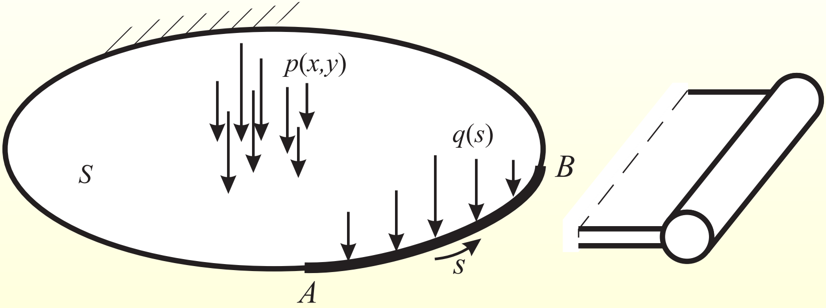

Figure: Left: plate with part of its boundary contour supported by beam AB. Right: detail of the beam support.

A cantilever plate

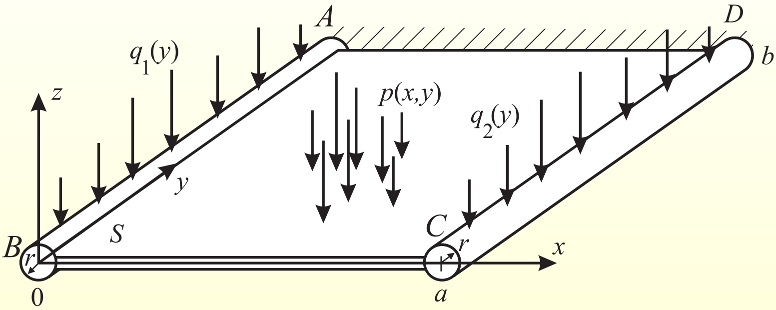

Figure: A cantilever plate supported by two straight beams.

Energy

The total potential energy functional of the plate with two reinforcement beams is

$$\begin{aligned}\mathcal{E}(w,\vartheta_1,\vartheta_2,u_1,u_2,\psi_1,\psi_2) &= \iint_S W\,dx\,dy - \iint_S pw\,dx\,dy \\ &+ \mathcal{E}_{b1}(u_1) + \mathcal{E}_{b2}(u_2) + \mathcal{E}_{t1}(\psi_1) + \mathcal{E}_{t2}(\psi_2).\end{aligned} \tag{36}$$

Here the energy functionals for beam bending for AB and CD are

$$\mathcal{E}_{b1}(u_1) = \frac{1}{2}\int_0^b E_1 I_1 (u_1''(y))^2\,dy - \int_0^b q_1(y)u_1(y)\,dy,$$

$$\mathcal{E}_{b2}(u_2) = \frac{1}{2}\int_0^b E_2 I_2 (u_2''(y))^2\,dy - \int_0^b q_2(y)u_2(y)\,dy,$$

where the E_α are Young's moduli, I_α are the moments of inertia of the beams, and $u_\alpha(y)$ are the vertical beam deflections. The energy functionals for torsion in the beams AB and CD are

$$\mathcal{E}_{t1}(\psi_1) = \frac{1}{2}\int_0^b D_{T1}(\psi_1'(y))^2\,dy, \quad \mathcal{E}_{t2}(\psi_2) = \frac{1}{2}\int_0^b D_{T2}(\psi_2'(y))^2\,dy.$$

Kinematic boundary conditions

Kinematic boundary conditions are the equations that describe rigid clamping of the plate along AD, clamping of the beams at points A and D, and the equality of the twisting angle to zero at A and D:

$$\begin{aligned} & w\big|_{y=b} = 0 = \vartheta_1\big|_{y=b} = \vartheta_1\big|_{y=b}\,, \\ & u_1\big|_{y=b} = u_2\big|_{y=b} = 0 = \psi_1\big|_{y=b} = \psi_2\big|_{y=b}. \end{aligned} \tag{37}$$

Kinematic compatibility conditions

Kinematic compatibility of deformation for the plate and beams requires equality between the defections of the plate edges and the beams,

$$u_1(y) = w(0,y) - r\psi_1(y), \quad u_2(y) = w(a,y) + r\psi_2(y), \tag{38}$$

and equality of the corresponding rotation angles:

$$\psi_1(y) = \vartheta_1(0,y), \quad \psi_2(y) = \vartheta_1(a,y). \tag{39}$$

The kinematic compatibility conditions (38) describe coupling between a plate and a pair of beams having circular cross sections of radius r as in Fig. 2. Clearly this is not the only way to fix beams to a plate. For beams of more complicated cross section, the kinematic compatibility conditions can differ from (38); however, the analysis will be similar.

Energy functional

By (38) and (39), the energy functional takes the form

$$\begin{aligned}\mathcal{E}(w,\vartheta_1,\vartheta_2) &= \iint_S W\,dx\,dy - \iint_S pw\,dx\,dy \\ &+ \mathcal{E}_{b1}(w(0,y) - r\vartheta_1(0,y)) + \mathcal{E}_{b2}(w(a,y) + r\vartheta_1(a,y)) \\ &+ \mathcal{E}_{t1}(\vartheta_1(0,y)) + \mathcal{E}_{t2}(\vartheta_1(a,y)). \qquad (40)\end{aligned}$$

Natural boundary conditions

Natural boundary conditions for the plate follow from the condition $\delta\mathcal{E} = 0$. We have

$$\begin{aligned}
0 = \delta\mathcal{E} = & -\int_0^a \int_0^b \left[(M_{\alpha\beta,\alpha} - Q_\beta)\varphi_\beta + (Q_{\alpha,\alpha} + p)\varphi_0 \right] dx\, dy \\
& + \int_{\partial S_2} (n_\alpha M_{\alpha\beta}\varphi_\alpha + n_\alpha Q_\alpha \varphi_0)\, ds \\
& + \int_0^b [E_1 I_1 (w_{yy} - r\vartheta_{1yy})(\varphi_{0yy} - r\varphi_{1yy}) - q_1(\varphi_0 - r\varphi_1)] \Big|_{x=0} dy \\
& + \int_0^b [E_2 I_2 (w_{yy} + r\vartheta_{1yy})(\varphi_{0yy} + r\varphi_{1yy}) - q_2(\varphi_0 + r\varphi_1)] \Big|_{x=a} dy \\
& + \int_0^b D_{T1}\vartheta_{1y}\varphi_{1y}\Big|_{x=0} dy + \int_0^b D_{T2}\vartheta_{1y}\varphi_{1y}\Big|_{x=a} dy
\end{aligned}$$

after use of integration by parts.

Natural boundary conditions

Finally, we get the following set of natural boundary conditions:

$$\begin{aligned}
AB&: \quad M_{11} - D_{T1}\vartheta_{1yy} + rQ_1 = 0, \quad M_{12} = 0,\\
&\quad\ Q_1 + E_1I_1(w_{yyyy} - r\vartheta_{1yyyy}) - q_1 = 0,\\
BC&: \quad M_{21} = 0, \quad M_{22} = 0, \quad Q_2 = 0,\\
CD&: \quad M_{11} - D_{T2}\vartheta_{1yy} - rQ_1 = 0, \quad M_{12} = 0,\\
&\quad\ Q_1 + E_2I_2(w_{yyyy} + r\vartheta_{1yyyy}) - q_2 = 0.
\end{aligned}$$

At the corner $B = (0, 0)$ the conditions

$$w_{yy} - r\vartheta_{1yy} = w_{yyy} - r\vartheta_{1yyy} = 0, \quad \vartheta_{1y} = 0$$

hold, at the corner $C = (a, 0)$

$$w_{yy} + r\vartheta_{1yy} = w_{yyy} + r\vartheta_{1yyy} = 0, \quad \vartheta_{1y} = 0$$

hold, while at $A = (0, b)$ and $D = (a, b)$ we have

$$w_{yy} - r\vartheta_{1yy} = 0 \quad \text{and} \quad w_{yy} + r\vartheta_{1yy} = 0.$$

Another Example

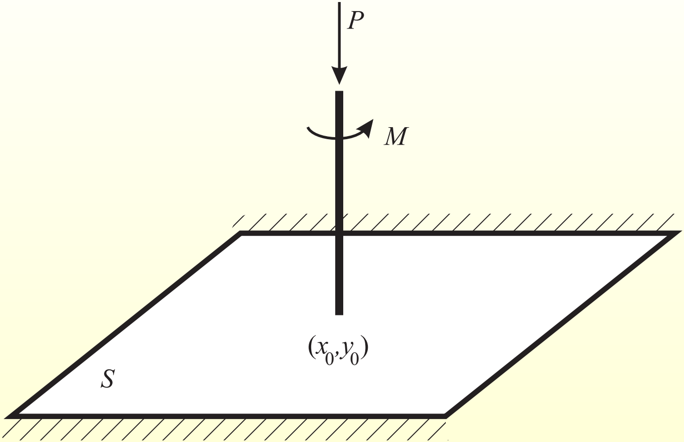

Figure: Plate with a vertical rod.

Plate with a vertical rod

The energy functional is

$$\mathcal{E}(w, \vartheta_1, \vartheta_2, y, \psi) = \iint_S W\, dS + \frac{1}{2}\int_0^a \left(EA(u')^2 + D_T(\psi')^2\right) ds \\ - Pu(a) - M\psi(a), \tag{41}$$

where $u(s)$ is the longitudinal displacement along the rod and $\psi(s)$ is the twisting angle. Taking into account the kinematic compatibility conditions $u(0) = w(x_0, y_0)$, we get a natural boundary condition that corresponds to the action of a point force at (x_0, y_0):

$$Q_{1,1} + Q_{2,2} + EAu'(0)\,\delta(x - x_0, y - y_0) = 0.$$

There is no problem with this physically.

Problems

But if we consider the influence of the drilling moment M, we see that there is no kinematic compatibility condition relating ψ, w, and ϑ_α. Correspondingly, the torsion problem for the rod yields the natural conditions

$$D_T\psi'(0) = 0, \qquad D_T\psi'(a) = M.$$

Hence the torsion in the rod does not appear to be affected by clamping to the plate: the lower end of the rod appears to be free!

Clearly, this strange conclusion must come from physical assumptions hidden in the model. The Reissner–Mindlin plate theory is derived under assumptions in which the drilling moment does not enter as a load.

Thank you for your attention!!!

Further questions:

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