13/12/2023

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INTEGRALI (INDEFINITI)

OSSERVAZIONE: SE $F(x)$ CON DERIVATA $F'(x)=f(x)$, ALLORA $F(x)+C$ HA ANCORA DERIVATA $[F(x)+c]' = F'(x)+c' = f(x)+0$ $\forall c \in \mathbb{R}$ NUMERO REALE COSTANTE.

ESEMPIO:

$$F(x)= x^2+\sin x \qquad F(x)+13 = x^2+\sin x+13$$

$$F'(x)= 2x+\cos x \qquad D[F(x)+13] = 2x+\cos x$$

DEF: UNA FUNZIONE $F(x)$ SI DICE PRIMITIVA DELLA FUNZIONE $f(x)$ DEFINITA IN $D \subseteq \mathbb{R}$ SE $F(x)$ É DERIVABILE IN D E HA DERIVATA PARI $f(x)$, OSSIA

$$F'(x) = f(x)$$

ES: $F(x)= x^2+3$ É UNA PRIMITIVA DI $f(x)=2x$

L'INSIEME DELLE PRIMITIVE DI $f(x)$ É CHIAMATO INTEGRALE INDEFINITO DI $f(x)$, E SCRIVEREMO

$$\int f(x)dx = F(x)+c$$

QUINDI L'INTEGRALE É L'OPERAZIONE INVERSA DELLA DERIVATA.

ESEMPI: $F(x) = x^2 + \operatorname{sen} x$ $\quad f(x) = F'(x) = 2x + \cos x$

$$\Rightarrow \int (2x + \cos x)\, dx = x^2 + \operatorname{sen} x + c$$

TABELLA DEGLI INTEGRALI ELEMENTARI

$f(x)$	$F(x) = \int f(x)\,dx$
0	c
1	x
x	$\frac{x^2}{2}$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$

$$\left[D\left[\frac{x^{n+1}}{n+1}\right] = \frac{1}{n+1} D\left[x^{n+1}\right] = \frac{1}{\cancel{n+1}} \cdot \cancel{(n+1)} \cdot x^n = x^n \right]$$

$f(x)$	$F(x)$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\frac{1}{\cos^2 x}$	$\operatorname{tg} x$

ES: VERIFICARE QUALI DEI SEGUENTI INTEGRALI SONO CORRETTI

1) $$\int \underbrace{(x^3+3)}_{f(x)}\,dx = \underbrace{\frac{x^3}{3}+3x}_{F(x)}+c$$ FALSO

$$D\left[\frac{x^3}{3}+3x+c\right] = \frac{1}{3}\cdot 3x^2+3\cdot 1+0 = x^2+3$$

2) $$\int \ln x \, dx = \frac{1}{x}+c$$ FALSO

$$D\left[\frac{1}{x}+c\right] = \frac{0\cdot x-1\cdot 1}{x^2}+0 = -\frac{1}{x^2}$$

3) $$\int \ln x \, dx = x\cdot \ln x - x + c$$ VERO

$$D[x\ln x - x + c] = 1\cdot \ln x + x\cdot\frac{1}{x} - 1 + 0$$

$$= \ln x + 1 - 1 = \ln x$$

4) $$\int x\cdot \sin x^2 \, dx = -\frac{1}{2}\cos x^2 + c$$ VERO

$$D\left[-\frac{1}{2}\cos x^2 + c\right] = -\frac{1}{\cancel{2}}\cdot(-\sin x^2)\cdot \cancel{2}x + 0$$

$$= +x\sin x^2$$

5) $$\int (3x^2-1)\,dx = x^3+c$$ FALSO

$$D\left[x^3+c\right] = 3x^2+0 = 3x^2$$

PROPRIETÀ DEGLI INTEGRALI:

1) SOMMA: $\int \left(f(x)+g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx$

SOTTRAZIONE: $\int \left(f(x)-g(x)\right)dx = \int f(x)\,dx - \int g(x)\,dx$

2) PRODOTTO PER UNA COSTANTE $k \in \mathbb{R}$:

$$\int k\cdot f(x)\,dx = k\cdot \int f(x)\,dx$$

ESEMPI:

1) $\int (1-x)\,dx \overset{(1)}{=} \int 1\,dx - \int x\,dx = x - \frac{x^2}{2} + c$

PROVA: $D\left[x - \frac{x^2}{2} + c\right] = 1 - \frac{1}{2}\cdot 2x + 0 = 1 - x.$

2) $\int 3e^x\,dx \overset{(2)}{=} 3\int e^x\,dx = 3\cdot e^x + c$

PROVA: $D\left[3e^x + c\right] = 3e^x + 0 = 3e^x.$

3) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$$= \frac{2}{3}\sqrt{x^3} + c$$

PROVA: $D\left[\frac{2}{3}x^{\frac{3}{2}} + c\right] = \frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}\,x^{\frac{3}{2}-1} + 0 = x^{\frac{1}{2}} = \sqrt{x}$

PROVA: $D\left[\frac{2}{3}x^{\frac{3}{2}}+c\right] = \cancel{\frac{2}{3}}\cdot\cancel{\frac{3}{2}}\,x^{\frac{3}{2}-1} + 0 = x^{\frac{1}{2}} = \sqrt{x}$

4) $$\int \frac{x^4+x^3-2x-4}{x^3}\,dx = \int\left(\frac{x^4}{x^3}+\frac{x^3}{x^3}-\frac{2x}{x^3}-\frac{4}{x^3}\right)dx$$

$$= \int\left(x+1-\frac{2}{x^2}-\frac{4}{x^3}\right)dx$$

$$\overset{(1)}{=} \int x\,dx + \int 1\,dx - \int \frac{2}{x^2}\,dx - \int \frac{4}{x^3}\,dx$$

$$\overset{(2)}{=} \int x\,dx + \int 1\,dx - 2\int \underbrace{\frac{1}{x^2}}_{=x^{-2}}dx - 4\int \underbrace{\frac{1}{x^3}}_{=x^{-3}}dx$$

$$= \frac{x^2}{2} + x - 2\cdot\frac{x^{-2+1}}{-2+1} - 4\cdot\frac{x^{-3+1}}{-3+1} + c$$

$$= \frac{x^2}{2} + x - 2\,\frac{x^{-1}}{-1} - \cancel{4}^{\,2}\cdot\frac{x^{-2}}{-\cancel{2}} + c$$

$$= \frac{x^2}{2} + x + \frac{2}{x} + \frac{2}{x^2} + c$$

PROVA: $D\left[\frac{x^2}{2}+x+\frac{2}{x}+\frac{2}{x^2}+c\right] =$

$$= \frac{1}{2}\cdot 2x + 1 + \frac{0\cdot x - 1\cdot 2}{x^2} + \frac{0\cdot x^2 - 2\cdot 2x}{x^4} + 0$$

$$= x + 1 - \frac{2}{x^2} - \frac{4x}{x^4} = x + 1 - \frac{2}{x^2} - \frac{4}{x^3}.$$

5) $\int (\sin x + 2x)\,dx$

6) $\int \frac{5}{x}\,dx$

7) $\int (x - 2x^3 + 3x^5)\,dx$

8) $\int \sqrt[5]{x^2}\,dx$

9) $\int \frac{x^3}{\sqrt{x}}\,dx$

10) $\int \frac{3x^2+2}{3x}\,dx$

11) $\int e^x(1 - 2xe^{-x})\,dx$

12) $\int \frac{1-8\cos^3 x}{\cos^2 x}\,dx$

5) $\int (\sin x + 2x)\,dx = \int \sin x\,dx + \int 2x\,dx$

$= \int \sin x\,dx + 2\int x\,dx$

$= -\cos x + \cancel{2}\frac{x^2}{\cancel{2}} + c = -\cos x + x^2 + c$

$\left[D[-\cos x + x^2 + c] = -(-\sin x) + 2x = \sin x + 2x \, . \right]$

6) $\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln|x| + c$

$\Big[D[5\ln|x| + c]$

<u>CASO $x>0$</u>: $D[5\ln x + c] = 5\cdot\frac{1}{x} + 0 = \frac{5}{x}$

<u>CASO $x<0$</u>: $D[5\ln(-x) + c] = 5\cdot\frac{1}{-x}\cdot(-1) + 0 = \frac{5}{x} \Big]$

7) $\int (x - 2x^3 + 3x^5)\,dx = \int x\,dx - 2\int x^3\,dx + 3\int x^5\,dx$

x^2 x^4 x^6 x^2 x^4 x^6

$$= \frac{x^2}{2} - \cancel{2}^{1} \cdot \frac{x^4}{\cancel{4}_{2}} + \cancel{3}^{1} \frac{x^6}{\cancel{6}_{2}} = \frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{2} + c$$

$$\left\ulcorner D\left[\frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{2}\right] = \frac{1}{2}\cdot 2x - \frac{1}{2}\cdot 4x^3 + \frac{1}{2}\cdot 6x^5\right.$$

$$= x - 2x^3 + 3x^5 \lrcorner$$

8) $$\int \sqrt[5]{x^2}\,dx = \int x^{\frac{2}{5}}\,dx = \frac{x^{\frac{2}{5}+1}}{\frac{2}{5}+1} = \frac{x^{\frac{7}{5}}}{\frac{7}{5}} = \frac{5}{7}\sqrt[5]{x^7} + c$$

$$\ulcorner D\left[\frac{5}{7}x^{\frac{7}{5}}\right] = \frac{\cancel{5}}{\cancel{7}}\cdot\frac{\cancel{7}}{\cancel{5}}x^{\frac{7}{5}-1} = x^{\frac{2}{5}} = \sqrt[5]{x^2}. \lrcorner$$

9) $$\int \frac{x^3}{\sqrt{x}}\,dx = \int \frac{x^3}{x^{\frac{1}{2}}}\,dx = \int x^{3-\frac{1}{2}}\,dx = \int x^{\frac{5}{2}}\,dx$$

$$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} = \frac{2}{7}x^{\frac{7}{2}} = \frac{2}{7}\sqrt{x^7} + c$$

$$\ulcorner D\left[\frac{2}{7}x^{\frac{7}{2}}\right] = \frac{\cancel{2}}{\cancel{7}}\cdot\frac{\cancel{7}}{\cancel{2}}x^{\frac{7}{2}-1} = x^{\frac{5}{2}}. \lrcorner$$

10) $$\int \frac{3x^2+2}{3x}\,dx = \int\left(\frac{3x^2}{3x} + \frac{2}{3x}\right)dx = \int\left(x + \frac{2}{3x}\right)dx$$

$$= \int x\,dx + \int \frac{2}{3x}\,dx = \int x\,dx + \frac{2}{3}\int \frac{1}{x}\,dx$$

$$= \frac{x^2}{2} + \frac{2}{3}\ln|x| + c$$

$$\ulcorner D\left[\frac{x^2}{2} + \frac{2}{3}\ln|x| + c\right] = \frac{1}{2}\cdot 2x + \frac{2}{3}\cdot\frac{1}{x} + 0 = x + \frac{2}{3x}$$

$$\left\lceil D\left[\frac{x^2}{2}+\frac{2}{3}\ln|x|+c\right]=\frac{1}{2}\cdot 2x+\frac{2}{3}\cdot\frac{1}{x}+0=x+\frac{2}{3x}\right\rfloor$$

11) $$\int e^x(1-2xe^{-x})\,dx=\int e^x-2x\underbrace{e^x\cdot e^{-x}}_{=e^0=1}\,dx$$

$$=\int(e^x-2x)\,dx=\int e^x\,dx-2\int x\,dx$$

$$=e^x-2\frac{x^2}{2}+c=e^x-x^2+c$$

$$\left\lceil D[e^x-x^2+c]=e^x-2x\right\rfloor$$

12) $$\int\frac{1-8\cos^3x}{\cos^2x}\,dx=\int\frac{1}{\cos^2x}-\frac{8\cos^3x}{\cos^2x}\,dx$$

$$=\int\left(\frac{1}{\cos^2x}-8\cos x\right)dx=\int\frac{1}{\cos^2x}\,dx-8\int\cos x\,dx$$

$$=\operatorname{tg}x-8\sin x+c$$

$$\left\lceil D[\operatorname{tg}x-8\sin x+c]=\frac{1}{\cos^2x}-8\cdot\cos x\right\rfloor$$

NOTA: L'INTEGRALE INDEFINITO HA COME RISULTATO UNA FUNZIONE! VEDREMO PROSSIMAMENTE CHE ESISTONO INTEGRALI CHE HANNO COME RISULTATO UN NUMERO!