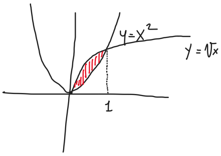
1)

$$\iint_D xy \, dx\,dy$$

$$D = \left\{ (x,y) \in \mathbb{R}^2 : \ x^2 \leq y \leq \sqrt{x} \right\}$$

$$0 \leq x \leq 1$$

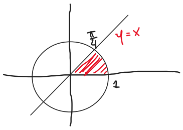


$$\int_0^1 \left(\int_{x^2}^{\sqrt{x}} xy \ dy \right) dx =$$

$$= \int_0^1 \left[\frac{x \cdot y^2}{2} \right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left(x \cdot \frac{(\sqrt{x})^2}{2} - \frac{x \cdot (x^2)^2}{2} \right) dx$$

$$= \int_0^1 \frac{x^2}{2} - \frac{x^5}{2} \ dx = \dots$$

$$\iint_D (x+y) \, dx\,dy \qquad D = \left\{ (x,y) \in \mathbb{R}^2 : \ x^2+y^2 \leq 1, \ 0 \leq y \leq x \right\}$$



$$\begin{cases} x = \rho\cos\theta & \theta \in (0, \frac{\pi}{4}) \\ y = \rho \,\mathrm{sen}\,\theta & \rho \in (0,1) \end{cases}$$

$$\int_0^1 \int_0^{\frac{\pi}{4}} (\rho\cos\theta + \rho\,\mathrm{sen}\,\theta) \cdot \rho \ d\rho \, d\theta$$

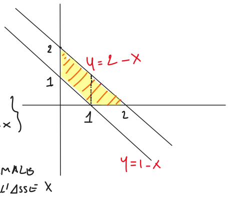
$$= \int_0^1 \left(\int_0^{\frac{\pi}{4}} (\rho^2 \cos\theta + \rho^2 \mathrm{sen}\,\theta) \ d\theta \right) d\rho = \int_0^1 \left[\rho^2 \cdot \mathrm{sen}\,\theta - \rho^2 \cos\theta \right]_0^{\frac{\pi}{4}} d\rho$$

$$= \int_0^1 \left(\rho^2 \left(\frac{\sqrt{2}}{2} - 0 \right) - \rho^2 \left(\frac{\sqrt{2}}{2} - 1 \right) \right) d\rho = \int_0^1 \left(\frac{\sqrt{2}}{2} \rho^2 - \rho^2 \frac{\sqrt{2}}{2} + \rho^2 \right) d\rho$$

$$= \dots$$

$$\iint_D 2x \ dx\,dy \qquad D = \{ 1 \leq x+y \leq 2, \quad x \geq 0, \ y \geq 0 \{$$

$$x+y \geq 1 \qquad y \geq 1-x$$
$$x+y \leq 2 \qquad y \leq 2-x$$



$$D = \left\{ \begin{array}{c} 0 \leq x \leq 1 \\ 1-x \leq y \leq 2-x \end{array} \right\} \cup \left\{ \begin{array}{c} 1 \leq x \leq 2 \\ 0 \leq y \leq 2-x \end{array} \right\}$$

↙ D1

DOMINIO NORMALE
RISPETTO ALL' ASSE X

↙ D2

DOMINIO NORMALE
RISPETTO ALL'ASSE X

$$\iint_D 2x \, dx\, dy = \iint_{D_1} 2x \, dx\, dy + \iint_{D_2} 2x \, dx dy$$

$$= \int_0^1 \left(\int_{1-x}^{2-x} 2x \, dy \right) dx + \int_1^2 \left(\int_0^{2-x} 2x \, dy \right) dx$$

$$= \int_0^1 \left[2xy \right]_{1-x}^{2-x} dx + \int_1^2 \left[2xy \right]_0^{2-x} dx =$$

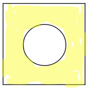
$$= \int_0^1 \left(2x(2-x) - 2x(1-x) \right) dx + \int_1^2 2x(2-x) \, dx =$$

$$= \int_0^1 4x - 2x^2 - 2x + 2x^2 \, dx + \int_1^2 4x - 2x^2 \, dx =$$

$$= \left[\frac{4x^2}{2} - \frac{2x^3}{3} - x^2 + \frac{2x^3}{3} \right]_0^1 + \left[2x^2 - \frac{2}{3} x^3 \right]_1^2 =$$

$$\frac{4}{2} - \frac{2}{3} - 1 + \frac{2}{3} + 8 - \frac{16}{3} - \left(2 - \frac{2}{3} \right) = \dots$$

$$\iint_D 1 \, dx\, dy = \iint_{[-2,2]\times[-2,2]} 1 \, dx\, dy - \iint_{x^2+y^2\le 1} 1 \, dx dy$$

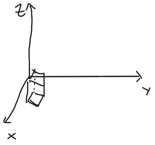


$$= \int_{-2}^{2} \int_{-2}^{2} 1 \, dx\, dy - \int_0^1 \int_0^{2\pi} 1 \cdot \rho \, d\theta\, d\rho =$$

$$= \int_{-2}^{2} \left[x \right]_{-2}^{2} dy - \int_0^1 \left(\rho\theta \right]_0^{2\pi} d\rho = \int_{-2}^{2} 4 \, dy - \int_0^1 \rho \, 2\pi \, d\rho$$

$$= \left[4y \right]_{-2}^{2} - \left[2\pi \frac{\rho^2}{2} \right]_0^1 = 16 - \pi$$

$$\iiint_T xy^2z^3 \, dx dy dz \qquad T = [2,3] \times [1,2] \times [0,1]$$



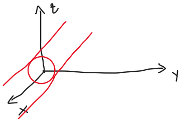
$$\int_2^3 \left(\int_1^2 \left(\int_0^1 xy^2z^3 \, dz \right) dy \right) dx$$

$$= \int_2^3 \int_1^2 \left[xy^2 \frac{z^4}{4} \right]_0^1 dy \, dx =$$

$$= \int_2^3 \int_1^2 \frac{xy^2 \cdot 1}{4} \, dy \, dx = \int_2^3 \left[\frac{x}{4\cdot 3} y^3 \right]_1^2 dx = \int_2^3 \frac{x}{4\cdot 3} (8-1) \, dx$$

$$\frac{7}{4\cdot 3} \left[\frac{x^2}{2} \right]_2^3 = \frac{7}{4\cdot 3} \cdot \left(\frac{9}{2} - \frac{4}{2} \right) = \dots$$

$$\iiint_T 1 \, dx\, dy\, dz \qquad T = \{ \overbrace{y^2+z^2 \le 1}^{D}, \; 0 \le x \le 2-y-z \}$$



$$\iint_D \left(\int_0^{2-y-z} 1 \, dx \right) dy\, dz$$

$$\begin{cases} x = x \\ y = \rho \cos\theta \\ z = \rho \,\text{sen}\,\theta \end{cases}$$

$|J| = |\rho|$

$\rho \in [0,1], \; \theta \in [0,2\pi] \qquad 0 \le x \le 2 - \rho\cos\theta - \rho\,\text{sen}\,\theta$

$$\int_0^1 \left(\int_0^{2\pi} \left(\int_0^{2-\rho\cos\theta-\rho\,\text{sen}\,\theta} \rho \, dx \right) d\theta \right) d\rho = \int_0^1 \left(\int_0^{2\pi} \left[\rho x \right]_0^{2-\rho\cos\theta-\rho\,\text{sen}\,\theta} \right) d\theta \Big) d\rho$$

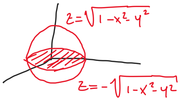
$$= \int_0^1 \left(\int_0^{2\pi} \rho \left(2 - \rho\cos\theta - \rho\,\text{sen}\,\theta \right) d\theta \right) d\rho =$$

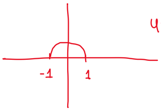
$$\int_0^1 \left[2\rho\theta - \rho^2 \,\text{sen}\,\theta + \rho^2 \cos\theta \right]_0^{2\pi} d\rho = \int_0^1 \left(2\rho \, 2\pi + \cancel{\rho^2} - \cancel{\rho^2 \cdot 1} \right) d\rho$$

$$= \int_0^1 4\pi\rho \; d\rho = \left[2\pi\rho^2 \right]_0^1 = 2\pi$$

$$D = \{ x^2+y^2 \leq 1 \}$$

$$f = \sqrt{1-x^2-y^2}$$





$$y = \sqrt{1-x^2}$$

$$\iint_{x^2+y^2\leq 1} \sqrt{1-x^2-y^2}\; dx\,dy - \iint_{x^2+y^2\leq 1} -\sqrt{1-x^2-y^2}$$

$$= 2\iint_{x^2+y^2\leq 1} \sqrt{1-x^2-y^2}\; dx\,dy$$

---

lunghezza di una curva: $L(\gamma) = \int_a^b \| \gamma'(t) \| \, dt = \int_a^b \sqrt{x_\gamma'^2 + y_\gamma'^2}\; dt$

$$\gamma = \begin{cases} x = e^t \cdot \operatorname{sen} t \\ y = e^t \cdot \cos t \end{cases} \qquad t \in [0,1]$$

$$\gamma' = \begin{cases} x' = e^t \cdot \operatorname{sen} t + e^t \cos t \\ y' = e^t \cos t - e^t \operatorname{sen} t \end{cases}$$

$$x_\gamma'^2 + y_\gamma'^2 = \left(e^{2t} \operatorname{sen}^2 t + e^{2t} \cos^2 t + 2e^{2t} \operatorname{sen} t \cos t \right) + \left(e^{2t}\cos^2 t + e^{2t} \operatorname{sen}^2 t - 2e^{2t} \operatorname{sen} t \cos t \right)$$

$$= e^{2t} \left(1 + 1 + 0 \right) = 2e^{2t}$$

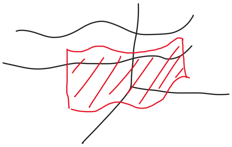
$$\| \gamma'(t) \| = \sqrt{2e^{2t}}$$

$$L(\gamma) = \int_0^1 \sqrt{2e^{2t}}\; dt = \int_0^1 \sqrt{2}\, e^t \; dt =$$

$$\sqrt{2}\left[e^t \right]_0^1 = \sqrt{2}\left(e^1 - e^0 \right) = \sqrt{2}\,(e - 1)$$

integrale di I specie

$$\int_\gamma f \, ds = \int_a^b f(\gamma(t)) \cdot \| \gamma'(t) \| \; dt$$



$$f(x,y) = -y \qquad \gamma(t) = (\cos t, \operatorname{sen} t) \quad t \in [0, 2\pi]$$

$$\int_\gamma -y \; ds = \int_0^{2\pi} - \operatorname{sen} t \cdot 1 \; dt$$

$$\gamma'(t) = (-\operatorname{sen} t, \cos t)$$

$$\| \gamma'(t) \| = \sqrt{1} = 1$$

$$= \left[\cos t \right]_0^{2\pi} = 0$$

integrale di II specie

$$F(x,y) = \left(F_1(x,y), F_2(x,y) \right) \qquad \gamma : [a,b] \to \mathbb{R}^2$$

$$\int_a^b F \cdot T_\gamma \; ds = \int_a^b (F_1, F_2) \cdot \frac{\gamma'(t)}{\cancel{\|\gamma'(t)\|}} \cdot \|\cancel{\gamma'(t)}\| \; dt = \int_a^b (F_1, F_2) \cdot \gamma'(t) \; dt$$

$$\gamma = \left(\overset{x}{\cos\theta}, \overset{y}{\operatorname{sen}\theta}, \overset{z}{0} \right) \qquad F(x,y,z) = \left(y + z^2, x^2 + y, \operatorname{arctg}\left(x^2+y^2+z \right) \right)$$

$$\theta \in [0, 2\pi]$$

$$\int_0^{2\pi} \left(\operatorname{sen}\theta, \cos^2\theta + \operatorname{sen}\theta, \overset{\frac{\pi}{4}}{\operatorname{arctg}(1)} \right) \cdot \frac{\left(-\operatorname{sen}\theta, \cos\theta, 0 \right)}{\cancel{\sqrt{\operatorname{sen}^2\theta + \cos^2\theta + 0^2}}} \cdot \cancel{\sqrt{\operatorname{sen}^2\theta + \cos^2\theta + 0^2}} \; d\theta$$

$$= \int_0^{2\pi} -\operatorname{sen}^2\theta + \cos^3\theta + \operatorname{sen}\theta \cdot \cos\theta + 0 \quad d\theta = \ldots$$

---

data una curva chiusa γ, $\oint_\gamma \langle F, T_\gamma \rangle \, ds$ si dice circuitazione (in $\mathbb{R}^2$)

in $\mathbb{R}^3$ la curva γ deve essere chiusa e piana

DEF Un campo vettoriale $F : A^{\subseteq \mathbb{R}^n} \to \mathbb{R}^n$ si dice conservativo se esiste una funzione differenziabile $f : A \to \mathbb{R}$ detta potenziale tale che $F = \nabla f = (f_x, f_y)$

se F ha un potenziale f, allora $f + k$ sono potenziali

$F = (y, x)$ campo vettoriale è conservativo:

$$f(x,y) = xy \qquad \nabla f = (y, x) = F$$

Teorema Dato un campo conservativo F $A^{\subseteq \mathbb{R}^2} \to \mathbb{R}^2$ continuo, e una curva regolare (a tratti) γ orientata, tale che $\gamma(a) = (x_1, y_1)$ $\gamma(b) = (x_2, y_2)$, $\Rightarrow$

$$\int_\gamma F \cdot T_\gamma \, ds = f(x_2, y_2) - f(x_1, y_1) \qquad \left(\begin{matrix} f \text{ potenziale} \\ \text{di } F \end{matrix}\right)$$

$$\int_\gamma \overset{(F_1, F_2)}{F(x,y)} \cdot \overset{(\gamma_x', \gamma_y')}{T_\gamma} \, ds = \int_a^b F_1 \cdot \gamma_x' + F_2 \cdot \gamma_y' \, dt$$

$$F \text{ conservativo} \Rightarrow F = \nabla f \qquad (F_1, F_2) = (f_x, f_y)$$

$$= \int_a^b f_x \cdot \gamma_x' + f_y \cdot \gamma_y' \, dt = \int_a^b \frac{df}{dt} \, dt = f(\gamma(b)) - f(\gamma(a))$$

derivata composta di $(f \circ \gamma)$

$\square$

1) la circuitazione di un campo conservativo $\left(\begin{matrix}\text{curva} \\ \text{chiusa}\end{matrix}\right)$ è sempre nulla

2) se la circuitazione di un campo è $\neq 0$ $\Rightarrow$ non è conservativo

3) se l'integrale curvilineo di un campo cambia in base alla curva (mantenendo lo stesso punto iniziale e finale) $\Rightarrow$ non è conservativo

esempio campo conservativo (ammette una primitiva. per il teorema visto, la circuitazione è nulla: proviamo a calcolarla)

$$\theta \in (0, 2\pi) \qquad P_1 = (1,0) \quad P_2 = (1,0)$$

$$F = (y, x) \qquad \gamma = \begin{cases} \cos\theta = x \\ \operatorname{sen}\theta = y \end{cases} \qquad f = xy \qquad f_x = y \qquad f_y = x$$

è un potenziale

$$\int_\gamma (y, x) \cdot T \, ds = \int_0^{2\pi} \overset{f_x \quad f_y}{(\operatorname{sen}\theta, \cos\theta)} \cdot \frac{(-\operatorname{sen}\theta, \cos\theta)}{\|\gamma'(t)\|} \cdot \|\gamma'(t)\| \cdot d\theta$$

$$= \int_0^{2\pi} -\operatorname{sen}^2\theta + \cos^2\theta \, d\theta = \int_0^{2\pi} -\operatorname{sen}^2\theta + 1 - \operatorname{sen}^2\theta \, d\theta = \int_0^{2\pi} -2\operatorname{sen}^2\theta + 1 \, d\theta$$

$$= \int_0^{2\pi} -2\left[\frac{1}{2}\left(1 - \cos(2\theta)\right)\right] + 1 \, d\theta = \int_0^{2\pi} -\cancel{1} + \cos(2\theta) + \cancel{1} \, d\theta$$

$$f(P_2) - f(P_1) = 1 \cdot 0 - 1 \cdot 0 = 0 \qquad = \left[\frac{\operatorname{sen}(2\theta)}{2}\right]_0^{2\pi} = 0$$