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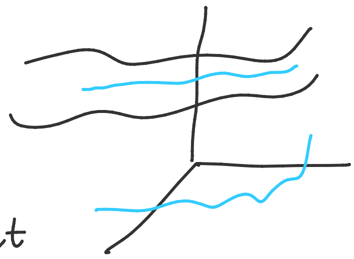
2 variabili: curve in $\mathbb{R}^2$

$f(x,y) = x^2 y$



$$\gamma : \begin{cases} x = 2\cos\theta \\ y = 2\,\text{sen}\,\theta \end{cases} \quad \theta \in [0, 2\pi)$$

integrale curvilineo di I specie

$$\int_\gamma f\, ds = \int_0^{2\pi} 4\cos^2\theta \cdot 2\,\text{sen}\,\theta \cdot \|\gamma'(t)\|\, dt$$

$$\gamma' = \begin{cases} x' = -2\,\text{sen}\,\theta \\ y' = 2\cos\theta \end{cases} \qquad \sqrt{4\,\text{sen}^2\theta + 4\cos^2\theta} = \sqrt{4} = 2$$

$$= \int_0^{2\pi} 16\cos^2 \text{sen}\,\theta\, d\theta = \left[16\,\frac{\cos^3\theta}{3}\right]_0^{2\pi} = 0$$

$$\gamma = \begin{cases} x = t \\ y = \frac{1}{2\sqrt{t}} = \frac{1}{2} t^{-\frac{1}{2}} \end{cases} \qquad f(x,y) = y$$

$$t \in [1,2]$$

$$\gamma' = \begin{cases} x' = 1 \\ y' = \frac{1}{2}\left(-\frac{1}{2}\right) \cdot t^{-3/2} = -\frac{1}{4} t^{-3/2} \end{cases} \qquad \|\gamma'(t)\| = \sqrt{1 + \frac{1}{16} t^{-3}}$$

$$= \sqrt{1 + \frac{1}{16t^3}}$$

$$\int_\gamma f\, ds = \int_1^2 \frac{1}{2\sqrt{t}} \cdot \underbrace{\sqrt{1 + \frac{1}{16t^3}}}_{\|\gamma'(t)\|}\, dt$$

$$= \frac{1}{2}\int_1^2 \frac{1}{\sqrt{t}} \cdot \sqrt{\frac{1}{16t^2}\left(16t^2 + \frac{1}{t}\right)}\, dt = \frac{1}{2}\int_1^2 \frac{1}{\sqrt{t}} \cdot \frac{1}{4t}\sqrt{16t^2 + \frac{1}{t}}\, dt$$

$= \dots$

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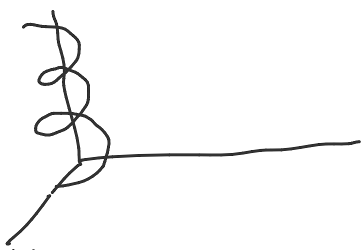
3 variabili: curve in $\mathbb{R}^3$

$f(x,y,z) = z \qquad \gamma : (\cos t, \text{sen}\, t, t) \quad t \in [0, 2\pi]$

$$\int_\gamma f\, ds = \int_0^{2\pi} t \cdot \|\gamma'(t)\|\, dt$$



$$\gamma'(t) = (-\text{sen}\, t, \cos t, 1)$$

$$\|\gamma'(t)\| = \sqrt{\text{sen}^2 t + \cos^2 t + 1^2} = \sqrt{1+1} = \sqrt{2}$$

$$= \int^{2\pi} t \cdot \sqrt{2}\, dt = \sqrt{2}\left[\frac{t^2}{2}\right]_0^{2\pi} = \sqrt{2}\,\frac{4\pi^2}{2} = 2\sqrt{2}\,\pi^2$$

$$= \int_0^{2\pi} t \cdot \sqrt{2}\, dt = \sqrt{2}\left[\frac{t^2}{2}\right]_0^{2\pi} = \sqrt{2}\,\frac{4\pi}{2} = 2\sqrt{2}\,\pi$$

<u>proprietà</u>

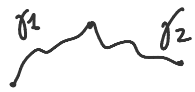
γ curva in $\mathbb{R}^n$ f, g definite su $\mathbb{R}^n$ $\alpha, \beta \in \mathbb{R}$ (numeri!)

- $\displaystyle\int_\gamma (\alpha f + \beta g)\, ds = \alpha \int_\gamma f\, ds + \beta \int_\gamma g\, ds$

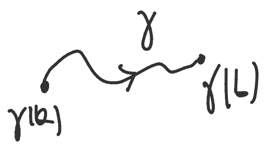
- $\displaystyle f \leq g \Rightarrow \int_\gamma f\, ds \leq \int_\gamma g\, ds$

- γ_1, γ_2 $\gamma_1 : [a,b] \to \mathbb{R}^n$ $\gamma_2 : [b,c] \to \mathbb{R}^n$ $\gamma_1(b) = \gamma_2(b)$

$$\int_{\gamma_1 \cup \gamma_2} f\, ds = \int_{\gamma_1} f\, ds + \int_{\gamma_2} f\, ds$$



- $\displaystyle\int_{-\gamma} f\, ds = \int_\gamma f\, ds$



nota: se γ si può esprimere in forma cartesiana,

$\|\gamma'(t)\| = \sqrt{1 + f'(t)^2}$ $\gamma : \begin{cases} x = t \\ y = f(t) \end{cases}$

integrali di II specie

$F(x,y)$ funzione vettoriale si dice campo vettoriale se

$F : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ $F(x,y)$ viene considerato come un vettore di coordinate $(F_1(x,y), F_2(x,y))$ applicato nel punto (x,y)

esempio

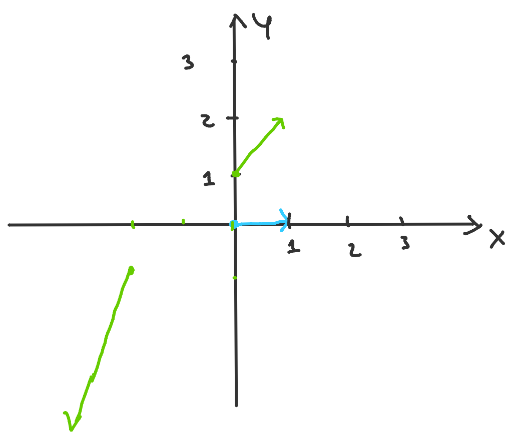
$F : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \to (\underbrace{x+1}_{F_1}, \underbrace{y+x}_{F_2})$

$(0,0)$ $F(0,0) = (1,0)$

$(0,1)$ $F(0,1) = (1,1)$

$(-2,-1)$ $F(-2,-1) = (-1,-3)$



$F(x,y) = (x+1)\,\bar{i} + (y+x)\,\bar{j} = (F_1, F_2)$

integrale di II specie di un campo vettoriale F lungo una curva γ

$(F: \mathbb{R}^n \to \mathbb{R}^m \,/\, \gamma: I \to \mathbb{R}^n)$

lavoro di F lungo γ è $\displaystyle\int_\gamma F(x,y)\cdot T(x,y)\,ds$ prodotto scalare

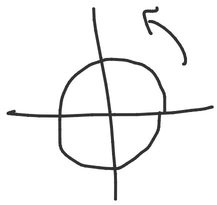
$$= \int_a^b F(x(t), y(t)) \cdot \frac{\gamma'(t)}{\cancel{\|\gamma'(t)\|}} \cdot \cancel{\|\gamma'(t)\|}\,dt =$$

$$= \int_a^b F(x(t), y(t)) \cdot \gamma'(t)\,dt$$

$\gamma'(t) = (-\text{sen}\,t, \cos t)$

esempio $F(x,y) = xy\,\bar{i} + y\,\bar{j} = (xy, y)$ $\quad \gamma(t) = (\cos t, \text{sen}\,t)$

$t \in [0, 2\pi]$



$$\int_\gamma F(x,y)\cdot T_\gamma\,ds = \int_0^{2\pi} (\cos t\cdot\text{sen}\,t, \text{sen}\,t)\cdot\frac{(-\text{sen}\,t, \cos t)}{\cancel{\|\gamma'(t)\|}}\cdot\cancel{\|\gamma'(t)\|}\,dt$$

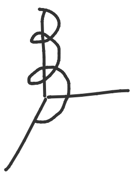
$$= \int_0^{2\pi} (-\cos t\cdot\text{sen}^2 t + \text{sen}\,t\cdot\cos t)\,dt$$

$$= \left[-\frac{\text{sen}^3 t}{3}\right]_0^{2\pi} + \left[\frac{\text{sen}^2 t}{2}\right]_0^{2\pi} = 0$$

$\gamma := \begin{cases} x = \cos t \\ y = \text{sen}\,t \\ z = t \end{cases}$ $\quad t \in [0, 2\pi]$ $\quad F(x,y,z) = (xz, x, z^2)$

$\gamma' = \begin{cases} x' = -\text{sen}\,t \\ y' = \cos t \\ z' = 1 \end{cases}$

$$\int_\gamma F\cdot T_\gamma\,ds = \int_0^{2\pi} (\cos t\cdot t, \cos t, t^2)\cdot\frac{(-\text{sen}\,t, \cos t, 1)}{\cancel{\|\gamma'(t)\|}}\cdot\cancel{\|\gamma'(t)\|}\,dt$$



$$= \int_0^{2\pi} -t\cos t\,\text{sen}\,t + \cos^2 t + t^2\,dt = \dots$$

$$\dots \int \overset{f}{t}\,\cos t\cdot\overset{g'}{\text{sen}\,t}\,dt = \overset{f\cdot g}{t\cdot\frac{\text{sen}^2 t}{2}} - \overset{\int f'\cdot g}{\int 1\cdot\frac{\text{sen}^2 t}{2}\,dt} = \dots$$

$g = \frac{\text{sen}^2 t}{2}$

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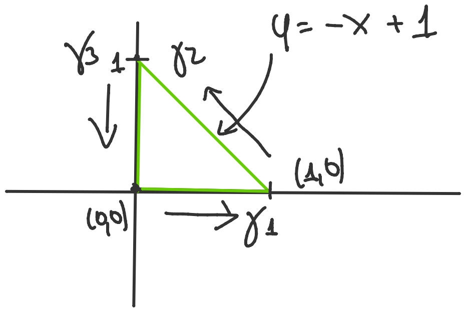
proprietà

$$\int_{-\gamma} F\cdot T_{-\gamma}\,ds = -\int_\gamma F\cdot T\,ds$$



$$F(x,y) = (e^x, y)$$

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

$$\gamma_1 \begin{cases} x = t \\ y = 0 \end{cases} \quad t \in [0,1] \qquad \gamma_1' \begin{cases} x' = 1 \\ y' = 0 \end{cases} \qquad (1,0)$$

$$-\gamma_2 \begin{cases} x = t \\ y = -t+1 \end{cases} \quad t \in [0,1] \qquad -\gamma_2' = \begin{cases} x' = 1 \\ y' = -1 \end{cases} \qquad (1,-1)$$

$$-\gamma_3 : \begin{cases} x = 0 \\ y = t \end{cases} \quad t \in [0,1] \qquad -\gamma_3' \begin{cases} x' = 0 \\ y' = 1 \end{cases} \qquad (0,1)$$

$$\int_\gamma F \cdot T_\gamma \, ds = \int_{\gamma_1} F \cdot T_{\gamma_1} \, ds + \int_{\gamma_2} F \cdot T_{\gamma_2} \, ds + \int_{\gamma_3} F \cdot T_{\gamma_3} \, ds$$

$$= \int_{\gamma_1} F \cdot T_{\gamma_1} \, ds - \int_{-\gamma_2} F \cdot T_{-\gamma_2} \, ds - \int_{-\gamma_3} F \cdot T_{-\gamma_3} \, ds$$

$$= \int_0^1 (e^t, 0) \cdot \frac{(1,0)}{\|(1,0)\|} \cdot \|(1,0)\| \, dt - \int_0^1 (e^t, -t+1) \cdot \frac{(1,-1)}{\|(1,-1)\|} \|(1,-1)\| \, dt$$

$$- \int_0^1 (e^0, t) \cdot \frac{(0,1)}{\|(0,1)\|} \cdot \|(0,1)\| \, dt =$$

$$=$$