MAT 3 30/11/23

giovedì 30 novembre 2023 11:04

lato dx di un ellisse

$$D=\{ 9x^2+4y^2 \le 36 \ ; \ x \ge 0 \} \qquad f(x,y) = xy^2$$

$$J = \begin{array}{cc} \rho & \theta \end{array}$$
$$J = \begin{pmatrix} 2\cos\theta & -2\rho \,\text{sen}\,\theta \\ 3\,\text{sen}\,\theta & 3\rho\cos\theta \end{pmatrix} = 6\rho\cos^2\theta + 6\rho\,\text{sen}^2\theta = 6\rho$$

$$\frac{9x^2}{36} + \frac{4y^2}{36} \le 1$$

$$\frac{x^2}{4}+\frac{y^2}{9} \le 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1$$

$$\begin{cases} x = \rho\cos\theta \cdot 2 \\ y = \rho\,\text{sen}\,\theta \cdot 3 \end{cases} \qquad 0 \le \rho \le 1 \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$T = [0,1] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

oppure:

$$x^2 = 4 - \frac{4}{9}y^2 \qquad \text{lato dx } (x \ge 0) \qquad a \le y \le b$$

$$x = \pm\sqrt{4-\frac{4}{9}y^2} \qquad x = \sqrt{4-\frac{4}{9}y^2} \qquad 0 \le x \le \sqrt{4-\frac{4}{9}y^2}$$

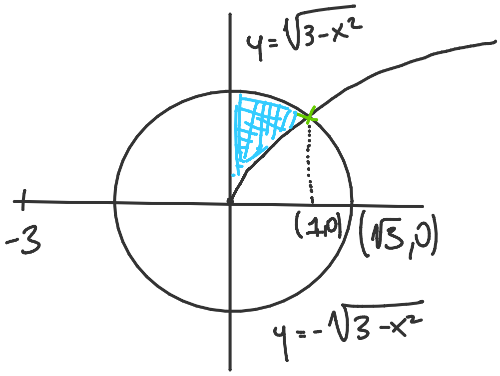
$$\iint_D xy^2 \, dx\,dy = \iint_T (2\rho\cos\theta)(3\rho\,\text{sen}\,\theta)^2 |6\rho| \, d\theta\, d\rho$$

$$= \iint_T 6 \cdot 18 \, \rho^4 \cos\theta \cdot \text{sen}^2\theta \, d\theta\, d\rho =$$

$$6\cdot 18 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos\theta\,\text{sen}^2\theta\right) d\theta \cdot \int_0^1 \rho^4 \, d\rho = 6\cdot 18 \left[\frac{\text{sen}^3\theta}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\rho^5}{5}\right]_0^1 =$$

$$= 6 \cdot 18 \left(\frac{1+1}{3}\right)\left(\frac{1}{5}\right) = \frac{6\cdot 18 \cdot 2}{15}$$

$$D = \{ x^2+y^2 \le 3 \ , \ y \ge \sqrt{2x} \} \qquad f = 3y$$



$$\{ 0 \le x \le 1 \ , \ \sqrt{2x} \le y \le \sqrt{3-x^2} \}$$

$$\begin{cases} y = \sqrt{3-x^2} \\ y = \sqrt{2x} \end{cases} \Rightarrow$$

$$3 - x^2 = 2x$$

$$x^2 + 2x - 3 = 0$$

$$x = \frac{-2 \pm \sqrt{4+12}}{2} = \begin{cases} \dfrac{-2-4}{2} = -\dfrac{6}{2} = -3 \\ \dfrac{-2+4}{2} = 1 \longrightarrow \text{è quello giusto perché } x \ge 0 \end{cases}$$

$$\iint 3y \, dx\,dy = \int^1 \left(\int_{\sqrt{\ }}^{\sqrt{3-x^2}} 3y \, dy \right) dx =$$

$$\iint_D 3y\,dx\,dy = \int_0^1 \left(\int_{\sqrt{2x}}^{\sqrt{3-x^2}} 3y\,dy \right) dx =$$

$$= \int_0^1 \left(\left[\frac{3y^2}{2}\right]_{\sqrt{2x}}^{\sqrt{3-x^2}} \right) dx = \int_0^1 \frac{3}{2}\left(3-x^2-2x\right) dx$$

$$= \frac{3}{2}\int_0^1 3-x^2-2x \quad dx = \frac{3}{2}\left[3x-\frac{x^3}{3}-x^2\right]_0^1 =$$

$$= \dots$$

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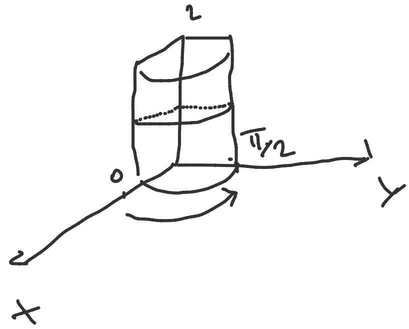
funzioni in 3 variabili:
integrali tripli: il dominio adesso
è un "volume" tridimensionale

$$f(x,y,z) = x^2+y^2$$

base circolare

$$D = \left\{ x^2+y^2 \leq 1 \;,\; 0 \leq z \leq 2 \;,\; x,y \geq 0 \right\}$$



$$\iiint_D (x^2+y^2)\,dx\,dy\,dz$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = z \end{cases}$$

cordinate cilindriche

$$[0,1] \times \left[0,\frac{\pi}{2}\right] \times [0,2] = T$$

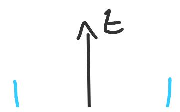
$$J = \begin{array}{c} \begin{array}{ccc} \rho & \theta & z \end{array} \\ \begin{pmatrix} \cos\theta & -\rho\,\text{sen}\,\theta & 0 \\ \text{sen}\,\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} = \begin{vmatrix} \cos\theta & -\rho\,\text{sen}\,\theta \\ \text{sen}\,\theta & \rho\cos\theta \end{vmatrix} = \rho\cos^2\theta + \rho\,\text{sen}^2\theta = \rho$$

$$\iiint_D (x^2+y^2)\,dx\,dy\,dz = \iiint_T \rho^2 \cdot |\rho| \; d\rho\,d\theta\,dz =$$

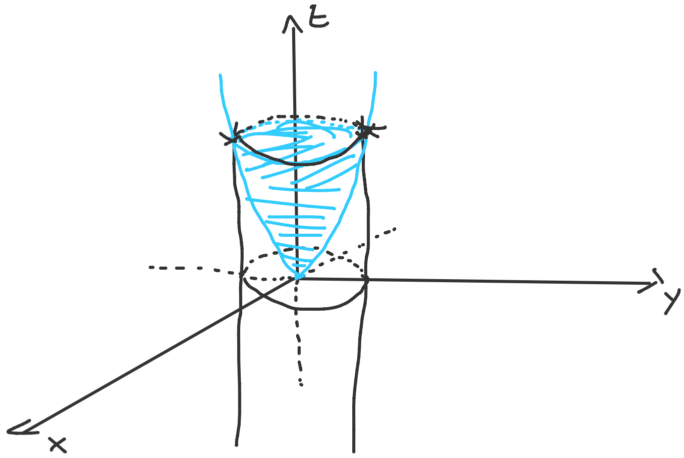
$$= \int_0^2 \left(\int_0^{\frac{\pi}{2}} \left(\int_0^1 \rho^3\,d\rho \right) d\theta \right) dz = \int_0^2 1\,dz \cdot \int_0^{\frac{\pi}{2}} 1\,d\theta \cdot \int_0^1 \rho^3\,d\rho =$$

$$= [z]_0^2 \cdot [\theta]_0^{\frac{\pi}{2}} \cdot \left[\frac{\rho^4}{4}\right]_0^1 = 2 \cdot \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{4}$$

$$D = \left\{ x^2+y^2 \leq 1 ,\; z \geq x^2+y^2 \right\}$$



$$D = \{ x^2+y^2 \leq 1, \quad z \geq x^2+y^2 \}$$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = z \end{cases} \qquad \begin{aligned} &0 \leq \rho \leq 1 \\ &0 \leq \theta \leq 2\pi \\ &\rho^2 \leq z \leq 1 \end{aligned}$$

$f = xyz$

$$\iiint_D xyz \, dx\,dy\,dz = \int_0^1 \left(\int_0^{2\pi} \left(\int_{\rho^2}^1 \rho\cos\theta\, \rho\,\text{sen}\,\theta\, z \cdot \rho \, dz \right) d\theta \right) d\rho$$

$$= \int_0^1 \left(\int_0^{2\pi} \left[\rho^3 \cos\theta\,\text{sen}\,\theta \, \frac{z^2}{2} \right]_{\rho^2}^1 d\theta \right) d\rho = \int_0^1 \left(\int_0^{2\pi} \rho^3 \cos\theta\,\text{sen}\,\theta \left(\frac{1}{2} - \frac{\rho^4}{2} \right) d\theta \right) d\rho$$

$$= \int_0^1 \left(\int_0^{2\pi} \frac{1}{2}\rho^3 \cos\theta\,\text{sen}\,\theta - \frac{\rho^7}{2} \cos\theta\,\text{sen}\,\theta \, d\theta \right) d\rho$$

$$= \frac{1}{2}\int_0^1 \int_0^{2\pi} \rho^3 \cos\theta\,\text{sen}\,\theta \, d\theta\, d\rho - \frac{1}{2} \int_0^1 \int_0^{2\pi} \rho^7 \cos\theta\,\text{sen}\,\theta \, d\theta\, d\rho =$$

$$= \frac{1}{2} \int_0^1 \rho\, d\rho \int_0^{2\pi} \cos\theta\,\text{sen}\,\theta \, d\theta - \frac{1}{2} \int_0^1 \rho^7 \, d\rho \cdot \int_0^{2\pi} \cos\theta\,\text{sen}\,\theta \, d\theta =$$

$$= \frac{1}{2} \left[\frac{\rho^4}{4} \right]_0^1 \left[\frac{\text{sen}^2\theta}{2} \right]_0^{2\pi} - \frac{1}{2} \left[\frac{\rho^8}{8} \right]_0^1 \left[\frac{\text{sen}^2\theta}{2} \right]_0^{2\pi} = \ldots$$

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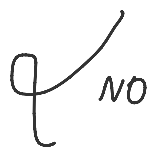
integrali curvilinei

<u>curva</u>: è una applicazione $\varphi: I \to \mathbb{R}^n$ continua.

si dice <u>semplice</u> se $\varphi(t_1) \neq \varphi(t_2) \quad \forall\, t_1 \neq t_2$



esempio di curva:

$$\begin{cases} x = \cos\theta \\ y = \text{sen}\,\theta \end{cases} \quad \theta \in (0, 2\pi) \rightsquigarrow \varphi: (0,2\pi) \to \mathbb{R}^2, \quad \theta \mapsto (\cos\theta, \text{sen}\,\theta)$$

si dice regolare se: componenti derivabili e $\varphi'(t) \neq (0,0)$

si dice <u>regolare</u> se: componenti derivabili, e $\varphi'(t) \neq (0,0)$

<u>esempio:</u> $\varphi(\theta) = (\cos\theta, \operatorname{sen}\theta)$ $\quad \varphi'(\theta) = (-\operatorname{sen}\theta, \cos\theta)$

è sempre $\neq (0,0)$

<u>esempio</u>

$$\varphi(\theta) = (\theta^2, \theta^2)$$

$$\varphi'(\theta) = (2\theta, 2\theta) \qquad \varphi'(0) = (2\cdot 0, 2\cdot 0) = (0,0)$$

non è regolare

$$\varphi(\theta_1) = \varphi(\theta_2) \qquad \underbrace{(\theta_1^2, \theta_1^2)}_{\varphi(\theta_1)} = \underbrace{(\theta_2^2, \theta_2^2)}_{\varphi(\theta_2)}$$

$$\begin{cases} \theta_1^2 = \theta_2^2 \\ \theta_1^2 = \theta_2^2 \end{cases}$$

$$\begin{cases} \theta_1 = \pm\theta_2 \\ \theta_1 = \pm\theta_2 \end{cases}$$

θ_1 e $-\theta_1$ sono due parametri diversi per cui $\varphi(\theta_1) = \varphi(\theta_2)$

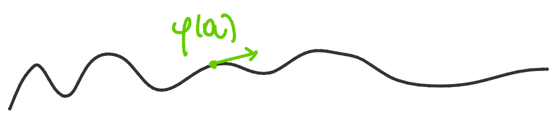
$\Rightarrow$ la curva non è semplice

φ si dice <u>regolare a tratti</u> se I può essere diviso in $I_1, \dots, I_n$ in cui le curve $\varphi_i : \begin{array}{l} I_i \to \mathbb{R}^n \\ t \mapsto \varphi(t) \end{array}$ sono regolari.

$$\varphi: \begin{array}{l} [0, 2\pi] \longrightarrow \mathbb{R}^2 \\ \theta \longmapsto (f(\theta), g(\theta)) \end{array} \qquad \theta_1 = \frac{\pi}{2} \qquad \theta_2 = \frac{3}{2}\pi$$

$$\varphi'(\theta_1) = (0,0) \qquad \varphi'(\theta_2) = (0,0)$$

$$[0, \tfrac{\pi}{2}] \cup [\tfrac{\pi}{2}, \tfrac{3}{2}\pi] \cup [\tfrac{3}{2}\pi, 2\pi]$$



<u>versore tangente</u> nel punto $t_0 \in I$ $\quad \dfrac{\varphi'(t_0)}{\|\varphi'(t_0)\|} = T(t_0)$

$$\begin{cases} x = \cos\theta \\ y = \operatorname{sen}\theta \end{cases} \quad \begin{array}{l} \theta \in [0, 2\pi] \\ \theta \in [0, 4\pi] \end{array}$$

due curve diverse

$\{ x = \cos(2\theta) \qquad \theta \in [0, \pi]$

attenzione alla velocità

$$\begin{cases} x = \cos(2\theta) \\ y = \text{sen}(2\theta) \end{cases} \qquad \theta \in [0, \pi]$$

attenzione alla velocità della curva!
in questo caso il parametro non corrisponde all'angolo percorso!

$$\varphi : [a,b] \longrightarrow \mathbb{R}^2$$

$$L(\varphi) = \int_a^b \| \varphi'(\theta) \| \, d\theta$$ lunghezza di una curva

esempio

$\varphi(\theta) = (\cos\theta, \text{sen}\theta) \qquad \theta \in (0, 2\pi)$

$\varphi'(\theta) = (-\text{sen}\theta, \cos\theta)$

$\| \varphi'(\theta) \| = \sqrt{(-\text{sen}\theta)^2 + (\cos\theta)^2} = \sqrt{1} = 1$

$$L(\varphi) = \int_0^{2\pi} 1 \, d\theta = [\theta]_0^{2\pi} = 2\pi$$

$\varphi(\theta) = (3\cos(2\theta), 3\,\text{sen}(2\theta)) \qquad \theta \in (0,\pi)$

$\varphi'(\theta) = (-6\,\text{sen}(2\theta), 6\cos\theta)$

$\| \varphi'(\theta) \| = \sqrt{36} = 6$

$$L(\varphi) = \int_0^{\pi} 6 \, d\theta = [6\theta]_0^{\pi} = 6\pi$$

$\varphi(t) = (e^t, e^{2t}) \qquad t \in [0,1] \qquad \varphi' = (e^t, 2e^{2t})$

$$\| \varphi'(t) \| = \sqrt{e^{2t} + 4e^{4t}} = \sqrt{e^{2t} + 4e^{2t} \cdot e^{2t}} = \sqrt{e^{2t}(1+4e^{2t})}$$

$$= e^t \sqrt{1+4e^{2t}}$$

$$L(\varphi) = \int_0^1 e^t \sqrt{1+4e^{2t}} \, dt = \ldots\ldots$$

integrale curvilineo di I specie di f esteso alla curva φ:

$f(x,y)$ definita su $B \subseteq \mathbb{R}^2$, $\varphi(t)$, $t \in I = [a,b]$, $\varphi(t) \subseteq B \quad \forall t \in I$.

$$\int_\varphi f \, ds = \int_a^b f(\varphi(t)) \cdot \| \varphi'(t) \| \, dt$$