MAT 3 29/11/23

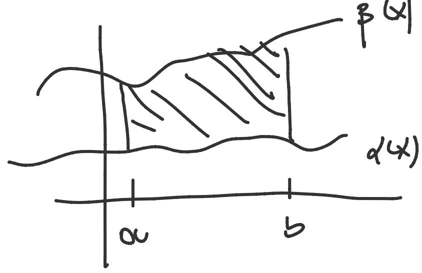
mercoledì 29 novembre 2023 09:09

$D = \{ a \le x \le b \wedge \alpha(x) \le y \le \beta(x) \}$ $\quad f(x,y)$

$$\iint_D f \, dx\, dy = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy \right) dx$$

$f(x) = 1 \quad \int_a^b 1 \, dx = b - a \quad$ misura$([a,b])$

$$f(x,y) = 1 \quad D \quad \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} 1 \, dy \right) dx = \int_a^b \left([y]_{\alpha(x)}^{\beta(x)} \right) dx =$$



$$= \int_a^b (\beta(x) - \alpha(x))\, dx$$

$D = \{ 0 \le x \le 2 \wedge 0 \le y \le 2x - x^2 \}$

$f = x^2(y-1)$

$$\int_0^2 \left(\int_0^{2x-x^2} x^2 (y-1)\, dy \right) dx = \int_0^2 \left(\left[x^2 \left(\frac{y^2}{2} - y \right) \right]_0^{2x-x^2} \right) dx$$

$$= \int_0^2 \left[x^2 \frac{(2x-x^2)^2}{2} - x^2(2x-x^2) - \left(x^2 (0-0) \right) \right] dx$$

$$= \int_0^2 \left[\frac{x^2 x^2 (2-x)^2}{2} - x^2 (2x - x^2) \right] dx =$$

$$= \int_0^2 \left[\frac{x^4 (4 - 4x + x^2)}{2} - 2x^3 + x^4 \right] dx = \dots$$

$D = \{ -1 \le y \le 1 \wedge 0 \le x \le \sqrt{1-y^2} \}$ $\quad f = x$

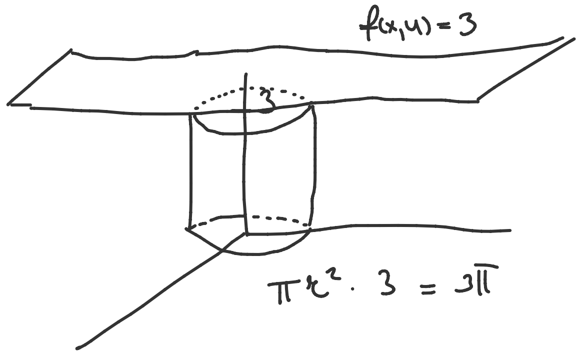
$$\iint_D x \, dx\, dy = \int_{-1}^{1} \left(\int_0^{\sqrt{1-y^2}} x \, dx \right) dy =$$

$$\int_{-1}^{1} \left[\frac{x^2}{2} \right]_0^{\sqrt{1-y^2}} dy = \int_{-1}^{1} \left(\frac{(1-y^2)}{2} - \frac{0}{2} \right) dy =$$

$$= \frac{1}{2} \int_{-1}^{1} (1-y^2)\, dy = \frac{1}{2} \left[y - \frac{y^3}{3} \right]_{-1}^{1} = \frac{1}{2} \left[1 - \frac{1}{3} + 1 - \frac{1}{3} \right] = \frac{1}{2} \left[2 - \frac{2}{3} \right]$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$D = \{ x^2 + y^2 \leq 1 \}$ $\quad f(x,y) = 3$



$$\iint_D 3 \, dx\, dy = 3 \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \right) dx$$

$$= 3 \int_{-1}^{1} \left[y \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx =$$

$$= 3 \int_{-1}^{1} \left(\sqrt{1-x^2} + \sqrt{1-x^2} \right) dx =$$

$$= 3 \cdot 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx = \dots$$

$\{ -1 \leq x \leq 1 \ \wedge -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \}$

Teorema Dato un dominio D e una trasformazione $\begin{cases} x = a(v,t) \\ y = b(v,t) \end{cases}$, detto T il dominio in cui variano v e t

$$\iint_D f(x,y)\, dx\, dy = \iint_T f(x(v,t), y(v,t)) \left| \det \frac{\partial(x,y)}{\partial(v,t)} \right| dv\, dt$$

$$J = \frac{\partial(x,y)}{\partial(v,t)} = \begin{pmatrix} \nabla a \\ \nabla b \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial v} & \frac{\partial a}{\partial t} \\ \frac{\partial b}{\partial v} & \frac{\partial b}{\partial t} \end{pmatrix}$$

$D = \{ x^2 + y^2 \leq 1 \}$ $\quad \begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$ $\quad \begin{matrix} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq 2\pi \end{matrix}$ $\quad T = [0,1] \times [0, 2\pi]$

$$J = \begin{pmatrix} \cos\theta & -\rho \operatorname{sen}\theta \\ \operatorname{sen}\theta & \rho\cos\theta \end{pmatrix}$$

$$\det J = \rho \cos^2\theta + \rho \operatorname{sen}^2\theta = \rho(\cos^2\theta + \operatorname{sen}^2\theta) = \rho$$

$$\iint_D 3\,dx\,dy = \iint_T 3\cdot|\rho|\,d\rho\,d\theta = \iint_T 3\cdot\rho\,d\rho\,d\theta = 3\int_0^1\left(\int_0^{2\pi}\rho\,d\theta\right)d\rho$$

$$=3\int_0^1\left(\left[\rho\cdot\theta\right]_0^{2\pi}\right)d\rho = 3\int_0^1\left[2\pi\rho\right]d\rho = 3\left[\frac{2\pi\rho^2}{2}\right]_0^1 = 3(1\pi-0\pi)$$

$$=3\pi$$

$D=\{x^2+y^2\leq 1\}$ $\qquad f(x,y)=\sqrt{x^2+y^2}$ $\qquad \begin{cases} x=\rho\cos\theta \\ y=\rho\,\mathrm{sen}\,\theta\end{cases}$ $\qquad T=[0,1]\times[0,2\pi]$

$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \iint_T \sqrt{(\rho\cos\theta)^2+(\rho\,\mathrm{sen}\,\theta)^2}\cdot\rho\,d\rho\,d\theta$$

$$=\int_0^1\left(\int_0^{2\pi}\sqrt{\rho^2}\,\rho\,d\theta\right)d\rho = \int_0^1\left(\int_0^{2\pi}|\rho|\cdot\rho\,d\theta\right)d\rho = \int_0^1\left(\int_0^{2\pi}\rho^2\,d\theta\right)d\rho =$$

$$\int_0^1\left[\rho^2\theta\right]_0^{2\pi}d\rho = \int_0^1\rho^2 2\pi\,d\rho = \left[2\pi\cdot\frac{\rho^3}{3}\right]_0^1 = \frac{2\pi}{3}$$

$D=\{3x^2+2y^2\leq 1\}$ $\qquad f(x,y)=x$

$3x^2+2y^2\leq 1$ $\qquad (\sqrt{3}x)^2+(\sqrt{2}y)^2\leq 1$ $\qquad \begin{cases} v=\sqrt{3}\,x \\ t=\sqrt{2}\,y\end{cases}$ $\qquad T=\{v^2+t^2\leq 1\}$

$v^2+t^2\leq 1$ $\qquad \begin{cases} x=\frac{v}{\sqrt{3}} \\ y=\frac{t}{\sqrt{2}}\end{cases}$

$$J=\begin{array}{cc} v & t \\ \multicolumn{2}{c}{\begin{pmatrix}\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}}\end{pmatrix}}\end{array}$$

$\det J=\frac{1}{\sqrt{6}}$ $\qquad \iint_D x\,dx\,dy = \iint_T\left(\frac{v}{\sqrt{3}}\right)\cdot\frac{1}{\sqrt{6}}\,dv\,dt$

$$=\frac{1}{3\sqrt{2}}\iint_T v\,dv\,dt \qquad T=\{v^2+t^2\leq 1\}$$

$\begin{cases} v=\rho\cos\theta & 0\leq\rho\leq 1 \\ t=\rho\,\mathrm{sen}\,\theta & 0\leq\theta\leq 2\pi\end{cases}$

$$=\frac{1}{3\sqrt{2}}\iint_H \rho\cos\theta\cdot\rho\,d\rho\,d\theta =$$

$H=[0,1]\times[0,2\pi]$

$$=\frac{1}{3\sqrt{2}}\int_0^1\left(\int_0^{2\pi}\rho^2\cos\theta\,d\theta\right)d\rho$$

$$=\frac{1}{\ldots}\int^1\left(\left[\rho^2\,\mathrm{sen}\,\theta\right]^{2\pi}\right)d\rho = \frac{1}{\ldots}\int^1\rho^2\cdot 0\,d\rho = 0$$

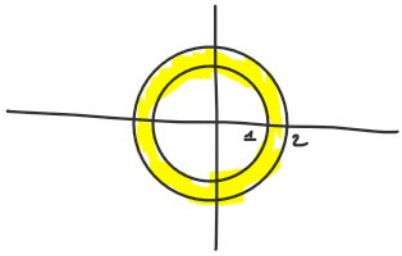
$$= \frac{1}{3\sqrt{2}} \int_0^1 \left(\left[\rho^2 \operatorname{sen}\theta \right]_0^{2\pi} \right) d\rho = \frac{1}{3\sqrt{2}} \int_0^1 \rho^2 \cdot 0 \, d\rho = 0$$

oppure

$$3x^2 + 2y^2 \le 1$$

$$(\sqrt{3}\,x)^2 + (\sqrt{2}\,y)^2 \le 1 \rightsquigarrow \begin{cases} \sqrt{3}\,x = \rho\cos\theta \\ \sqrt{2}\,y = \rho \operatorname{sen}\theta \end{cases} \qquad \begin{cases} x = \dfrac{\rho\cos\theta}{\sqrt{3}} \\ y = \dfrac{\rho \operatorname{sen}\theta}{\sqrt{2}} \end{cases}$$

$$D = \{ 1 \le x^2 + y^2 \le 4 \}$$



$$\iint_D \sqrt{x^2+y^2}\, dx\, dy$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\cos\theta \end{cases} \qquad \begin{matrix} 1 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \end{matrix} \qquad T = [1,2] \times [0, 2\pi]$$

$$\iint_D \sqrt{x^2+y^2}\, dx\, dy = \iint_T \sqrt{\rho^2}\, |\rho|\, d\rho\, d\theta =$$

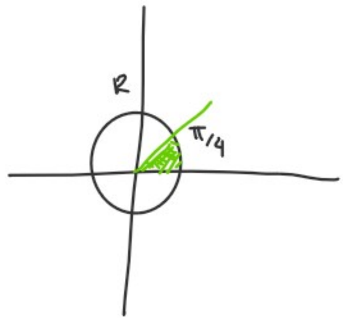
$$\int_1^2 \left(\int_0^{2\pi} \rho^2 \, d\theta \right) d\rho = \int_1^2 \left[\rho^2 \theta \right]_0^{2\pi} d\rho = \int_1^2 2\pi \rho^2 \, d\rho$$

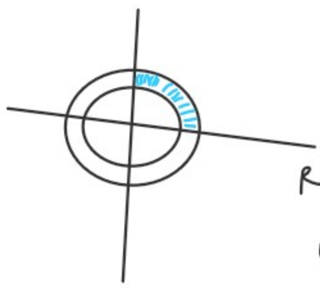
$$= \left[2\pi \frac{\rho^3}{3} \right]_1^2 = \frac{2\pi}{3} (8 - 1) = \frac{14}{3}\pi$$

$$T = [0, R] \times \left[0, \frac{\pi}{4}\right] \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$$

$$0 \le \rho \le R$$

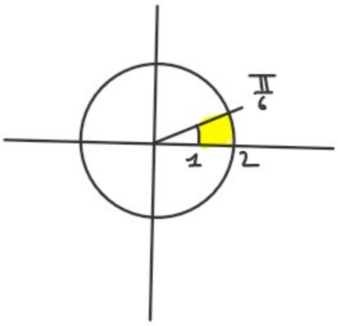
$$0 \le \theta \le \frac{\pi}{4}$$





$$R_1 \le \rho \le R_2 \qquad T = [R_1, R_2] \times \left[0, \frac{\pi}{2}\right]$$

$$0 \le \theta \le \frac{\pi}{2}$$



$$\iint_D e^x \cdot y \; dx\, dy$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \end{cases} \qquad 1 \le \rho \le 2 \qquad 0 \le \theta \le \frac{\pi}{6} \qquad \det(J) = \rho \qquad T = [1,2] \times \left[0, \frac{\pi}{6}\right]$$

$$\iint_D e^x y \, dx\, dy = \iint_T e^{\rho\cos\theta} \cdot \rho\,\text{sen}\,\theta\, |\rho| \, d\rho\, d\theta$$

$$= \int_1^2 \left(\int_0^{\frac{\pi}{6}} e^{\rho\cos\theta} \cdot \text{sen}\,\theta \cdot \rho^2 \, d\theta \right) d\rho = -\int_1^2 \left(\int_0^{\frac{\pi}{6}} \rho \left(-e^{\rho\cos\theta} \cdot \rho\,\text{sen}\,\theta\right) d\theta \right) d\rho$$

$$= -\int_1^2 \left(\rho \left[e^{\rho\cos\theta} \right]_0^{\frac{\pi}{6}} \right) d\rho = -\int_1^2 \rho \left(e^{\rho\frac{\sqrt{3}}{2}} - e^{\rho} \right) d\rho$$

$f \quad g' \qquad\qquad f \cdot g \quad - \int f' \cdot g$

$$\int_1^2 \rho\, e^{a\rho} \, d\rho = \frac{\rho\, e^{a\rho}}{a} - \int 1 \cdot \frac{e^{a\rho}}{a} \, d\rho = \left[\frac{\rho\, e^{a\rho}}{a} - \frac{e^{a\rho}}{a^2} \right]_1^2$$

$$= \left(\frac{2e^{2a}}{a} - \frac{e^{2a}}{a^2} \right) - \left(\frac{e^{a}}{a} - \frac{e^{a}}{a^2} \right) = e^{2a}\left(\frac{2}{a} - \frac{1}{a^2} \right) + e^{a}\left(\frac{1}{a^2} - \frac{1}{a} \right) = h(a)$$

$$= -\int_1^2 \rho\, e^{\rho\frac{\sqrt{3}}{2}} \, d\rho + \int_1^2 \rho\, e^{\rho} \, d\rho =$$

$$= -h\left(\frac{\sqrt{3}}{2}\right) + e^2(2-1) + e(1-1) = -h\left(\frac{\sqrt{3}}{2}\right) + e^2$$

$$D = \left\{ (x-1)^2 + y^2 \le 9 \right\} \qquad f(x,y) = x^2 y$$

$$\begin{cases} x - 1 = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \end{cases} \qquad 0 \le \theta \le 2\pi \qquad 0 \le \rho \le 3 \qquad \begin{cases} x = \rho\cos\theta + 1 \\ y = \rho\,\text{sen}\,\theta \end{cases} \qquad \det(J) = \rho$$

$$T = [0,3] \times [0, 2\pi]$$

$\Big| \; y = \rho \operatorname{sen}\theta$ $\qquad 0 \le \rho \le 3$ $\qquad \Big| \; y = \rho \operatorname{sen}\theta$

$T = [0,3] \times [0,2\pi]$

$$\iint_D x^2 y \, dx\, dy = \iint_T (\rho\cos\theta + 1)^2 \cdot \rho \operatorname{sen}\theta \, |\rho| \, d\rho\, d\theta =$$

$$= \int_0^3 \left(\int_0^{2\pi} (\rho^2 \cos^2\theta + 1 + 2\rho\cos\theta) \cdot \rho^2 \operatorname{sen}\theta \, d\theta \right) d\rho =$$

$$= \int_0^3 \left(\int_0^{2\pi} \rho^4 \cos^2\theta \operatorname{sen}\theta + \rho^2 \operatorname{sen}\theta + 2\rho^3 \cos\theta \operatorname{sen}\theta \, d\theta \right) d\rho =$$

$$= \int_0^3 \left[\frac{-\rho^4 \cos^3(\theta)}{3} - \rho^2 \cos\theta + \rho^3 \operatorname{sen}^2\theta \right]_0^{2\pi} d\rho$$

$$\int_0^3 \left(-\frac{\rho^4}{3}(1-1) - \rho^2(1-1) + 0 \right) d\rho = 0$$