MAT 3

giovedì 19 ottobre 2023 10:59

funzioni composte

$f(x,y)$ differenziabile, $g(t) = \left(t^2 + \ln(t),\ e^{t^2}\right)$ $\quad (g_1(t), g_2(t))$

$f(g(t))$ $\quad$ $g_1, g_2$ derivabili

$f(x,y) = x+y$

$f(g(t)) = f\left(\overset{x}{t^2 + \ln t},\ \overset{y}{e^{t^2}}\right) = t^2 + \ln t + e^{t^2}$

$(f \circ g)'(t) = H'(t) = \nabla f(g(t)) \cdot (g_1'(t),\ g_2'(t))$

$\nabla f = (1,1)$ $\quad = (1,1) \cdot \left(2t + \frac{1}{t},\ 2t\, e^{t^2}\right)$

$f$ differenziabile $\quad g(v,t) = (v^2 + \ln t,\ e^v + t)$

$f = x^2 + y^2 \quad \nabla f = (2x, 2y)$

$f(g(v,t))$

$\frac{\partial (f \circ g)}{\partial v}(v,t) = \nabla f(g(v,t)) \cdot (g_{1v}(v,t),\ g_{2v}(v,t))$

$= \left(2(v^2 + \ln t),\ 2(e^v + t)\right) \cdot (2v,\ e^v) =$

$= 4v(v^2 + \ln t) + 2e^v(e^v + t)$

$(f_x, f_y) \cdot (2v, e^v) = f_x \cdot 2v + f_y \cdot e^v$

$\frac{\partial^2 (f \circ g)}{\partial v \partial t}(v,t) = (f_x \cdot 2v)_t + (f_y \cdot e^v)_t =$

$2v \cdot \left((f_{xx}, f_{xy}) \cdot \left(\frac{1}{t}, 1\right)\right) + e^v \left((f_{yx}, f_{yy}) \cdot \left(\frac{1}{t}, 1\right)\right)$

$2v\, f_{xx} + 2v\, f_{xy} + e^v f_{yx} + e^v f_{yy}$

$$= \frac{2v}{t} f_{xx} + 2v f_{xy} + \frac{e^v}{t} f_{yx} + e^v f_{yy}$$

SIMULAZIONE

$f(x,y) = e^{x^2+y^2+x}$  $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$  $D: \mathbb{R}^2$

$$\nabla f = \left( e^{x^2+y^2+x} \cdot (2x+1),\ e^{x^2+y^2+x} \cdot 2y \right)$$

$$\begin{cases} \cancel{e^{x^2+y^2+x}} (2x+1) = 0 \\ \cancel{e^{x^2+y^2+x}}\ 2y = 0 \end{cases} \quad \begin{cases} 2x+1 = 0 \\ 2y = 0 \end{cases} \quad \begin{cases} x = -\frac{1}{2} \\ y = 0 \end{cases}$$

$x^2+y^2 \leq 1$  $\left(-\frac{1}{2}\right)^2 + 0^2 = \frac{1}{4} < 1$  sta nel dominio A

$f_{xx} = e^{x^2+y^2+x}(2x+1)^2 + e^{x^2+y^2+x} \cdot 2$

$f_{xy} = (2x+1)\ e^{x^2+y^2+x} \cdot 2y$

$f_{yy} = e^{x^2+y^2+x} \cdot (2y)^2 + e^{x^2+y^2+x} \cdot 2$

$f_{xx}\left(-\frac{1}{2},0\right) = e^{-\frac{1}{4}} \cdot 0 + e^{-\frac{1}{4}} \cdot 2 = \frac{2}{e^{\frac{1}{4}}}$

$f_{xy}\left(-\frac{1}{2},0\right) = 0$

$f_{yy}\left(-\frac{1}{2},0\right) = \frac{2}{e^{\frac{1}{4}}}$

$$Hf\left(-\frac{1}{2},0\right) = \begin{pmatrix} \frac{2}{e^{\frac{1}{4}}} & 0 \\ 0 & \frac{2}{e^{\frac{1}{4}}} \end{pmatrix}$$

due autovalori positivi $\Rightarrow$ definita positiva $\Rightarrow$ minimo relativo

$$\begin{cases} e^{x^2+y^2+x}(2x+1) = 2\lambda x \\ e^{x^2+y^2+x}(2y) = 2\lambda y \\ x^2+y^2-1 = 0 \end{cases}$$

$x, y \neq 0$
$$\begin{cases} e^{x^2+y^2+x} \frac{(2x+1)}{2x} = \lambda \\ e^{x^2+y^2+x} = \lambda \\ x^2+y^2-1 = 0 \end{cases} \quad \begin{cases} e^{x^2+y^2+x} \frac{(2x+1)}{2x} = e^{x^2+y^2+x} \\ x^2+y^2-1 = 0 \end{cases}$$

$\frac{2x+1}{2x} = 1$  $2x+1 = 2x$  $1=0$ IMPOSSIBILE

$$\begin{cases} \frac{2x+1}{2x} = 1 \\ x^2+y^2-1=0 \end{cases} \qquad 2x+1 = 2x \qquad 1=0 \text{ IMPOSSIBILE}$$

$$y=0 \qquad e^{x^2+x} \cdot 0 = 0$$

$$\nexists \qquad x^2-1=0 \qquad x^2=1 \begin{cases} x=-1 \\ x=1 \end{cases} \qquad (-1,0) \quad (1,0)$$

$$x=0 \qquad e^{y^2} \cdot 1 = 0 \qquad \text{IMPOSSIBILE}$$

$$\left(-\tfrac{1}{2},0\right) \qquad (-1,0) \qquad (1,0)$$

$$f\left(-\tfrac{1}{2},0\right) = e^{\frac{1}{4}+0-\frac{1}{2}} = e^{-\frac{1}{4}} = \frac{1}{e^{\frac{1}{4}}} \qquad \text{MIN ASSOLUTO}$$

$$f(-1,0) = e^{1+0-1} = e^0 = 1$$

$$f(1,0) = e^{1+0+1} = e^2 \qquad \text{MAX ASSOLUTO}$$

2) $f(x,y) = |x|y$ in $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{|h|\cdot 0 - 0}{h} = \frac{0}{h} \to 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{|0|\cdot(0+h)-0}{h} = 0$$

$$\nabla f(0,0) = (0,0) \qquad (0,0)$$

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k)-f(0,0)-\nabla f(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k)\to(0,0)} \frac{|h|\,k}{\sqrt{h^2+k^2}}$$

$$h=0 \qquad \lim \frac{0}{\sqrt{\ }} = 0 \qquad k=0 \qquad \to 0$$

$h=0 \qquad \lim_{k\to 0} \frac{0}{\sqrt{k^2}} = 0 \qquad\qquad k=0 \qquad \to 0$

$h=k \qquad \lim_{k\to 0} \frac{|k|\cdot k}{\sqrt{2k^2}} = \lim_{k\to 0} \frac{\cancel{|k|}\, k}{\cancel{2}|k|} = 0 \qquad\qquad |h| = \sqrt{h^2}$

$$\left|\frac{|h|\cdot k}{\sqrt{h^2+k^2}}\right| = \frac{|h|}{\sqrt{h^2+k^2}}\cdot|k| = \frac{\sqrt{h^2}}{\sqrt{h^2+k^2}}\cdot|k| = \underbrace{\sqrt{\frac{h^2}{h^2+k^2}}}_{\leq 1}\cdot|k| \leq |k| \xrightarrow[(h,k)\to 0]{} 0$$

la funzione è differenziabile in $(0,0)$.

$f(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)(x-x_0, y-y_0)$

$z = 0 + 0 \qquad z=0$

3) $\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^2+y^2}$

$x=0 \quad \to 0$

$y=0 \quad \to 0$

$x=y \quad \lim_{x\to 0} \frac{2x^4}{2x^2} = 0$

$$\left|\frac{2x^2y^2}{x^2+y^2}\right| = 2x^2\cdot\underbrace{\frac{y^2}{x^2+y^2}}_{\leq 1} \leq 2x^2 \xrightarrow[(x,y)\to(0,0)]{} 0$$

4) $f(x,y) \quad C^\infty(\mathbb{R}^2)$
$$\begin{cases} x = t\cos v \\ y = t^2 - v^2 \end{cases}$$

$H_t = f_x \cdot x_t + f_y \cdot y_t$

$= f_x \cdot \cos v + f_y\, 2t$

$H_v = f_x \cdot x_v + f_y \cdot y_v \quad = -t\,\mathrm{sen}\,v\, f_x - 2v f_y$

$$H_{tt} = (f_x \cdot \cos v)_t + (f_y \cdot 2t)_t$$

$$= \cos v\,(f_{xx} \cdot x_t + f_{xy} \cdot y_t) + (f_{yx} \cdot x_t + f_{yy} \cdot y_t) \cdot 2t + f_y \cdot 2$$

$$= \cos v\,(f_{xx} \cos v + f_{xy} \cdot 2t) + (f_{yx} \cos v + f_{yy} \cdot 2t) \cdot 2t + f_y \cdot 2$$

$$= \cos^2 v\, f_{xx} + 2t \cos v\, f_{xy} + 2t \cos v\, f_{yx} + 4t^2 f_{yy} + 2 f_y$$

$$H_{tv} = (f_x \cdot \cos v)_v + (f_y \cdot 2t)_v$$

$$= (f_{xx} \cdot x_v + f_{xy} \cdot y_v) \cdot \cos v + f_x(-\operatorname{sen} v) + 2t\,(f_{yx} \cdot x_v + f_{yy} \cdot y_v)$$

$$= -t \operatorname{sen} v\, f_{xx} \cdot \cos v - 2v\, f_{xy} \cos v - \operatorname{sen} v\, f_x - 2t^2 \operatorname{sen} v\, f_{yx} - 4vt\, f_{yy}$$

$$H_{vv} = (-t \operatorname{sen} v\, f_x)_v + (-2v\, f_y)_v$$

$$= -t\left[\cos v \cdot f_x + \operatorname{sen} v\,(f_x \cdot x_v + f_y \cdot y_v)\right] - 2\left[f_y + v\,(f_{yx} \cdot x_v + f_{yy} \cdot y_v)\right]$$

$$= -t \cos v\, f_x - t \operatorname{sen} v\,(-t \operatorname{sen} v\, f_x - 2v f_y) - 2f_y - 2v\,(-t \operatorname{sen} v\, f_{yx} - 2v f_{yy})$$

$$= -t \cos v\, f_x + t^2 \operatorname{sen}^2 v\, f_x + 2vt\, f_y - 2f_y + 2vt \operatorname{sen} v\, f_{yx} + 4v^2 f_{yy}$$

$$\lim_{(x,y)\to(0,0)} \frac{h^2}{\sqrt{h^2+k^2}}$$

$$\left|\frac{h^2}{\sqrt{h^2+k^2}}\right| = \frac{|h|\,|h|}{\sqrt{h^2+k^2}} = \frac{\sqrt{h^2}\ |h|}{\sqrt{h^2+k^2}} = \sqrt{\frac{h^2}{h^2+k^2}}\ |h| \leq |h| \to 0$$

---

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y^2}{x^4+y^4}$$

$$(x+y)^2 = x^2+y^2+2xy \geq 0$$

$$x^2+y^2 \geq -2xy$$

$$|2xy| \leq |x^2+y^2|$$

$$|xy| \leq \left|\frac{x^2+y^2}{2}\right|$$

$$\left|\frac{x^3y^2}{x^4+y^4}\right| = \frac{|x|\ |x^2y^2|}{|x^4+y^4|} \leq \frac{|x|\left(\frac{x^4+y^4}{2}\right)}{x^4+y^4}$$

$$= \frac{|x|}{2} \to 0$$

---

$$\begin{cases} x = \rho\cos\theta \\ y = \rho \,\text{sen}\,\theta \end{cases} \qquad \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+2y^2}$$

$$\left|\frac{xy^2}{x^2+2y^2} - \ell\right| \leq g(\rho) \to 0$$

$$\left| \frac{\rho^3 \cos\theta\,\text{sen}^2\theta}{\rho^2\cos^2\theta + 2\rho^2\text{sen}^2\theta}\right| = \left| \frac{\rho^3\cos\theta\,\text{sen}^2\theta}{\rho^2+\rho^2\text{sen}^2\theta}\right| = \left| \frac{\rho\cos\theta\,\text{sen}^2\theta}{1+\text{sen}^2\theta}\right|$$

$$\underbrace{\rho^2\cos^2\theta + \rho^2\text{sen}^2\theta}_{\rho^2} + \rho^2\text{sen}^2\theta$$

$$= |\rho|\,\underbrace{|\cos\theta|}_{\leq 1}\,\underbrace{\left|\frac{\text{sen}^2\theta}{1+\text{sen}^2\theta}\right|}_{\leq 1} \leq |\rho| \to 0$$

---

$$\lambda = (1,1) \qquad \frac{\lambda}{\|\lambda\|} = \frac{(1,1)}{\sqrt{2}} = \left(\underset{\lambda_1}{\frac{1}{\sqrt{2}}}, \underset{\lambda_2}{\frac{1}{\sqrt{2}}}\right)$$

$$\frac{\partial f}{\partial\lambda}(x,y) = \lim_{h\to 0}\frac{f(x+\lambda_1 h, y+\lambda_2 h) - f(x,y)}{h}$$

$$f(x,y) = (x+y)^2 \qquad \lambda = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \qquad (0,0)$$

$$\frac{\partial f}{\partial\lambda}(x,y) = \lim_{h\to 0}\frac{\left(\frac{h}{\sqrt{2}}+\frac{h}{\sqrt{2}}\right)^2 - 0}{h} = \lim_{h\to 0}\frac{(\sqrt{2}h)^2}{h} = \lim_{h\to 0}\frac{2h^{\cancel{2}}}{\cancel{h}} = 0$$

differenziabile $\Rightarrow$ esistono tutte le derivate direzionali

$$\frac{\partial f}{\partial\lambda}(x,y) = \nabla f(x,y)\cdot\lambda \qquad |a\cdot b| \leq |a|\cdot|b|$$

$$\left|\frac{\partial f}{\partial\lambda}(x,y)\right| = |\nabla f(x,y)\cdot\lambda| \leq \|\nabla f(x,y)\|\cdot\underbrace{\|\lambda\|}_{1} = \|\nabla f(x,y)\|$$

il gradiente di $f$ in un punto indica la direzione di massima crescita

---

$$F(x,y) = (\underset{F_1}{x^2-y}, \underset{F_2}{y+2x})$$

$$F(x,y) = (\overset{F_1}{x^2 - y}, \overset{F_2}{y + 2x})$$

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2x + 1$$

$$\operatorname{rot}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

la divergenza di una funzione vettoriale è una funzione scalare

il rotore di una funzione vettoriale è una funzione vettoriale

$f(x,y)$ scalare

$$\Delta f(x,y) = \nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)$$ laplaciano

si dice che una funzione $f$ è armonica se

$\Delta f(x,y) = 0 \quad \forall (x,y) \in$ (campo di esistenza)