$f(x,y)$ è differenziabile in (x_0,y_0) $\qquad \|(h,k)\| = \sqrt{h^2+k^2}$

- è derivabile in (x_0,y_0)
- $\lim_{(h,k)\to(0,0)} \frac{f(x_0+h,y_0+k) - f(x_0,y_0) - \nabla f(x_0,y_0)(h,k)}{\sqrt{h^2+k^2}} = 0$

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \nabla f(x_0,y_0)(h,k) + o(\sqrt{h^2+k^2})$$
$$(h,k)\to(0,0)$$

$x_0 + h = x \quad y_0 + k = y$

$$f(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0) + o(\dots)$$

---

derivabile con continuità $\Rightarrow$ differenziabile $\Rightarrow$ derivabile, $\Rightarrow$ continua

esempio $f(x,y) = x^2 + e^{x+y}$
derivabile con continuità (in finite volte)
$\Rightarrow$ differenziabile in tutti i punti del campo di esistenza ($\mathbb{R}^2$)

il piano tangente ad $f(x,y)$ in $(1,2)$

$$f(x,y) = z = \underset{f(1,2)}{1+e^3} + (2+e^3,\ e^3)\cdot(x-1, y-2) = 1+e^3 + (2+e^3)(x-1) + e^3 y - 2e^3$$

$$\nabla f(x,y) = (2x + e^{x+y},\ e^{x+y}) \qquad \nabla f(1,2) = (2+e^3,\ e^3)$$

$$z = 1 + \cancel{e^3} + x(2+e^3) - 2 - \cancel{e^3} + e^3 y - 2e^3$$
$$z = -1 - 2e^3 + x(2+e^3) + e^3 y$$

una funzione si dice $C^0(A)$ se è continua su A;
si dice di classe $C^k(A)$ se è derivabile k-volte su A con continuità

$$C^\infty(A) \subseteq \dots \subseteq C^k(A) \subseteq \dots \subseteq C^1(A)$$

---

funzioni composte

$f: \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(x,y) = f(x,y)$
$(x,y) \longmapsto f(x,y)$

$g: \mathbb{R} \to \mathbb{R}^2$ $\qquad g(t) = (\overset{g_1(t)}{t^2}, \overset{g_2(t)}{2t+1}) = (g_1(t), g_2(t))$
$t \mapsto (t^2, 2t+1)$

$$H(t) = (f\circ g)(t) = f(g(t)) = f(\overset{x}{t^2}, \overset{y}{2t+1}) = f(t^2, 2t+1)$$

$$H'(t) = \frac{dH}{dt}(t) = \frac{1}{t^2+2t+1}\cdot(2t+2)$$

Teorema
se g_1, g_2 sono derivabili in $\mathbb{R}$, e se $f(x,y)$ è differenziabile
$\Rightarrow g(t) = (g_1(t), g_2(t))$

$$H'(t) = \frac{d}{dt}(f\circ g)(t) = \nabla f(g(t))\cdot(g_1'(t), g_2'(t))$$

$f(x,y) = f(x,y) \qquad g(t) = (\overset{x}{\overset{g_1(t)}{t^2}}, \overset{y}{\overset{g_2(t)}{2t+1}})$

$\nabla f(x,y) = \left(\frac{1}{x+y}, \frac{1}{x+y}\right)$

$\nabla f(g(t)) = \left(\frac{1}{t^2+2t+1}, \frac{1}{t^2+2t+1}\right)$

$g_1'(t) = 2t \qquad g_2'(t) = 2$

$$H'(t) = \left(\frac{1}{t^2+2t+1}, \frac{1}{t^2+2t+1}\right)\cdot(2t, 2) =$$
$$= \frac{2t+2}{t^2+2t+1}$$

---

f differenziabile $\qquad g(\theta) = (\overset{x}{\cos\theta}, \overset{y}{\sin\theta})$ $\qquad \nabla f(x,y) = (f_x, f_y)$

$f(x,y) \qquad f(g(\theta)) = f(\cos\theta, \sin\theta)$

$$\frac{d(f\circ g)}{d\theta}(\theta) = \frac{dH}{d\theta}(\theta) = H'(\theta) = (f_x(\overset{x}{\cos\theta}, \overset{y}{\sin\theta}), f_y(\cos\theta, \sin\theta))\cdot(-\sin\theta, \cos\theta)$$
$$= f_x\cdot(-\sin\theta) + f_y(\cos\theta) =$$
$$-\sin\theta\, f_x + \cos\theta\, f_y$$

---

f differenziabile $\qquad g(t) = (3t-1, \ln(t))$ $\qquad \begin{cases} x = 3t-1 \\ y = \ln(t) \end{cases}$

$$\frac{d(f\circ g)}{dt}(t) = H'(t) = \nabla f(\overset{x}{3t-1}, \overset{y}{\ln(t)})\cdot(3, \tfrac{1}{t})$$
$$= (f_x, f_y)\cdot(3, \tfrac{1}{t}) = 3f_x + \frac{f_y}{t}$$

$f, \qquad g(s,v) = (s^2 - v, v + 2s)$ $\qquad \begin{cases} x = s^2 - v \\ y = v + 2s \end{cases}$

$$f(g(s,v)) = f(s^2 - v, v+2s) = H(s,v)$$

$$\frac{\partial H}{\partial s}(s,v) = \nabla f(g(s,v))\cdot(g_{1s}(s,v), g_{2s}(s,v)) = (f_x, f_y)\cdot(2s, 2) = 2s f_x + 2f_y$$

$$\frac{\partial H}{\partial v}(s,v) = \nabla f(g(s,v))\cdot(g_{1v}(s,v), g_{2v}(s,v)) = (f_x, f_y)\cdot(-1, 1) = -f_x + f_y$$

$$(2s)\cdot f_x + 2f_y$$

$$H_{ss} = 2\cdot f_x + 2s\,(f_x)_s + 2(f_y)_s$$
$$= 2f_x + 2s\left[(f_{xx}, f_{xy})\cdot(2s, 2)\right] + 2\left[(f_{yx}, f_{yy})\cdot(2s, 2)\right]$$
$$= 2f_x + 4s^2 f_{xx} + 4s f_{xy} + 4s f_{yx} + 4 f_{yy}$$

$H_{sv} = 2s f_x + 2f_y \qquad q = (s^2 - v, v + 2s)$

$$H_{sv} = (2s f_x + 2f_y)_v = (2s f_x)_v + (2f_y)_v =$$
$$= 2s(f_x)_v + 2(f_y)_v =$$
$$2s(\nabla f_x, f_{xy})\cdot(-1, 1) + 2(\nabla f_{yx}, f_{yy})\cdot(-1, 1)$$
$$= 2s(-f_{xx} + f_{xy}) + 2(-f_{yx} + f_{yy})$$

$$H_{vs} = (-f_x + f_y)_s = (-f_x)_s + (f_y)_s =$$
$$= -(-f_{xx}\cdot 2s + f_{xy}\cdot 2) + (f_{yx}\cdot 2s + f_{yy}\cdot 2)$$

$H_{vs} = -f_x + f_y$ $\qquad g_s = (2s, 2)$ $\qquad g_v = (-1, 1)$

$$H_{vv} = (-f_x + f_y)_v = -(f_x)_v + (f_y)_v =$$
$$= -(f_{xx}\cdot(-1) + f_{xy}\cdot 1) + (f_{yx}\cdot(-1) + f_{yy}\cdot 1)$$
$$\boxed{= f_{xx} - f_{xy} - f_{yx} + f_{yy}}$$

$f, \quad g = \begin{cases} x = \sin(k+v) \\ y = v^2 k \end{cases}$

$$H_k = f_x\cdot x_k + f_y\cdot y_k$$
$$= f_x(\cos(k+v)) + f_y(v^2) = \cos(k+v) f_x + v^2 f_y$$

$$H_v = f_x\cdot x_v + f_y\cdot y_v = f_x(\cos(k+v)) + f_y(2vk)$$

$$H_{kk} = [\cos(k+v)\cdot f_x]_k + [v^2 f_y]_k = -\sin(k+v)\cdot f_x + \cos(k+v)(f_x)_k + v^2 (f_y)_k$$

$$= -\sin(k+v) f_x + \cos(k+v)\left(f_{xx}\cdot x_k + f_{xy}\cdot y_k\right) + v^2\left(f_{yx}\cdot x_k + f_{yy}\cdot y_k\right)$$
$$= -\sin(k+v) f_x + \cos(k+v)\left(f_{xx}\cos(k+v) + f_{xy} v^2\right) + v^2\left(f_{yx}\cdot\cos(k+v) + f_{yy} v^2\right)$$

$f, \quad g = \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

$$H_\rho = f_x\cdot x_\rho + f_y\cdot y_\rho = \nabla f\cdot(x_\rho, y_\rho)$$
$$= f_x\cos\theta + f_y\sin\theta$$

$$H_{\rho\rho} = (f_x\cos\theta)_\rho + (f_y\sin\theta)_\rho = \cos\theta(f_x)_\rho + \sin\theta(f_y)_\rho$$
$$= \cos\theta(f_{xx}\cdot x_\rho + f_{xy}\cdot y_\rho) + \sin\theta(f_{yx}\cdot x_\rho + f_{yy}\cdot y_\rho)$$
$$= \cos\theta(f_{xx}\cos\theta + f_{xy}\sin\theta) + \sin\theta(f_{yx}\cos\theta + f_{yy}\sin\theta)$$
$$= \cos^2\theta\, f_{xx} + \cos\theta\sin\theta\, f_{xy} + \sin\theta\cos\theta\, f_{yx} + \sin^2\theta\, f_{yy}$$

$$H_{\rho\theta} = (f_x\cos\theta)_\theta + (f_y\sin\theta)_\theta =$$
$$(f_x)_\theta\cdot\cos\theta + f_x(-\sin\theta) + (f_y)_\theta\sin\theta + f_y\cos\theta$$
$$= (f_{xx}\cdot x_\theta + f_{xy}\cdot y_\theta)\cdot\cos\theta - f_x\sin\theta + \sin\theta(f_{yx}\cdot x_\theta + f_{yy}\cdot y_\theta) + f_y\cos\theta$$
$$= (-\rho\sin\theta f_{xx} + \rho\cos\theta f_{xy})\cos\theta - f_x\sin\theta + \sin\theta(-\rho\sin\theta f_{yx} + \rho\cos\theta f_{yy}) + f_y\cos\theta$$
$$= -\rho\sin\theta\cos\theta f_{xx} + \rho\cos^2\theta f_{xy} - f_x\sin\theta$$
$$+ f_y\cos\theta - \rho\sin^2\theta f_{yx} + \rho\cos\theta\sin\theta f_{yy}$$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\qquad \begin{cases} x = -\rho\sin\theta \\ y = \rho\cos\theta \end{cases}$

$$H_{\theta\theta} = f_x\cdot x_\theta + f_y\cdot y_\theta = -\rho\sin\theta f_x + \rho\cos\theta f_y$$

$$H_{\theta\theta} = (-\rho\sin\theta f_x)_\theta + (\rho\cos\theta f_y)_\theta =$$
$$= -\rho(\sin\theta f_x)_\theta + \rho(\cos\theta f_y)_\theta =$$
$$= -\rho\left[\cos\theta f_x + \sin\theta\cdot(f_x)_\theta\right] + \rho\left[-\sin\theta f_y + \cos\theta\cdot(f_y)_\theta\right] =$$
$$= -\rho\cos\theta f_x - \rho\sin\theta f_y - \rho\sin\theta(f_{xx}\cdot x_\theta + f_{xy}\cdot y_\theta)$$
$$+ \rho\cos\theta\left(f_{yx}\cdot x_\theta + f_{yy}\cdot y_\theta\right) =$$
$$= -\rho\cos\theta f_x - \rho\sin\theta f_y - \rho\sin\theta(-\rho\sin\theta f_{xx} + \rho\cos\theta f_{xy})$$
$$+ \rho\cos\theta(-\rho\sin\theta f_{yx} + \rho\cos\theta f_{yy})$$
$$= -\rho\cos\theta f_x - \rho\sin\theta f_y + \rho^2\sin^2\theta f_{xx} - \rho^2\sin\theta\cos\theta f_{xy}$$
$$- \rho^2\cos\theta\sin\theta f_{yx} + \rho^2\cos^2\theta f_{yy}$$