MAT 3 13/12

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esercitazione 8

1) γ: $(t, 2t^2-1)$ $t \in [1,2]$ $\gamma'(t)=(1,4t)$ $\|\gamma'(t)\| = \sqrt{1+16t^2}$

$L(\gamma) = \int_\gamma 1\, ds = \int_1^2 \sqrt{1+16t^2}\, dt$

$\cosh^2 y - \operatorname{senh}^2 y = 1$
$\cosh^2 y = 1 + \operatorname{senh}^2 y$

$16t^2 = \operatorname{senh}^2 y$
$4t = \operatorname{senh} y$
$t = \frac{\operatorname{senh} y}{4}$
$dt = \frac{\cosh y}{4}$

$= \int_1^2 \sqrt{1+\operatorname{senh}^2 y} \cdot \frac{\cosh y}{4}\, dy = \int_1^2 \frac{\cosh^2 y}{4}\, dy =$

$\int_1^2 \frac{(e^y + e^{-y})^2}{16}\, dy = ..$

$\cosh y = \frac{e^y + e^{-y}}{2}$

$\gamma(t) = \left(t, \frac{3t+1}{2}\right)$ $t \in [0,1]$

$\gamma'(t) = \left(1, \frac{3}{2}\right)$ $\|\gamma'(t)\| = \sqrt{1+\frac{9}{4}} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$

$L(\gamma) = \int_\gamma ds = \int_0^1 \frac{\sqrt{13}}{2}\, dt = \frac{\sqrt{13}}{2}(1-0)$

$\gamma(t) = (\ln(t^2), \ln(t))$ $t \in [1,2]$

$\gamma'(t) = \left(\frac{1}{t^2} \cdot (2t), \frac{1}{t}\right) = \left(\frac{2}{t}, \frac{1}{t}\right)$ $\|\gamma'(t)\| = \sqrt{\frac{4}{t^2} + \frac{1}{t^2}} = \sqrt{\frac{5}{t^2}} = \frac{\sqrt{5}}{|t|} = \frac{\sqrt{5}}{t}$

$t \in [1,2]$

$\int_1^2 \frac{\sqrt{5}}{t}\, dt = \sqrt{5}\left[\ln t\right]_1^2 = \sqrt{5} \cdot \ln 2$

$\cosh^2 t - \operatorname{senh}^2 t = 1$

γ: $(\cosh t, \operatorname{senh} t)$ $t \in [0,1]$

$\gamma'(t) = (\operatorname{senh} t, \cosh t)$ $t \in [0,1]$ $\|\gamma'(t)\| = \sqrt{\operatorname{senh}^2 t + \cosh^2 t} = \sqrt{2\operatorname{senh}^2 t + 1}$

$\int_0^1 \sqrt{2\operatorname{senh}^2 t + 1}\, dt$

2) $f(x,y) = x+1-y$ $\gamma = (\underset{x}{1-\cos t}, \underset{y}{1+\operatorname{sen} t})$ $t \in [\pi, 2\pi]$

$\gamma'(t) = (\operatorname{sen} t, \cos t)$ $\|\gamma'(t)\| = \sqrt{1} = 1$

$\int f\, ds$

$\gamma'(t) = (\text{sen}\, t, \cos t)$ $\quad \|\gamma'(t)\| = \sqrt{\ }$

$\int_\gamma f \, ds$

$= \int_\pi^{2\pi} [(1-\cos t) + 1 - (1+\text{sen}\, t)] \, dt = \int_\pi^{2\pi} -\cos t - \text{sen}\, t + 1 \quad dt = \dots$

$f(x,y) = y \qquad \gamma(t) = (t, \frac{1}{2}\cos(2t)) \qquad t \in [-1,1]$

$\gamma'(t) = (1, -\text{sen}(2t)) \qquad \|\gamma'(t)\| = \sqrt{1+\text{sen}^2(2t)}$

$\int_\gamma f \, ds = \int_{-1}^{1} \sqrt{1+\text{sen}^2(2t)} \cdot (\frac{1}{2}\cos(2t)) \, dt = \dots$

3) $\int_\gamma \sqrt{x+2y} \, ds \qquad \gamma$: segmento che unisce $(0,0)$ e $(2,4)$

$m = \frac{4-0}{2-0} = 2$

$\gamma: \begin{cases} x = t \\ y = 2t \end{cases} \quad t \in [0,2] \qquad y - y_0 = 2(x - x_0) \qquad y - 0 = 2(x-0)$, $y = 2x$

$\gamma' = \begin{cases} x = 1 \\ y = 2 \end{cases} \qquad \|\gamma'\| = \sqrt{1+4} = \sqrt{5}$

$\int_0^2 \sqrt{t+4t} \; \sqrt{5} \, dt = \sqrt{5}\int_0^2 \sqrt{5t} \, dt = \sqrt{5}\cdot\sqrt{5}\int_0^2 \sqrt{t} \, dt = 5\int_0^2 t^{\frac{1}{2}} dt$

$= \left[5 \frac{t^{3/2}}{3/2}\right]_0^2 = \frac{2 \cdot 5}{3}(\sqrt{8}) = \frac{10}{3} 2\cdot\sqrt{2} = \frac{20}{3}\sqrt{2}$

4) $\gamma \qquad x^2 + y^2 \leq 4^{r^2}$

$\frac{x^2}{4} + \frac{y^2}{4} \leq 1$

$\begin{cases} x = 2\cos\theta \\ y = 2\,\text{sen}\,\theta \end{cases} \qquad \theta \in [0, \frac{\pi}{2}]$

...

5) $y = \sqrt{x} \qquad x \in [1,2]$

$\begin{cases} x = t \\ y = \sqrt{t} \end{cases} \quad t \in [1,2] \qquad \gamma' = \begin{cases} x' = 1 \\ y' = \frac{1}{2\sqrt{t}} \end{cases} \qquad \|\gamma'(t)\| = \sqrt{1 + \frac{1}{4t}}$

$L(\gamma) = \int_1^2 \sqrt{1+\frac{1}{4t}}$

$\int_\gamma y \, ds = \int_1^2 \sqrt{t}\sqrt{1+\frac{1}{4t}} \, dt = \int_1^2 \sqrt{t}\sqrt{\frac{1}{4t}(4t+1)} \, dt =$

$\int_\gamma \quad \int_1 \sqrt{t}\sqrt{1+\frac{1}{4t}}\,dt \quad \int_1$ [illegible]

$$= \int_1^2 \sqrt{t}\,\frac{1}{\sqrt{4t}}\sqrt{4t+1}\,dt = \int_1^2 \frac{1}{\sqrt{4}}\sqrt{4t+1}\,dt = \frac{1}{2}\left[\frac{(4t+1)^{3/2}}{3/2}\,\frac{1}{4}\right]_1^2 =$$

6) $F = (2xy, x^2)$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}\ ? \qquad \frac{\partial F_1}{\partial y} = \frac{\partial(2xy)}{\partial y} = 2x$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial(x^2)}{\partial x} = 2x$$

è irrotazionale.

$$\Gamma = \{\, x = \cos\theta,\ y = \theta,\ 0 < \theta \le \tfrac{\pi}{2} \,\} \qquad r' = \begin{cases} x' = -\operatorname{sen}\theta \\ y' = 1 \end{cases}$$

$$\int_\Gamma F\cdot T_\Gamma\,ds = \int_0^{\frac{\pi}{2}} (\underbrace{2\cos\theta\cdot\theta}_{F_1}, \underbrace{\cos^2\theta}_{F_2})\cdot\frac{(-\operatorname{sen}\theta, 1)}{\cancel{\|r'(\theta)\|}}\cdot\cancel{\|r'(\theta)\|}\,d\theta =$$

$$= \int_0^{\frac{\pi}{2}} [2\cos\theta\operatorname{sen}\theta\cdot\theta + \cos^2\theta]\,d\theta$$

$$-2\int_0^{\frac{\pi}{2}} \underbrace{\cos\theta\operatorname{sen}\theta}_{f'}\cdot\underbrace{\theta}_{g}\,d\theta + \int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$\left(\frac{\operatorname{sen}^2\theta}{2}\right)' = \frac{2}{2}\operatorname{sen}\theta\cos\theta$$

$$= -2\left[\underbrace{\frac{\operatorname{sen}^2\theta}{2}}_{f}\cdot\underbrace{\theta}_{g}\right]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} \underbrace{\frac{\operatorname{sen}^2\theta}{2}}_{f}\cdot\underbrace{1}_{g'}\,d\theta + \int_0^{\frac{\pi}{2}}\underbrace{\cos^2\theta}_{1-\operatorname{sen}^2\theta} =$$

7) $F = (x-2y, -(2x+y))$

il campo è definito su $\mathbb{R}^2$, s.c., quindi se è irrotazionale è anche conservativo

$$\frac{\partial F_1}{\partial y} = -2 \qquad \frac{\partial F_2}{\partial x} = -2$$

è irrotazionale e quindi conservativo

$$(f_x, f_y) = (F_1, F_2) \qquad \begin{aligned} f_x &= x - 2y \\ f_y &= -(2x+y) \end{aligned}$$

$$f = \int (x-2y)\,dx = \frac{x^2}{2} - 2xy + \varphi(y)$$

$$f_y = -2x + \varphi'(y) = -(2x+y)$$

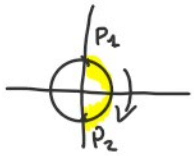
$$-2x+\varphi'(y) = -2x-y$$

$$\varphi'(y) = -y \rightsquigarrow \varphi(y) = \int -y\,dy = -\frac{y^2}{2}+C$$

$$f = \frac{x^2}{2} - 2xy + \varphi(y) = \frac{x^2}{2} - 2xy - \frac{y^2}{2} + C$$

$$f_x = x-2y \qquad f_y = -2x-y$$

$\gamma = \{ x^2+y^2=4$ in senso orario $x \geq 0 \}$



$\gamma : \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}$ $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

senso antiorario

$$\int_{-\gamma} (x-2y, -2x-y)\cdot T\,ds = -\int_\gamma (x-2y, -2x-y)\cdot(-2\sin\theta, 2\cos\theta)\,dt$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta - 4\sin\theta, -4\cos\theta - 2\sin\theta)\cdot(-2\sin\theta, 2\cos\theta)\,dt$$

$$= \dots$$

oppure $= f(P_2) - f(P_1) = -\left[f(\gamma(\tfrac{\pi}{2})) - f(\gamma(-\tfrac{\pi}{2}))\right]$

$$= f(0,-1) - f(0,1) = \frac{0^2}{2} - 2\cdot 0\cdot(-1) - \frac{(-1)^2}{2} - \left(\frac{0^2}{2} - 2\cdot 0\cdot(1) - \frac{(1)^2}{2}\right) = \dots$$

$$F = (2x+y^3, 3xy^2)$$

$g : \nabla g = F \qquad (g_x, g_y) = (2x+y^3, 3xy^2)$

$$g_x = 2x+y^3 \Rightarrow g = \int (2x+y^3)\,dx = x^2 + y^3x + \varphi(y)$$

$$g_y = 0 + 3y^2x + \varphi'(y) = 3xy^2 \Rightarrow \varphi'(y) = 0 \quad \varphi(y) = C$$

$$g = x^2 + y^3x + C$$

$$g_x = 2x+y^3 \qquad g_y = 0 + 3y^2x$$

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sia Γ superficie di parametrizzazione $\varphi : D^{\subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^3$
$(v,t) \longmapsto (\varphi_1(v,t), \varphi_2(v,t), \varphi_3(v,t))$

$g : \mathbb{R}^3 \to \mathbb{R}$

$(v,t) \longmapsto (\varphi_1(v,t), \varphi_2(v,t), \varphi_3(v,t))$

$g: \mathbb{R}^3 \to \mathbb{R}$

data una funzione g continua, l'integrale superficiale di g esteso a Σ

$$\iint_\Sigma g\, d\sigma = \iint_D g(\varphi(v,t)) \cdot |\varphi_v \wedge \varphi_t|\, dv\, dt$$

Σ

$$\varphi: [0,1] \times [2,3] \longrightarrow \mathbb{R}^3$$
$$(v,t) \longmapsto (v^2+t^2,\ 2v-t,\ e^v)$$

$\varphi_v = (2v, 2, e^v)$

$\varphi_t = (2t, -1, 0)$

$$\varphi_v \wedge \varphi_t = \begin{vmatrix} i & j & k \\ 2v & 2 & e^v \\ 2t & -1 & 0 \end{vmatrix} =$$

$$(e^v)\bar{i} + (e^v 2t)\bar{j} + \bar{k}(-2v-4t)$$
$$= (e^v, e^v 2t, -2v-4t)$$

$$|\varphi_v \wedge \varphi_t| = \sqrt{(e^v)^2 + (e^v 2t)^2 + (-2v-4t)^2}$$

$f(x,y,z) = \cos(x+y+z)$

$$\int_\Sigma f\, d\sigma = \int_0^1\int_2^3 \cos(v^2+t^2+2v-t+e^v) \cdot \sqrt{(e^v)^2+(e^v 2t)^2+(-2v-4t)^2}\, dv\, dt = \ldots$$

Σ $\quad \varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(x,y) \longmapsto (x, y, f(x,y))$

$\varphi_x \wedge \varphi_y = (-f_x, -f_y, 1)$

$\|\varphi_x \wedge \varphi_y\| = \sqrt{f_x^2 + f_y^2 + 1}$

$$\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x,y) \longmapsto (x, y, x^2+y^2)$$

calcolo l'elemento d'area

$\varphi_x = (1, 0, 2x)$

$\varphi_y = (0, 1, 2y)$

$$\varphi_x \wedge \varphi_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = i(-2x) + \bar{j}(-2y) + \bar{k}(1)$$
$$= (-2x, -2y, 1)$$

$$\|\varphi_x \wedge \varphi_y\| = \sqrt{4x^2+4y^2+1}$$

calcoliamo l'integrale di superficie di $f(x,y,z) = z$ (con $(x,y) \in [0,1] \times [0,1]$)

$$\int_\mu f \, d\sigma = \int_0^1 \int_0^1 (x^2+y^2) \cdot \sqrt{4x^2+4y^2+1} \; dx\,dy = \dots$$

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integrale doppio

$$\int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy \right) dx \qquad \{a \le x \le b,\ \alpha(x) \le y \le \beta(x)\} = D$$

(calcolo un volume)

se integro $f = 1$ trovo l'area di D.

$$\int_a^b \left(\int_c^d \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\, dz \right) dy \right) dx \qquad \{a \le x \le b,\ c \le y \le d,\ \alpha(x,y) \le z \le \beta(x,y)\} = T$$

se integro $f=1$ trovo il volume di T

$$\int_a^b \| \varphi'(t) \| \, dt$$ lunghezza di una curva $\varphi(t)$ $t \in [a,b]$

int. curvilineo I specie

$$\int_a^b f(\varphi(t)) \cdot \| \varphi'(t) \| \, dt$$ calcolare l'area della regione curvilinea della funzione

II specie

$$\int F(\varphi(t)) \cdot T_\gamma \, dt$$ lavoro lungo una curva di un campo

il segno dipende dall'orientazione della curva

se il campo è conservativo dipende solo dal punto finale e dal punto iniziale

integrale di superficie

$$\int_D f(\varphi(v,t))\, d\sigma$$

$\varphi : D \to \mathbb{R}^3$ superficie

$f : \mathbb{R}^3 \to \mathbb{R}$

$$= \iint_D f(\varphi(v,t)) \cdot \| \varphi_v \wedge \varphi_t \| \, dv\,dt$$

se $f = 1$ si trova l'area della superficie.