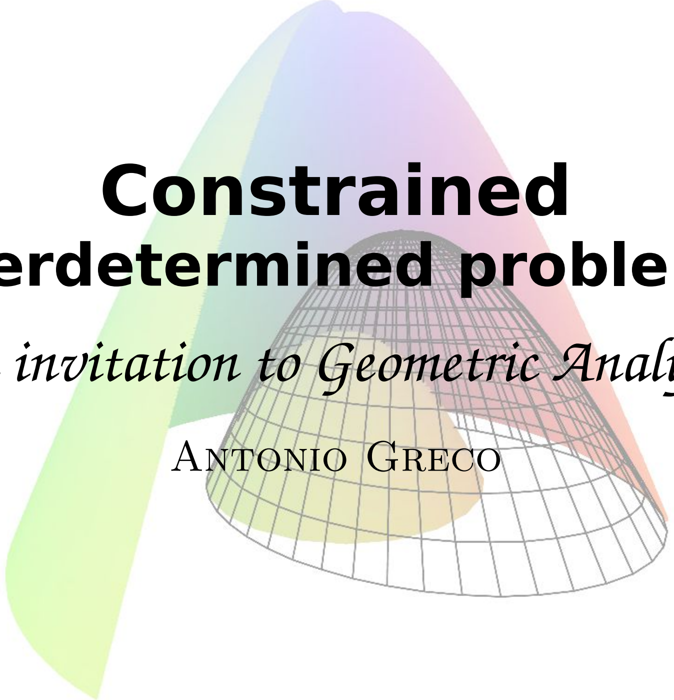


Constrained overdetermined problems

An invitation to Geometric Analysis

Antonio Greco

Symmetry

1. Aleksandrov (1962):

   the only closed surface with constant mean curvature is the sphere

Now let f be smooth, and let Ω be bounded and smooth. Consider *any positive* solution of $-\Delta u = f(u)$ in Ω, vanishing on $\partial\Omega$.

2. Serrin (1971):

   if u satisfies $|Du| = constant$ on $\partial\Omega$, then Ω is a ball

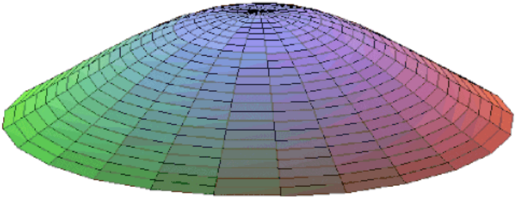
3. Gidas, Ni, and Nirenberg (1979):

   if Ω is convex and symmetric with respect to a hyperplane, then u is also symmetric

Convexity

First eigenfunction of the Laplacian in the disc

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\,\varphi_1 & \text{in } \Omega \\ \varphi_1 = 0 & \text{on } \partial\Omega \end{cases}$$

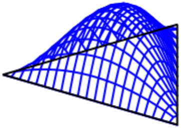


φ_1 is not concave!

But $\log\varphi_1$ is concave for *every* convex Ω!

Torsion function of the equilateral triangle

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



$\sqrt{u}$ is concave for every convex Ω!

The beginning of this story

The beginning of this story

Advances in Mathematical Sciences and Applications
Gakkōtosho, Tokyo, Vol. 9, No. 2(1999), pp. 737-747

OVERDETERMINED PROBLEMS ON RING SHAPED DOMAINS

A. HENROT, G. A. PHILIPPIN AND H. PRÉBET

Theorem 1. *Let $u(\mathbf{x}) \in H^2(\Omega)$ be the solution of the electrostatics problem*

$$\begin{aligned} \Delta u = 0 \quad &\text{in} \quad \Omega := \Omega_1 \setminus \overline{\Omega}_0\,, \\ u = 0 \quad &\text{on} \quad \Gamma_0 := \partial\Omega_0\,, \\ u = a \quad &\text{on} \quad \Gamma_1 := \partial\Omega_1\,, \end{aligned} \tag{1}$$

where a is a positive constant, *overdetermined by the condition*

$$|\nabla u| = q(r) \qquad on\ \Gamma_1\,.$$

Assume Γ_1 to be Lipschitz and $q(r)$ to be positive and nondecreasing. If Ω_0 is a ball centered at the origin, then Ω_1 is a concentric ball and $u = u(r)$.

First improvement

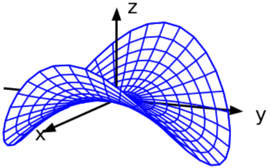
MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2001; **24**:103–115

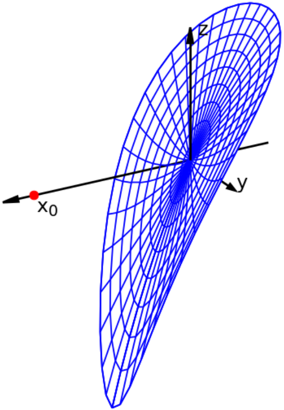
Radial symmetry and uniqueness for an overdetermined problem

Antonio Greco

If $r^{N-1}q(r)$ is non-decreasing, then Ω_1 must be a concentric ball.

The maximum principle





A function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ harmonic in a bounded domain Ω attains its maximum and minimum at the boundary

The *comparison* principle

Theorem. If *two* functions $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$:

- are harmonic in a bounded domain Ω, and
- satisfy $u \geq v$ on $\partial\Omega$, then

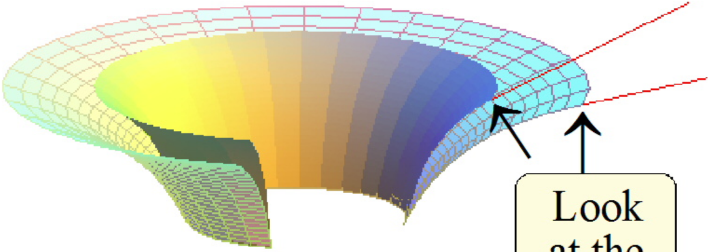
$$u \geq v \text{ in the whole domain.}$$

Proof. The difference $w = u - v$ is harmonic in Ω and satisfies $w \geq 0$ on $\partial\Omega$, hence $w \geq 0$ in Ω by the maximum principle.

Corollary. If, furthermore, $u = v$ at some $P \in \partial\Omega$ admitting an outer normal ν, then

$$\frac{\partial u}{\partial n}(P) \leq \frac{\partial v}{\partial n}(P)$$

The basic observation



Look at the slopes!

In the **radial** case, the gradient on the exterior boundary decreases faster than R_1^{1-N}.

Analytical formulation: completion

Lemma 2.1. Let $\Omega_0 = B(0, R_0)$, $\Omega_1 = B(0, R_1)$, $R_0 < R_1$ and let $u = u(r)$ be the (radially symmetric) solution of (1). Then, we have:

(1) The quantity $R_1^{N-1} |\nabla u(R_1)|$ is a strictly decreasing function of R_1.

(2) The quantity $R_0 |\nabla u(R_0)|$ is a strictly increasing function of R_0.

Proof. The solution u is given by $u(r) = c \int_{R_0}^{r} \frac{\mathrm{d}t}{t^{N-1}}$

where c is the constant such that $u(R_1) = a$. By computation we find

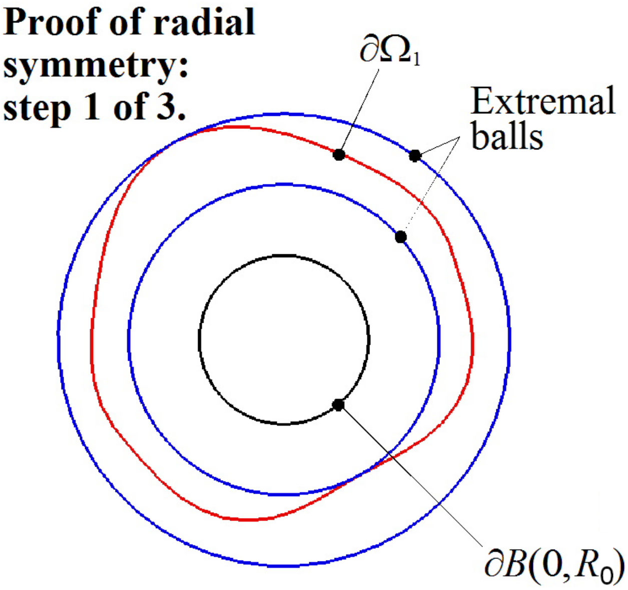
$$|\nabla u(r)| = \frac{a}{r^{N-1} \int_{R_0}^{R_1} (\mathrm{d}t / t^{N-1})}$$

The first claim follows immediately. Furthermore, if we let $r = R_0$ in the previous equality and compute the integral, then we obtain

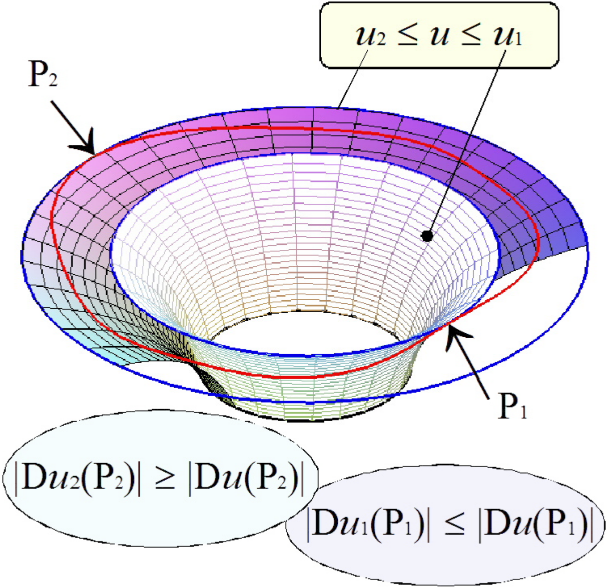
$$|\nabla u(R_0)| = \begin{cases} \dfrac{a}{R_0(\log R_1 - \log R_0)}, & N = 2 \\ \dfrac{(N-2)\, a\, R_1^{N-2}}{R_0(R_1^{N-2} - R_0^{N-2})}, & N > 2 \end{cases} \tag{3}$$

The second claim also becomes apparent and the proof is complete. □

The *extremal* balls



Proof of radial symmetry: step 2 of 3



Proof of radial symmetry: last step

Let $r_1 := |P_1| \le |P_2| =: r_2$.
We have:

$$r_1^{N-1}|Du_1(r_1)| \le r_1^{N-1} q(r_1)$$

by the basic observation $\quad \lor\!| \qquad\qquad |\!\land \quad$ by assumption on q

$$r_2^{N-1}|Du_2(r_2)| \ge r_2^{N-1} q(r_2)$$

The conclusion follows.

Vice versa

Interior boundary

If the solution u to:

$$\begin{cases} \Delta u = 0 & \text{in } B(0,R)\setminus\overline{\Omega}_0 \\ u = 0 & \text{on } \partial\Omega_0 \\ u = 1 & \text{on } \partial B(0,R) \end{cases}$$

satisfies the overdet. condition

$$|\mathrm{D}u(x)| = p(|x|) \quad \text{on } \partial\Omega_0$$

where ***rp(r) is non-increasing***,
then Ω_0 is a ball centred at 0.

Unsatisfying generalization

Both boundaries

Let $0 \in \Omega_0$. **If** the solution u to:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_1 \setminus \bar{\Omega}_0 \\ u = 0 & \text{on } \partial\Omega_0 \\ u = 1 & \text{on } \partial\Omega_1 \end{cases}$$

satisfies the overdet. conditions

$$|Du(x)| = q(|x|) \quad \text{on } \partial\Omega_1$$

$$|Du(x)| = p(|x|) \quad \text{on } \partial\Omega_0$$

where ***$r^{N-1} q(r)$ is non-decreasing*** and ***$r^{N-1} p(r)$ is non-increasing***, **then** Ω_0 and Ω_1 are balls centred at 0.

What happened next

- Condenser-type problems
- Non-autonomous Serrin-type problems
- Partially overdetermined problems in a conical domain

What happened next

- Condenser-type problems
  - 1999. Homofocal ellipsoids (A. Henrot, G. A. Philippin and H. Prébet)
  - 2001. Improvement on homofocal ellipsoids
  - 2003. Approximate radial symmetry (A. Henrot and G. A. Philippin)
  - 2003. Two free boundaries (does not contain the case of one free b.)
  - 2013. Improvement on approximate radial symmetry
  - 2017. Infinity-Laplacian (convex cavity)
  - 2019. Infinity-Laplacian (cavity with positive reach)
  - Finsler p-Laplacian (B. Mebrate, submitted)

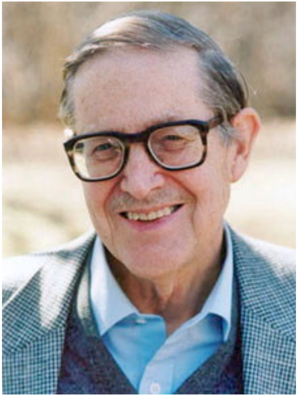
What happened next

- Non-autonomous Serrin-type problems
  - 2003. Beginning: ball, ellipsoid, prescribed curvature, p-Laplacian...
  - 2016. Riemannian geometry (G. Ciraolo and L. Vezzoni)
  - 2017. Infinity-Laplacian
  - 2019. Fractional Laplacian (with V. Mascia, R. Servadei)
  - 2021. p-Laplacian (with F. Pisanu)
  - Normalized p-Laplacian (with L. Cadeddu and B. Mebrate)

What happened next

- Partially overdetermined problems in a conical domain
  - 2021. Finsler Laplacian (with G. Ciraolo)
  - Finsler p-Laplacian (with B. Mebrate, in preparation)

Serrin's symmetry theorem (1971)



$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |Du| = \text{const.} & \text{on } \partial\Omega \end{cases}$$

Let Ω be a smooth bounded domain. The problem above is solvable if and only if Ω is a ball.

Proof based on the *moving plane method* and on Serrin's *corner lemma*.

Remark:

Serrin's problem is invariant under translations

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |Du| = \text{const.} & \text{on } \partial\Omega. \end{cases}$$

Questions:

Which conditions force Ω to be a ball **centred at a prescribed point?**

If $|Du(x)| = q(|x|)$ on $\partial\Omega$, which assumptions on q imply that Ω is a ball **centred at the origin?**

Theorem

Suppose that the ratio

$$\frac{q(r)}{r} \text{ is non-decreasing in } r.$$

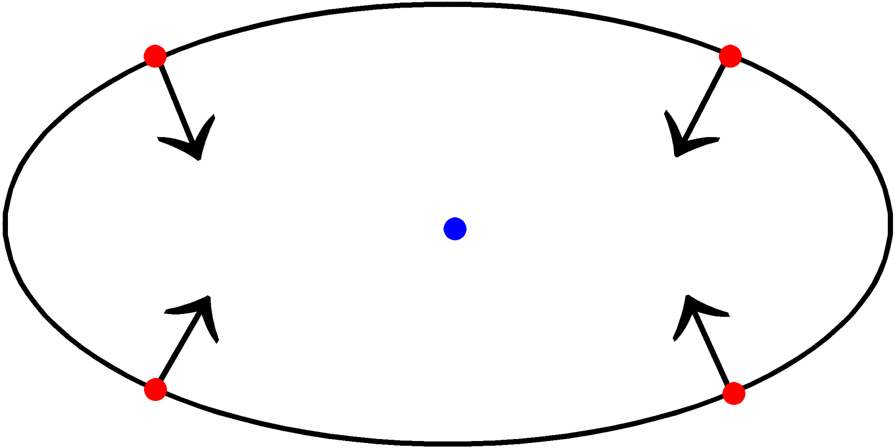
The overdetermined problem

$$\begin{cases} \Delta u = -1 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ |Du(x)| = q(|x|) \text{ for } x \in \partial\Omega \end{cases}$$

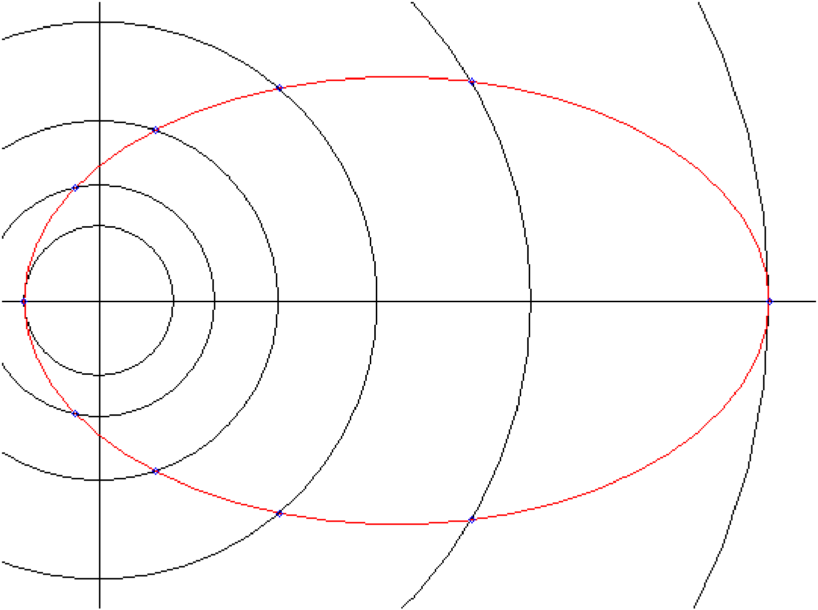
where Ω is a bounded domain of class C^1 containing the origin, is solvable in $C^2(\Omega) \cap C^1(\overline{\Omega})$ only if Ω is a ball centred at the origin.

A. Greco, *Symmetry around the origin for some overdetermined problems,* Adv. Math. Sci. Appl. **13** (2003), 383–395.

Counterexample 1



Counterexample 2



The fractional Laplacian

Let Ω be a bounded domain containing the origin and satisfying the interor ball condition at every $z \in \partial\Omega$. Suppose there exists $u \in C^0(\mathbb{R}^N)$ satisfying

$$\begin{cases} (-\Delta)^s \, u = 1 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega; \\ \displaystyle\lim_{\Omega \ni x \to z} \frac{u(x)}{(\delta_\Omega(x))^s} = q(|z|) & \text{for every } z \in \partial\Omega. \end{cases}$$

If the ratio

$$\frac{q(r)}{r^s}$$

is non-decreasing, then Ω is a ball centred at 0.

This statement: 2016. Extension to $(-\Delta)^s \, u = f(|x|, u)$: 2019.

What happens with the p-Laplacian?

Let Ω be a bounded domain of class C^1 containing the origin, and let $u \in C^1(\overline{\Omega})$ satisfy (in the weak sense)

$$\begin{cases} \Delta_p u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

If $|Du(x)| = q(|x|)$ for all $x \in \partial\Omega$, and the ratio

$$\frac{q(r)}{r^{\frac{1}{p-1}}} = \frac{q(r)}{r^{p'-1}}$$

is non-decreasing, then Ω is a ball centred at 0.

This statement: 2003. Extensions to $-\Delta_p u = f(|x|, u)$: 2013, 2014, 2021.

What if $p = +\infty$?

The *classical* infinity-Laplacian: $\Delta_\infty u = u_{\nu\nu}\,|u_\nu|^2 = \sum_{i,j} u_{ij}\,u_i\,u_j$

The *normalized* infinity-Laplacian: $\Delta_\infty^N u = u_{\nu\nu} = \dfrac{1}{|Du|^2}\,\sum_{i,j} u_{ij}\,u_i\,u_j$

where $\nu = Du$. By the way, the *normalized* p-Laplacian is

$$\Delta_p^N u = \frac{1}{p\,|Du|^{p-2}}\,\mathrm{div}(|Du|^{p-2}\,Du)$$

The *classical* infinity-Laplacian

Let Ω be a bounded domain with a differentiable boundary and containing the origin, and let $u \in C^0(\overline{\Omega})$ be the viscosity solution of

$$\begin{cases} \Delta_\infty u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Then u is differentiable up to $\partial\Omega$. If $|Du(x)| = q(|x|)$ for all $x \in \partial\Omega$, and the ratio

$$\frac{q(r)}{r^{1/3}}$$

is strictly increasing, then Ω is a ball centred at 0.

This statement: 2017. Extension to the *normalized* infinity-Laplacian: submitted

The *normalized* p-Laplacian

Let Ω be a bounded domain with a differentiable boundary and containing the origin, and let $u \in C^0(\overline{\Omega})$ be the viscosity solution of

$$\begin{cases} \Delta_p^N u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

If u satisfies $-\frac{\partial u}{\partial n} = q(|x|)$ in the viscosity sense for all $x \in \partial\Omega$, where n denotes the outer normal, and if the ratio

$$\frac{q(r)}{r}$$

is strictly increasing, then Ω is a ball centred at the origin (no matter what $p \in [1, +\infty]$ is).

Cadeddu, Greco and Mebrate: submitted manuscript

The *Finsler* p-Laplacian

Let Ω be a bounded domain, and suppose that the problem

$$\begin{cases} \Delta_{F;p}\, u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for some $p \in (1, +\infty)$ has a weak solution $u \in C^1(\overline{\Omega})$ satisfying

$$F(Du(z)) = q(F^*(-z)) \quad \forall z \in \partial\Omega.$$

If the ratio

$$\frac{q(r)}{r^{\frac{1}{p-1}}} = \frac{q(r)}{r^{p'-1}}$$

is strictly increasing, then for some $R > 0$ we must have

$$\Omega = \{\, x : F^*(-x) < R \,\}$$

Greco and Mebrate: in preparation

Thank you

for your attention