On the curvature of functions blowing up on the boundary

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1. Content

The idea that a function v that becomes infinite at the boundary of a strictly convex domain Ω must be convex itself, at least near the boundary, goes back to a book by Kawohl [4]. Of course, additional assumptions are needed in order that the result hold. Such assumptions are often satisfied in applications to partial differential equations.

This note starts with a detailed proof of a known result of convexity near the boundary (see [1], [5], [3]). The result is of special interest because it does not make use of any equation. Then a counterexample is constructed that shows that the domain Ω must be convex in the strict sense for the conclusion to hold.

The *mean curvature* of the graph of v is also considered, and conditions are found that imply that it approaches the mean curvature of the cylinder $\partial\Omega \times \mathbb{R}$.

2. Curvature near the boundary

Some definitions to be used in the sequel are collected here.

Definitions.

(1) A domain Ω of class C^2 is told *strictly convex at* $P \in \partial\Omega$ if its boundary has positive principal curvatures at P (with respect to the *inner* normal $-n$).

(2) The domain Ω is *strictly convex* if it is strictly convex at each $P \in \partial\Omega$.

(3) A function $u \in C^2(\Omega)$ is told *strictly concave in the quasi-tangent directions near* $P \in \partial\Omega$ if there exists a positive ε such that $(\partial^2 u/\partial\xi^2)(x) < -\varepsilon$ for all $x \in B(P,\varepsilon) \cap \Omega$ and for every direction $\xi \in S^{N-1} \subset \mathbb{R}^N$ such that $\xi \cdot n < \varepsilon$, where n is the outer normal to $\partial\Omega$ at P.

(4) A function $v \in C^2(\Omega)$ is *strictly convex in the non-tangent directions near* $P \in \partial\Omega$ if there exists a positive ε such that $(\partial^2 v/\partial\xi^2)(x) > \varepsilon$ for all $x \in B(P,\varepsilon) \cap \Omega$ and all ξ satisfying $\xi \cdot n \geq \varepsilon$.

(5) The function v is *strictly convex near* $\partial\Omega$ if there exists a positive ε such that the least eigenvalue of the Hessian matrix $H[v]$ is larger than ε at each $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) < \varepsilon$.

Now we give a detailed proof of the following well-known result:

Theorem 2.1. *Let Ω be a strictly convex bounded domain of class C^2 in $\mathbb{R}^N$. If $u \in C^2(\overline{\Omega})$ satisfies $u > 0$ in Ω, $u|_{\partial\Omega} \equiv 0$, and $\nabla u(P) \neq 0$ for all $P \in \partial\Omega$, then the function $v = -\log u$ is strictly convex near $\partial\Omega$.*

Remark. Note that no assumption is made on the Hessian matrix of u at P. Just the gradient should not vanish. The theorem is consequence of three distinct factors:

a) The strict convexity of $\partial\Omega$ forces u to be strictly concave in the quasi-tangent directions near each $P \in \partial\Omega$;

b) The way $-\log u$ tends to $+\infty$ as $u \to 0$ makes v strictly convex in the non-tangent directions near each $P \in \partial\Omega$;

c) The compactness of $\partial\Omega$ makes the result to hold uniformly with respect to $P \in \partial\Omega$.

The following discussion reflects this structure, and it is divided into two lemmas and a final conclusion.

Lemma 2.2. *Let Ω be a domain of class C^2 in $\mathbb{R}^N$, strictly convex at $P \in \partial\Omega$. If $u \in C^2(\overline{\Omega})$ satisfies $u > 0$ in Ω, $u|_{\partial\Omega} \equiv 0$, and $\nabla u(P) \neq 0$, then u is strictly concave in the quasi-tangent directions near P.*

Proof. Without loss of generality assume that, close to P, $\partial\Omega$ is the graph of a function $x_N = x_N(x_1, \dots, x_{N-1})$ such that $(\partial x_N/\partial x_i)(P) = 0$ for $i = 1, \dots, N-1$. Let us choose the direction $e_N \in S^{N-1} \subset \mathbb{R}^N$ of the x_N-axis so that $e_N = -n$, which implies $(\partial u/\partial x_N)(P) > 0$.

By differentiating twice the equality $u(x_1, \dots, x_{N-1}, x_N(x_1, \dots, x_{N-1})) = 0$ we find, at P:

$$\frac{\partial^2 u}{\partial x_i\, \partial x_j} + \frac{\partial^2 x_N}{\partial x_i\, \partial x_j}\, \frac{\partial u}{\partial x_N} = 0, \qquad i, j = 1, \dots, N-1.$$

By assumption, the matrix $(\partial^2 x_N/\partial x_i\, \partial x_j)(P)$ is positive definite, hence by the equality above it follows that the second derivative $(\partial^2 u/\partial \xi^2)(P)$ is negative and far from zero for every tangent direction $\xi \in S^{N-2} = \{\, \xi \in S^{N-1} \mid \xi \cdot e_n = 0 \,\}$.

Recall that $(\partial^2 u/\partial \xi^2)$ is given by the quadratic form ${}^T\xi H \xi$, which is continuous with respect to both the vector ξ and the Hessian matrix $H = H[u]$. Since H is in turn a continuous function of x, and since S^{N-2} is compact, it follows that there exists a positive ε such that $(\partial^2 u/\partial \xi^2)(x) < -\varepsilon$ for all $x \in B(P, \varepsilon) \cap \Omega$ and all ξ satisfying $\xi \cdot e_N < \varepsilon$. $\square$

Of course, this reflects on $v = -\log u$. Indeed, by the chain rule we obtain

$$\frac{\partial^2 v}{\partial \xi^2} = -\frac{1}{u}\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{u^3}\left(\frac{\partial u}{\partial \xi}\right)^2.$$

This shows the well-known fact that if u is concave in a given direction then v is convex in the same direction. By the preceding lemma we conclude that all the second derivatives of v in the quasi-tangent directions are positive and far from zero in a neighborhood of P. Now we turn to examine non-tangent directions.

Lemma 2.3. *Let Ω be a domain of class C^2 in $\mathbb{R}^N$. If $u \in C^2(\overline{\Omega})$ satisfies $u > 0$ in Ω, $u|_{\partial\Omega} \equiv 0$ and $\nabla u(P) \neq 0$, then $v = -\log u$ is strictly convex in the non-tangent directions near P.*

Proof. Observe that by the smoothness of u we have $\partial^2 u/\partial \xi^2 \leq M$ in Ω. Furthermore, since $\nabla u(P) \neq 0$, we may assume that $|\nabla u| > \varepsilon$ in $B(P, \varepsilon)$. Hence the above formula implies that $\partial^2 v/\partial \xi^2 \geq -M/u + \varepsilon^4/u^3$ in $B(P, \varepsilon)$ and for all ξ such that $\xi \cdot n \geq \varepsilon$. Since u vanishes at P, $\partial^2 v/\partial \xi^2$ tends to $+\infty$ as $\varepsilon \to 0$, uniformly with respect to ξ. □

Proof of Theorem 2.1. By combining the two lemmas we see that v is strictly convex near each $P \in \partial\Omega$. Then the claim follows by the compactness of $\partial\Omega$. □

Remarks.

i) The last lemma does not require convexity of the domain, since it is a consequence of the qualitative properties of the logarithm. Indeed, the function $v = -\log u$ may be replaced by $v = g(u)$ with a suitable g. (**which?**)

ii) The proof of the lemma shows that the second derivatives of v in the non-tangent directions become infinite near P. It is surprising that *the curvatures of v in the non-tangent directions* vanish at P. More precisely, let $' = \partial/\partial\xi$ for shortness and denote by $k_{v,\xi} = v''/(1 + (v')^2)^{-3/2}$ the curvature of the restriction of v to a straight line having direction ξ, evaluated at $x \in \Omega$. By computation we find $k_{v,\xi} := (u^2\, u'' - u\,(u')^2)/(u^2 + (u')^2)^{-3/2}$. Since $u(P) = 0$ and if ξ is non-tangent, we have $u'(P) \neq 0$ and $k_{v,\xi} \to 0$ as $x \to P$.

iii) This suggests to investigate the *mean curvature* of the graph of v near the boundary, which is given by

$$\frac{Mv}{N} := \operatorname{div} \frac{\nabla v}{N\sqrt{1 + |\nabla v|^2}} = \frac{-\Delta u}{N\sqrt{u^2 + |\nabla u|^2}} + \frac{u\,|\nabla u|^2 + u_i\, u_j\, u_{ij}}{N\,(u^2 + |\nabla u|^2)^{3/2}}.$$

To evaluate this expression near $\partial\Omega$, recall that since u is constant on $\partial\Omega$ we have $\Delta u|_{\partial\Omega} = u_{nn} + (N-1)\,H\,u_n$, where H is the mean curvature of $\partial\Omega$ and $u_n = \partial u/\partial n$, n being the outer normal. Furthermore $u \to 0$, and $u_i\, u_j\, u_{ij} \to$

$u_n^2\, u_{nn}$ because ∇u is orthogonal to $\partial\Omega$. Finally, since ∇u is supposed not to vanish on $\partial\Omega$, the denominators in the above expression do not vanish there, and we find $Mv/N \to (1 - 1/N)\, H$. The convexity of Ω has played no role, therefore we have incidentally proved the following statement:

Theorem 2.4. *Let Ω be a (not necessarily bounded) domain of class C^2 in $\mathbb{R}^N$, and let $u \in C^2(\overline{\Omega})$ satisfy $u|_\Omega > 0$, $u|_{\partial\Omega} = 0$, $\nabla u|_{\partial\Omega} \neq 0$. Then the mean curvature of the graph of $v := -\log u$ approaches the mean curvature of the cylinder $\partial\Omega \times \mathbb{R}$.*

Remark. If Ω is a ball $B(0, R)$ then one may assert that, for any radially symmetric and smooth function v tending to $+\infty$ near $\partial\Omega$, there exists a sequence $r_i \to R$ such that $Mv(r_i)/N \to (1-1/N)/R$ (see [2], Corollary 3.3(1) for a proof). The quantity $(1 - 1/N)/R$ equals the mean curvature of the cylinder $\partial\Omega \times \mathbb{R}$.

3. Counterexamples

At this point it is natural to raise the following question:

Question: does Theorem 2.1 hold true even if the domain is only convex, not necessarily strictly convex?

We may also ask whether or not the theorem holds without the assumption that ∇u does not vanish at the boundary. This is a much easier question which will be shortly examined at the end of this section.

A glance to the proof of the theorem reveals that strict convexity of the domain transmits strict concavity in the quasi-tangent directions to the function u, and this determines strict convexity of $v = -\log u$ in those directions.

In case Ω is convex but not strictly convex, the same argument implies that the Hessian matrix $H[u]$ is negative semidefinite on $\partial\Omega$. Unfortunately v is infinite there, and negative semidefiniteness of $H[u]$ does not extend by continuity to a neighborhood of $\partial\Omega$.

Therefore one may suspect that there could be a case in which $H[u]$ is not negative semidefinite near $\partial\Omega$, and the corresponding v is not convex there. This suspicion is well-founded: indeed, the answer to the preceding question is negative, as the following example shows.

Example. Let ϕ be a C^∞ function on the interval $[-1, 1]$, positive in all of $[-1, 1]$, such that $\phi(-1) = \phi(1) = 1$, $\phi^{(k)}(-1) = \phi^{(k)}(1) = 0$ for $k = 1, 2, 3, \ldots$, and $\phi''(0) > 0$.

Let ψ be another C^∞ function on $[-1, 1]$, positive in the open interval $(-1, 1)$, symmetric and such that $\psi(-1) = \psi(1) = 0$, $\psi(0) = 1$, $\psi^{(k)}(0) = 0$ for $k = 1, 2, 3, \ldots$, $\psi'(x) \neq 0$ for $x \in [-1, 0) \cup (0, 1]$.

A possible choice of ϕ and ψ is given by

$$\phi(x) = \begin{cases} 1 - e^{(x-1)^{-2}} & \text{if } |x| < 1; \\ 1 & \text{if } |x| = 1; \end{cases} \qquad \psi(x) = \begin{cases} 1 - e^{1-x^{-2}} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

Denote by Q the square $\{ (x,y) \mid |x|,|y|<1 \} \subset \mathbb{R}^2$ and let Ω be the stadium-shaped domain $\Omega = \{ (x,y) \mid |y|<1,\ |x|<1+\psi^{-1}(1-\psi(y)) \} \subset \mathbb{R}^2$, where ψ^{-1} denotes the inverse of ψ after restriction to the interval $[0,1]$.

Define in Ω the function $u(x,y)$ as follows:

$$u(x,y) = \begin{cases} \phi(x)\,\psi(y) & \text{if } (x,y) \in Q; \\ \psi(|x|-1)+\psi(y)-1 & \text{if } (x,y) \in \Omega \setminus Q. \end{cases}$$

Then u satisfies all the hypotheses of the preceding lemma (see details ahead), Ω is convex, but for $v = \log u$ we have $v_{xx}(0,y) = \phi''(0)/\phi(0) > 0$. This quantity remains far from zero as $|y| \to 1$, hence u is not log-concave near the boundary of Ω.

Details. Let us check that u satisfies the hypotheses of the lemma.

(1) u is positive in Q since ϕ and ψ are, and it is positive in the rest of Ω since for $1 \le |x| < 1+\psi^{-1}(1-\psi(y))$ we have $\psi(|x|-1)+\psi(y)-1>0$.

(2) u is clearly of class $C^\infty(Q) \cap C^\infty(\Omega \setminus \overline{Q}) \cap C^0(\overline{\Omega})$ and it vanishes on $\partial\Omega$. It remains to check that u is smooth up to $\partial\Omega$ and on the two segments $|x|=1$, and that the gradient of u does not vanish on the boundary. We have, for $n+m>0$:

$$\frac{\partial^{n+m}u}{\partial x^n\,\partial y^m} = \begin{cases} \phi^{(n)}(x)\,\psi^{(m)}(y) & \text{in } Q; \\ \begin{cases} (\operatorname{sign}(x))^n\,\psi^{(n)}(|x|-1) & \text{if } m=0; \\ \psi^{(m)}(y) & \text{if } n=0; \\ 0 & \text{if } mn \neq 0; \end{cases} & \text{in } \Omega \setminus \overline{Q}. \end{cases}$$

It is easy to see that each derivative admits a continuous prolonguement to all of $\overline{\Omega}$, and that u_x and u_y do not vanish simultaneously on $\partial\Omega$. It also follows that Ω is of class C^∞.

Remark 1. The theorem is false without the assumption $\nabla u|_{\partial\Omega} \neq 0$. As a counterexample we may take the radial function

$$v(r) = \frac{1}{1-r}\left(2-\sin\frac{1}{1-r}\right).$$

Of course, v can be modified near the origin to obtain a function of class C^∞ in the ball $B(0,1) \subset \mathbb{R}^N$. Then the function $u = e^{-v}$ satisfies the assumptions of Theorem 2.1 excepted that $\nabla u|_{\partial\Omega} \equiv 0$.

It is easy to check that v near the boundary is not convex in the radial direction (i.e., the logarithm is not able to make it convex), nor in the tangent direction (i.e., v does not inherit the convexity of the domain). For instance, if we take $r_n = (n\pi)^{-1}$ then the second radial derivative v_{rr} is such that $v_{rr}(r_{2n}) \to -\infty$, and the second derivative in the tangent direction $v_{tt} = v_r/r$ is such that $v_{tt}(r_{2n+1}) \to -\infty$.

Remark 2. Nevertheless, the assumption $\nabla u|_{\partial\Omega} \neq 0$ in Theorem 2.1 is not necessary. Indeed, the function $u = e^{1/(r^2-1)}$ is smooth and positive in $B(0,1) \subset \mathbb{R}^N$, and may be prolongued to a function of class $C^\infty(\overline{B}(0,1))$ whose gradient vanishes at the boundary, but $v = -\log u = 1/(1-r^2)$ is a convex function.

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