On the Foundations of Geometry

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ON THE FOUNDATIONS OF GEOMETRY*

I

HILBERT'S publication on the foundations of geometry prompted me to express to the author my divergent views. An exchange of letters ensued which, unfortunately, came to an early end. I considered the questions raised in these letters to be of more general interest, however, and was thinking of subsequently publishing them. Hilbert, on the other hand, still withholds his consent because meanwhile his own views have changed. I regret this, because studying these letters would have been the easiest way for the reader to become acquainted with the questions raised. Also, I would have been spared the trouble of rewriting the discussion. But opinions expressed on this subject seem to be so divergent and distant from a solution that I think a public discussion is called for with a view to bringing about an agreement. I should therefore like to consider a number of questions of fundamental importance, and I should like to do this by giving a critical exposition of Hilbert's publication. In doing so it is of no importance whatever whether Hilbert has ceased to hold the disputed views.

To begin with, I should like to discuss two questions: What is an axiom? What is a definition and in what relation do they stand to each other?

For a long time an axiom has always been taken to be a thought whose truth is known without being susceptible of proof by a logical chain of reasoning. Logical laws, too, are of this kind. Yet not everyone would agree to calling these general laws of inference "axioms." Some would rather wish to confine the name "axiom" to the basic laws of a limited field, for example, the field of geometry. But this is a question of less

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consequence. Let us not now enter upon a discussion of why we are justified in predicating truth of an axiom. For the axioms of geometry the usual source given is intuition.

Definition in mathematics usually means a determination of the reference of a word or symbol. Definitions are distinct from all other mathematical propositions in containing a word or symbol which up to then has had no reference; the definition now supplies one. All other mathematical propositions (theorems and ones expressing axioms) must not contain proper names, concept words, relation words or functional symbols, whose reference is not already determined.[1]

Once a word or symbol has been assigned a reference by definition, we can form a self-evident proposition from the definition, which may be used in a proof in the same way as we use basic propositions.[2] Let us suppose, for example, the references of the plus sign, and of the symbols for the numbers three and one, to be known; then we can assign a reference to the symbol for the number four by means of the defining equation "$3 + 1 = 4$." Once this has been done the content of this equation is trivially true and does not require a proof. It would still, however, be inept to include definitions among the basic propositions. For, initially, definitions are arbitrarily laid down and are in this way distinct from statements. Even if we go on to assert whatever has been laid down by definition, it has no greater value as a contribution to knowledge than an example of the law of identity, $a=a$. By defining we do not create knowledge, and we can only say, therefore, that although definitions which have been made into statements formally play the part of basic propositions, they are not really such. For even if, at best, we could call the law of identity itself an axiom, we should hardly wish to give the status of an axiom to every single instance

[1] Letters as a rule have no reference. (There are some exceptions: "π," "e.") They do not designate anything; they only indicate something in order to give generality to a thought. As in the case of certain formal words, we cannot look for their reference. But it must be determined in what way they contribute to the expression of a thought. I have treated in great detail of the use of letters in Bk. I, Sections 8, 9, 17, 19, 24, 25 of my *Grundgesetze der Arithmetik* (Jena, Pohle, 1893).

[2] I will here call a basic proposition a proposition whose sense is an axiom.

of that law. Such a status really demands a greater cognitive value. No definition extends our knowledge. Definitions are only a means of reducing manifold contents to a concise word or symbol and in doing so making them easier to handle. This and only this is the use of definitions in mathematics.[3] A definition must never attempt more than that. And if, notwithstanding, it is yet meant to do more and to generate real knowledge and one wants to avoid having to give a proof, the definition degenerates into logical thimble-rigging. One feels tempted to write in the margin beside some of the definitions one finds in mathematical writings:

> "If you cannot give a demonstration,
> Regard it as an explanation."

One must simply never present as a definition that which requires a proof or an intuition to establish its truth. On the other hand, one can never expect basic propositions and theorems to determine the reference of a word or symbol. The rigor of mathematical investigations makes it absolutely imperative that we should not obscure the difference between definitions and all other propositions.

Axioms *do not* contradict each other because they are true; no proof is necessary to establish this fact. Definitions *must not* contradict each other. In defining we must formulate our basic propositions in such a way as to rule out any possibility of contradiction. Mainly we must avoid giving various explanations of the same symbol.[4]

The usage of the words "axiom" and "definition" as presented in this paper would seem to concur with that employed in works on the subject to date and also seems to be the most useful.

Now as far as Hilbert's publication is concerned, we are struck by a curious confusion of linguistic usage. If he says in the

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[3] It may be counted as part of the usefulness of a definition that it makes us more clearly conscious of a content only half-consciously associated with a word. This sometimes happens, but what is useful in this way is the act of defining rather than the definition. When the definition is once given, it is no matter in the sequel whether the defined sign is a newly invented one or already had a sense associated with it.

[4] Cf. my *Grundgesetze der Arithmetik*, II, Sections 56-57.

introduction, "For the cogent construction of geometry we require only few and simple basic facts, and these facts are called axioms of geometry," this is entirely in keeping with what has been expounded above; equally, when in Section 1, page 4, he says, "The axioms of geometry fall into five groups; each one of these groups expresses certain basic facts of our intuition that belong together."[5]

The following pronouncement (Section 3), however, seems to be based on a totally different conception: "The axioms of this group define the concept 'between.' " How can axioms define? Here we are burdening axioms with something which is really the job of definitions. The same observation is forced on us when we read in Section 6: "The axioms of this group define the concepts of congruence or movement."

Hilbert was good enough to enable me to state in what sense he has been using the word "axiom." For him axioms are components of his definitions.[6] Thus the axioms II 1 to II 5, for example, are components of the definition of "between." "Between" is therefore a relation for those points on a straight line to which the axioms II 1 to II 5 apply. II 1 is thus formulated in the *Festschrift*: "If A, B, C are points on a straight line and B lies between A and C, then B also lies between C and A."

Axioms specify the notes of concepts, which would otherwise be missing from the explanations. The explanation of Section 1 of the *Festschrift* also contains the definitions of the concepts point, straight line, plane, if one includes the axioms of the axiom groups I to V, whose exposition covers the whole of the first chapter. So the first definition is as long as that. Other definitions are embedded in it, for example, that of "between," and theorems, for example, theorems of congruence. It is therefore not easy to see what parts of the first chapter belong to that definition. At least it is difficult to believe that the theorems

[5] Whilst in the first sentence quoted above axioms are thoughts, in the second quotation they are expressions of thoughts, propositions.

[6] As the time when Hilbert was holding this view we must take the time of his writing his *Festschrift* and the date of his letter (29, XII, 99). [I.e., *Festschrift zur Feier der Enthüllung des Gaus-Weber-Denkmals in Göttingen*. Leipzig, 1899.—EDITOR]

should also be regarded as such components. This explains Hilbert's claim that axioms are defining something. But is this compatible with the view that axioms express basic facts of our intuition? If they do, then they assert something. But that means that every expression occurring in them must already be completely understood. Yet if the axioms are components of definitions, they will contain expressions, for example, "point," "straight line," whose references are not previously fixed but are just being determined. No single axiom is then independent, or conceivable at all, except in conjunction with the other axioms belonging to the same definition. If Hilbert's intention is to be met, the reference of the word "point" will only be determined on page 19 of the *Festschrift*. It is not until then that the axioms that have been set out express thoughts which are true by definition. But this is the very reason why they do not express basic facts of our intuition, for then their validity would be founded precisely on intuition. Let us take the following straightforward example: the definition of "a rectangle is a parallelogram with a right angle" may be rewritten in the following way:

"Explanation: we imagine plane figures which we call rectangles.

Axiom 1. All rectangles are parallelograms.

Axiom 2. In every rectangle there is a pair of sides which are perpendicular to each other."

These two axioms must be regarded as inseparable elements of the explanation. If we omitted axiom 1, for example, we should be giving a different reference to the word "rectangle," and the remaining second axiom would receive a different sense, if one gave it the status of a statement in the completed definition, from the sense it now has as a result of its conjunction with the first axiom; that is, axiom 2 would not even be the same proposition any more, at least if one regards the thought expressed in the proposition as an essential feature of the proposition. Once the explanation including both axioms has been given, we can assert them as true. Yet their truth is not based on intuition but on the definition. For this reason they do not contain any real knowledge, as axioms in the customary sense of the word undoubtedly do.

In Chapter 2 of the *Festschrift* Hilbert discusses the questions whether the axioms are independent of each other and are free from contradiction. How are we to understand this independence? In the above example does not each axiom require the other to be what it is? We notice this in other instances too. Only by stating all axioms which, according to Hilbert, belong, for example, to the definition of a point, does the word "point" receive a sense. Accordingly, only through the totality of axioms in which the word "point" occurs, does each one of them receive its full sense. It is impossible to separate the axioms so as to regard some of them as holding and others as not holding, because in doing so we would also alter the sense of the ones we wanted to count as holding good. Thus axioms belonging to the same definitions are dependent on each other, nor are they contradictory; if they were, the definition would be unjustified. Neither can one find out whether the axioms contradict each other before the definition is completed, because only the definition gives a sense to the axioms and it is pointless to inquire whether senseless propositions are contradictory.

How then are we to understand Hilbert's approach to this question? I think we can suppose that he is not concerned with axioms as wholes,[7] but only with those of their parts which express notes of the concept that is being defined. In our example these notes are *parallelogram* and *having two perpendicular sides*. If these were contradictory, there would be no object which could be found to have these properties; in other words, there would be no rectangle. Conversely, if we can produce a rectangle, it follows that these notes are not contradictory. And this is in fact roughly the way in which Hilbert proves that his axioms are free from contradiction. In reality, however, all that has been shown is that the notes of a concept are not contradictory. Similarly with independence. If from the fact that an object has a first property we can conclude that it also has a second one, we can say that the second is dependent on the first. And if these properties are notes of a concept, then the second note is dependent on the first. This is roughly the manner in which Hilbert proves the independence of his axioms (really notes of concepts). For the time being we may view the matter in this way.

[7] As is obvious, I am here employing Hilbert's usage, as I have already been doing earlier.

Yet it is not as simple as may seem to be the case so far. If we wish to fathom the real nature of the problem, we must take a closer look at the special nature of Hilbert's definitions. This we shall do in a succeeding essay.

II

Hilbert's definitions and explanations seem to be of a twofold nature. The first explanation of Section 4 explains the expressions "points on a straight line lying on the same side of a point" and "points on a straight line lying on different sides of a point." Once the expressions "point on a straight line *a*" and "a point lying between a point A and a point B" are understood, this explanation enables us to know quite precisely what these expressions refer to. The explanation of Section 9 is, however, of an entirely different kind. It says:

> Points on a straight line stand in a certain relation to each other, and the word "between" in particular is used to describe that relation.

It is obvious that thereby we are not furnished with the reference of the word "between." But the explanation is also incomplete. It must be supplemented by the following axioms:

> II 1. If A, B, C are points on a straight line, and B lies between A and C, then B also lies between C and A.
> II 2. If A and C are two points on a straight line, then there always is at least one point B, such that it lies between A and C, and at least one point D, such that C lies between A and D.
> II 3. Among any three points on a straight line there always is one and only one point that lies between the other two.
> II 4. Any four points A, B, C, D on a straight line may be ordered in such a way that B lies between A and C and also between A and D, and furthermore that C lies between A and D and also between B and D.

But does the above tell us when the relation of lying-in-between occurs? It does not; but, conversely, once we have understood the relation, we realize the truth of the axioms. If we take as our basis the Gaussean definition of the congruence of numbers, we can easily decide whether 2 and 8 are congruent *modulo* 3, or what sorts of investigation we should have to make to find out. All we have to know are the expressions occurring in the definition ("difference," "one number going into another").

Let us now contrast the above with a further explanation which I have written up following Hilbert's example:

All numbers stand in certain relations to each other, and the word "congruent" in particular is used to describe that relation.

Axiom 1. Every number is congruent to itself under any modulus.

Axiom 2. If any number is congruent to any other number and that number is congruent to a third number under the same modulus, then the first number is also congruent to the third number under that modulus.

Axiom 3. If a first number is congruent to a second and a third number is congruent to a fourth under the same modulus, then the sum of the first and third numbers is also congruent to the sum of the second and fourth numbers under the same modulus, and so forth.

But would such a definition tell us that 2 is congruent to 8 *modulo* 3? Hardly! And it must be noted that in the last example we have given we have a much more favorable case than Hilbert's explanation containing the words "point" and "straight line," whose references are still unknown to us at this stage. But even if we understand these words in the sense that has been given to them in Euclidean geometry, we cannot decide, given our explanation, which of the three points lying on a straight line lies between the other two.

If we survey the whole of Hilbert's explanations and axioms, it would appear comparable to a system of equations with several unknowns; for an axiom as a rule contains several unknown expressions such as "point," "straight line," "plane," "lying," "between," and so forth, and it is not sufficient to state some axioms or groups of axioms to determine the unknowns; only by stating all the axioms can we determine the unknowns. Yet is this totality sufficient for our purpose? Who is to tell us that the system is soluble for the unknowns and that these are uniquely determined? What indeed would a solution look like, if it were possible to give one? Each of the expressions "point," "straight line," and so forth, would have to be explained severally, in a sentence in which all other words are known and understood. If such a solution to the Hilbertean system of definitions and axioms were possible, it would have to be given. But it is certainly impossible. If we want to answer the question whether an object, for example, my watch, is a point, we are at once faced with the difficulty that the first axiom already talks of two points.[8] We must therefore already know an object as a point, in order to be able to answer the question whether my watch, for example, together with that point determines a straight line. This means, however, that we must know how to interpret the word "determine," and also what counts as a straight line. Hence this axiom does not get us any further forward. And we shall find that the same applies to all axioms. When and if we have finally got them all, we still do not know whether they hold good for my watch, so that they enable us to call it a point. Nor do we know what sorts of investigation we shall have to make to answer this question.

Axiom I 7 says: "Every straight line has at least two points." Compare this with the following:
"Explanation: We imagine objects which we call gods.
Axiom 1. Every god is omnipotent.
Axiom 2. There is at least one god."

If this procedure were legitimate, then the ontological proof of God would be brilliantly vindicated. And this takes us to the core of the problem. He who sees quite clearly why such a proof is mistaken also recognizes the fundamental error contained in Hilbert's definition. The error consists in failing to distinguish first- and second-level concepts, as I shall call them. It would appear true to say that it was I who first introduced such a rigid distinction, and Hilbert, when he wrote the *Festschrift*, must have been quite unfamiliar with my writings on this subject.[9] And many others must still be in the same position. But since,

[8] "Two distinct points A, B, always determine a straight line *a*."

[9] The *Foundations of Arithmetic, a logico-mathematical investigation concerning the Concept of Number*. Breslau, Köbner, 1884, Section 53, where I use "order" for "level." *Function and Concept*. Paper given before a meeting of the Society for Medicine and Natural Science at Jena. Jena, Pohle, 1891, p. 26. *Grundgesetze der Arithmetik derived in logical symbolism*, vol. I, Jena, Pohle, 1893, Section 21 seq.

on the other hand, any deeper insight in mathematics and logic without this distinction is impossible, I will try to show very briefly what it is all about.

Let us take the proposition "two is a prime number." Linguistically we distinguish two parts: a subject "two" and a predicative part "is a prime number." With the latter we usually associate assertive force. Yet this is not essential. If an actor utters a statement on the stage, it cannot be said that he *really* asserts anything, nor is he responsible for the truth of that statement. Let us eliminate the assertive force from the predicative part, as it is inessential. The two parts of the proposition will still remain as distinctly different as they are, and it is important to grasp the point that this difference really cuts deep and must not be blurred. The first part, "two," is a proper name of a certain number, designates an object, something complete that does not require a complement.[10] The predicative part, "is a prime number," on the other hand, does require a complement, and does not designate an object. I shall also call the first part *saturated* and the second part *unsaturated*. To this distinction among the symbols there naturally corresponds an analogous distinction in the realm of references: to a proper name corresponds an object, and to the predicative part corresponds what I will call a concept. This is not meant to be a definition. For the decomposition into saturated and unsaturated parts must be regarded as a primitive feature of logical structure, which must simply be recognized and accepted but which cannot be reduced to something more primitive.

I am well aware that expressions like "saturated" and "unsaturated" are figurative and only serve the purpose of pointing to what we have been meaning to talk about; here we must always count on the reader's willingness to meet us halfway. Notwithstanding, we may perhaps be able to show more intelligibly why these parts must be distinct. An object, for example, the number 2, cannot logically adhere to another object, for example, Julius Caesar, without some sort of liaison; and this liaison cannot be an object but must rather be un-

[10] Propositions with "all," "every," "some" are of a totally different kind and will not be considered here.

saturated. A logical combination into a whole can come about only if we saturate or complement an unsaturated part with one or more parts. This is somewhat similar to complementing "the capital of" with "Germany" or "Sweden"; or "half of" with "six."[11]

It now follows from the fundamental difference of objects from concepts that an object can never occur as a predicate or unsaturated expression and that a concept can never logically take the place of an object.[12] Figuratively, this point may be expressed in the following way: there are different logical places; some of them can be filled only by objects and not by concepts, and others only by concepts and not by objects.

Let us now consider the proposition "there is a square root of 4." Obviously we cannot be talking about a particular square root of 4; we are rather dealing with the concept. And here too it has preserved its predicative nature. That this is so can be seen from the fact that we can rewrite the proposition in the following way: "there is something which is a square root of 4," or "it is false that, whatever *a* may be, *a* is not a square root of 4." But in this case we obviously cannot split the proposition up so that the unsaturated part is a concept and the saturated part

[11] What one should regard linguistically as subject is determined by the form of the proposition. For logical analysis it is different. We can break down the proposition "$8 = 2^3$" into either "8" and "is 2 raised to the power of 3," or into "2" and "is something whose third power is 8," or into "3" and "is something such that raising 2 to that power has the result 8."

[12] B. Russell in Section 49 of *The Principles of Mathematics*, vol. 1 (Cambridge, 1903) does not wish to concede that there is a difference of kind between concepts and objects. He maintains that concepts, too, are always terms. He bases his argument on the fact that we are forced to use a concept substantively as a term if we want to say something about it, e.g., that it is not the case that it is a term. This necessity, it seems to me, is only founded in the nature of our language and thus is not genuinely logical. On the other hand, at the bottom of p. 508, Russell seems to lean toward my contention. I have treated of this difficulty in my essay "On Concept and Object." It is obvious that we cannot represent a concept as something independent in the way we can represent an object. A concept can only occur in a complex. One might say that a concept can be distinguished within, but not separated from, the complex in which it occurs. All seeming contradictions which we meet at this juncture result from the fact that one is tempted to treat a concept as an object contrary to its unsaturated nature, which, it is true to say, language sometimes compels us to do. But this is only a point of language.

an object. If we compare the proposition "there is something which is a prime number" with the proposition "there is something which is a square root of 4," we recognize that what they have in common is "there is something which" containing what would genuinely be called the logical predicate, whereas the parts that differ, despite their predicative, unsaturated nature, play a role analogous to that of the subject in other cases. Here there is something being predicated of a concept. But obviously there is a very great difference between the logical place of the number 2, if we predicate of it that it is a prime number, and the concept prime number, if we say that there is something which is a prime number. The first place can be filled only by objects, the second only by concepts. Not only is it linguistically improper to say "there is Africa," or "there is Charlemagne," but it is nonsensical. We may well say "there is something which is called Africa," and the words "is called Africa" signify a concept. Thus the expression *there is something which* is also unsaturated, but in a totally different way from *is a prime number*. In the first case we can saturate the expression only with a concept and in the second only with an object. We take account of the similarity and disparity of these cases by means of the following terminological distinction. In the proposition "2 is a prime number" we say that an object—2—falls *under* a first-level concept—prime number—whereas in the proposition "there is a prime number" we say that a first-level concept—prime number—falls *within* a certain second-level concept. Thus first-level concepts can stand to second-level concepts in a similar relation to that of objects to first-level concepts.

What applies to concepts also applies to their notes. For notes of a concept are concepts which are logical parts of the concept of which they are notes. Instead of saying "2 is the square root of 4, and 2 is positive" we can say "2 is a positive square root of 4," and then we have as the notes of the concept *is a positive square root of 4* the two partial concepts *is a square root of 4* and *is positive*. We may also call these properties of the number 2 and hence we can say: a note of a concept is a property which an object must possess if the object is to fall under the concept. Correspondingly, of course, with second-level concepts. From the above it is easy to see that first-level concepts can have only first-level notes, and second-level concepts only second-level notes. It is quite impossible to have a mixture of first- and second-level notes. This follows from the fact that the logical places of concepts do not serve for objects, nor the logical places of objects for concepts. From this it follows further that our explanation which began with the words "we imagine objects which we call gods" is inadmissible, because the note contained in the first axiom is of the first level, whereas the note contained in the second axiom is of the second level.

What consequences has all this for Hilbert's definitions? Apparently every single point is an object. From this it follows that the concept of point (*is a point*) is of the first level, and that all its notes must consequently be of the first level. But on a perusal of Hilbert's axioms, regarding them as parts of the definition of a point, we find that the notes occurring in these axioms are not of the first level, that is, properties which an object must have in order to be a point, but of the second level. If, therefore, they do define a concept, it can only be a concept of the second level. Whether a concept is really being defined must, however, seem doubtful, since it is not only the word "point" that occurs, but also the words "straight line" and "plane." Disregarding this difficulty for the moment, let us suppose that with his axioms Hilbert has defined a second-level concept. Then we would have to express the relation of the Euclidean concept of point, which is of the first level, to the Hilbertean concept, which is of the second level, by saying that, according to the convention we have adopted above, the Euclidean concept of point falls *within* the Hilbertean concept. It is then conceivable and even probable that this does not apply only to the Euclidean concept of point. Only compare what is said on page 20 (*Festschrift*): "We imagine a pair of numbers (x, y) of the domain omega as a point," and so forth. If the word "point" had already been given a reference by the definition and the axioms belonging to it, then this could not be done all over again at this juncture. I believe we shall have to conceive of the matter in the following way, that the concept *is a pair of numbers of the domain omega* is of the first level, and, in the same way as the Euclidean concept of point, is supposed to fall within the Hilbertean second-level concept (if this exists). But the use of the word "point" in both cases is somewhat disturbing, for it obviously has a different reference in each of them.

On these principles Euclidean geometry is represented as a special case of a more comprehensive system of knowledge which allows for innumerably many other special cases, innumerable geometries (if that word is still permissible). And in every one of those geometries there will be a concept of point (first-level) and each concept will fall within the same second-level concept. If one wanted to use the word "point" in every one of those geometries, it would become ambiguous, and in order to avoid this one would have to add the name of the particular geometry one was talking about, for example, "point of the A-geometry," "point of the B-geometry," and so forth. This will apply similarly to the words "straight line" and "plane." Looked at from this angle it would seem inevitable that under these circumstances we should have to re-examine the questions whether the axioms are free from contradiction and are independent of each other, and also the question of the non-provability of propositions from certain postulates. We shall then not simply be able to say "the axiom of parallels"; since there will be a different axiom of parallels in each different geometry. If we want to use the same terminology we can do so only by mistakenly calling the "straight line of the A-geometry," for example, simply "straight line," thus covering up the fact that the thought it really contains is different. But in doing so we cannot eliminate the difference.

Yet here we have already reached the beginning of a path leading to greater depths. Perhaps I may be allowed some time to pursue it further.

G. FREGE

Jena

GLOSSARY

Anschauung — intuition
Bedeutung — reference
Begriff — concept
Begriff erster Stufe — first-level concept
Behauptungssatz — statement
Bestandteil — part
Eigenschaft — property
Ergänzung — complement
Erkenntniswert — cognitive value
Festschrift — "Festschrift" or publication
Formwort — formal word
Grundgesetz — basic law
Grundsatz — basic proposition
Grundtatsache — basic fact
Grundtatsache unserer Anschauung — basic fact of our intuition
Lehrsatz — theorem
Merkmal — note
Satz — proposition
Urerscheinung — primitive feature
Zeichen — symbol