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Structural Stability and Limit Analysis of Structures

(Instabilità delle strutture e calcolo a rottura)

> Lezione 7

Variational approach to instability of beams

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Statement of the problem

Equilibrium equations

$$EIv'''' + Pv'' = 0,$$

or

$$\left(E(x)I(x)v''\right)'' + \left(P(x)v'\right)' = 0,$$

and boundary conditions:
kinematic $v = 0$, $v' = 0$ at some points, or
static (natural) $M = 0$, $V = 0$, $T = 0$, etc.

We are looking for such a value of $P = P_c$ that for this value the corresponding boundary-value problem has non-zero solution and related $v = v_c(x)$.

Integral identity

Let us assume that we know a critical force and the related solution. So we can provide the following procedure.
Let us multiply the equilibrium equation by $v(x)$ and integrate the result over $(0, l)$. So we get

$$\int_0^l \left[\left(E(x)I(x)v''(x) \right)'' v(x) + \left(Pv'(x) \right)' v(x) \right] dx = 0.$$

Integral identity. Continued

Integrating by parts with our homogeneous boundary conditions we came to

$$\int_0^l \left[E(x)I(x)v''(x)^2 - Pv'(x)^2\right] dx = 0.$$

As P is constant we can extract ii as follows

$$P = \frac{\int_0^l E(x)I(x)\left(v''(x)\right)^2 dx}{\int_0^l \left(v'(x)\right)^2 dx}.$$

This expression is called Rayleigh quotient.

Rayleigh quotient

$$R[v] \equiv \frac{\int\limits_0^l E(x)I(x)\left(v''(x)\right)^2\,dx}{\int\limits_0^l \left(v'(x)\right)^2\,dx}.$$

Results:

- If we know the buckling mode we can calculate the corresponding critical force.
- Rayleigh quotient can be used for numerical calculation of critical forces and related buckling modes.

Rayleigh variational principle

The eigenmodes, i.e. buckling modes, are the stationary points of the Rayleigh quotient $R[v]$ on kinematically admissible functions $v(x)$, i.e. satisfying kinematic boundary conditions.
Moreover, the first critical force can be determined through the minimization of $R[v]$

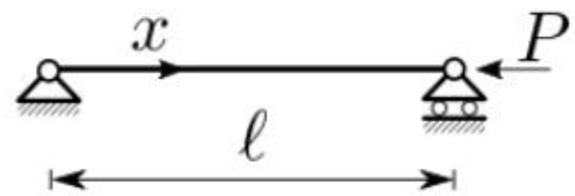
$$P_c = \min_{v(x)} \frac{\int\limits_0^l E(x)I(x)\left(v''(x)\right)^2\,dx}{\int\limits_0^l \left(v'(x)\right)^2\,dx}.$$

For any approximations we have the inequality $P_c \le P_{approx}$. So any P_{approx} gives an upper bound.

Examples

Bounds for a critical force

```
In[1]:= (*variational approcach*)
        (*Simply supported beam*)
        v = x * (1 - x)

Out[1]= (1 - x) x
```



$$v(0) = M(0) = 0$$
$$v(\ell) = M(\ell) = 0$$

$$\frac{\pi^2 EI}{\ell^2}$$

```
In[2]:= v1 = D[v, x]

Out[2]= 1 - 2 x

In[3]:= v2 = D[v1, x]

Out[3]= -2

In[4]:= P = Integrate[v2^2, {x, 0, 1}] / Integrate[v1^2, {x, 0, 1}] EI / l^2
```

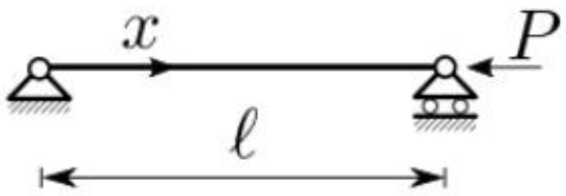
Out[4]= $\frac{12\ EI}{l^2}$

```
In[5]:= (*Exact value*)
        Pexact = Pi^2 EI / l^2
        Pexact = N[Pi^2 EI / l^2]
```

Out[5]= $\frac{EI\ \pi^2}{l^2}$

Out[6]= $\frac{9.8696\ EI}{l^2}$

```
In[7]:= (*Relative Error*)
        Error = (P - Pexact) / Pexact

Out[7]= 0.215854
```

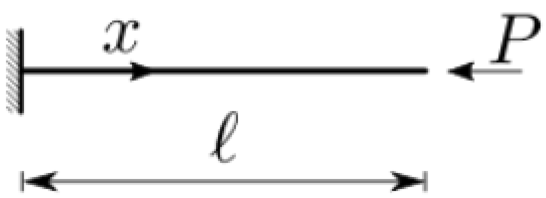


$$v(0) = M(0) = 0$$
$$v(\ell) = M(\ell) = 0$$

$$\frac{\pi^2 EI}{\ell^2}$$

```
(*Exact solution with zero error*)

In[19]:= v = Sin[Pi * x]
Out[19]=
Sin[π x]

In[20]:= v1 = D[v, x]
Out[20]=
π Cos[π x]

In[21]:= v2 = D[v1, x]
Out[21]=
-π^2 Sin[π x]

In[22]:= P = Integrate[v2^2, {x, 0, 1}] / Integrate[v1^2, {x, 0, 1}] EI / l^2
Out[22]=
```

$$\frac{\mathrm{EI}\,\pi^2}{l^2}$$



$$v\,(0) = v'\,(0) = 0$$
$$M\,(\ell) = V\,(\ell) = 0$$

$$\frac{\pi^2 EI}{4\ell^2}$$

$$2\,\ell$$

In[13]:=
```
(*variational approach*)
(*Cantilever beam*)
```

In[14]:=
```
v = x^2
```

Out[14]=

x^2

In[15]:=
```
v1 = D[v, x]
```

Out[15]=

$2\,x$

In[16]:=
```
v2 = D[v1, x]
```

Out[16]=

2

In[17]:=
```
P = Integrate[v2^2, {x, 0, 1}] / Integrate[v1^2, {x, 0, 1}] EI / l^2
```

Out[17]=

$$\frac{3\,\mathrm{EI}}{l^2}$$

In[18]:=
```
(*Exact value*)
Pexact = Pi^2 / 4 EI / l^2
Pexact = N[Pi^2 / 4 EI / l^2]
```

Out[18]=

$$\frac{\mathrm{EI}\,\pi^2}{4\,l^2}$$

Out[19]=

$$\frac{2.4674\,\mathrm{EI}}{l^2}$$

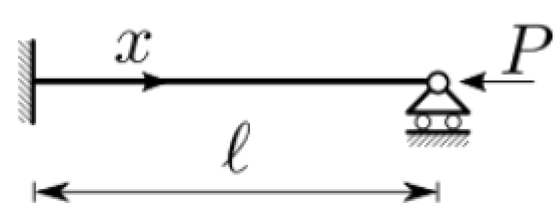
In[20]:=
```
(*Relative Error*)
Error = (P - Pexact) / Pexact
```

Out[20]=

0.215854

In[22]:=
```
(*variational approach*)
(*Clamped - simply supported beam*)

v = x^2 (1 - x)
v1 = D[v, x]
v2 = D[v1, x]
P = Integrate[v2^2, {x, 0, 1}] / Integrate[v1^2, {x, 0, 1}] EI / l^2
(*Exact value*)
Pexact = 2.05 * Pi^2 EI / l^2

(*Relative Error*)
Error = (P - Pexact) / Pexact
```

Out[22]= $(1 - x)\, x^2$



$$v(0) = v'(0) = 0$$
$$v(\ell) = M(\ell) = 0$$
$$2.05\, \frac{\pi^2 EI}{\ell^2}$$

Out[23]= $2\,(1 - x)\, x - x^2$

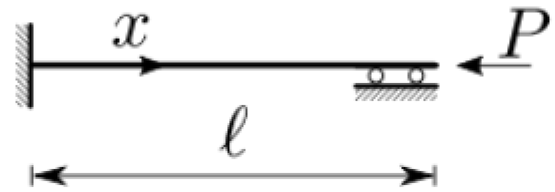
Out[24]= $2\,(1 - x) - 4\, x$

Out[25]= $\dfrac{30\, EI}{l^2}$

Out[26]= $\dfrac{20.2327\, EI}{l^2}$

Out[27]= 0.482749

In[28]:=
```
(*variational approach*)
(*Clamped - clamped beam*)
```



$$v(0) = v'(0) = 0$$
$$v(\ell) = v'(\ell) = 0$$

$$\frac{4\pi^2 EI}{\ell^2}$$

In[29]:=
```
v = x^2 (1 - x)^2
v1 = D[v, x]
v2 = D[v1, x]
P = Integrate[v2^2, {x, 0, 1}] / Integrate[v1^2, {x, 0, 1}] EI / l^2
(*Exact value*)
Pexact = 4 * Pi^2 EI / l^2

(*Relative Error*)
Error = N[(P - Pexact) / Pexact]
```

Out[29]=

$$(1-x)^2 x^2$$

Out[30]=

$$2 (1-x)^2 x - 2 (1-x) x^2$$

Out[31]=

$$2 (1-x)^2 - 8 (1-x) x + 2 x^2$$

Out[32]=

$$\frac{42\,\mathrm{EI}}{l^2}$$

Out[33]=

$$\frac{4\,\mathrm{EI}\,\pi^2}{l^2}$$

Out[34]=

0.0638724