Università degli Studi di Cagliari
**PhD program in Mathematics and Computer Science**

# Geometric Analysis

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# THE MAXIMUM PRINCIPLE

« AN HARMONIC FUNCTION IN A BOUNDED DOMAIN TAKES ITS MINIMUM AND ITS MAXIMUM AT THE BOUNDARY »

TO BE MORE SPECIFIC: LET $\Omega$ BE A BOUNDED OPEN SET IN $\mathbb{R}^N$ AND $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ SUCH THAT $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = 0$ IN $\Omega$. THEN

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$$

$$\min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x)$$

FOLLOWS IMMEDIATELY FROM GAUSS' THEOREM OF THE MEAN: LET $u \in C^2(\Omega)$ BE HARMONIC IN $\Omega$, I.E. LET $\Delta u = 0$ IN $\Omega$ AND PICK $x_0 \in \Omega$. LET $\overline{B}(x_0, R) \subset \Omega$. THEN:

(1) $$u(x_0) = ⨍_{B(x_0,R)} u(x)\,dx = \frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} u(x)\,dx = \frac{1}{\omega_N R^N}\int_{B(x_0,R)} u(x)\,dx$$

(2) $$u(x_0) = ⨍_{\partial B(x_0,R)} u(x)\,d\Sigma = \frac{1}{N\omega_N R^{N-1}}\int_{\partial B(x_0,R)} u(x)\,d\Sigma$$

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THE POINT IS THAT CONDITION $\Delta u = 0$ IS EQUIVALENT TO $\operatorname{div} \nabla u = 0$. THEREFORE IF $\Omega'$ IS SMOOTH AND $\overline{\Omega'} \subset \Omega$ WE HAVE

$$0 = \int_{\Omega'} \operatorname{div} \nabla u\,dx = \int_{\partial\Omega'} n \cdot \nabla u\,d\Sigma$$

$$= \int_{\partial\Omega'} \frac{\partial u}{\partial n}\,d\Sigma, \quad n = \text{OUTER UNIT NORMAL}$$

IN ORDER TO PROVE (2) I PLAN TO CHECK THAT $\frac{1}{N\omega_N R^{N-1}}\int_{\partial B(x_0,R)} u(x)\,d\Sigma$ IS CONSTANT IN $R$.

SECONDLY, I WILL CHECK THAT

$$\lim_{R \to 0^+} \frac{1}{N\omega_N R^{N-1}}\int_{\partial B(x_0,R)} u(x)\,d\Sigma = u(x_0)$$

IN ORDER TO CHECK THAT THE INTEGRAL $\frac{1}{N\omega_N R^{N-1}}\int_{\partial B_R} u(x)\,d\Sigma$ IS CONSTANT IN $R$, LET US COMPUTE $\frac{d}{dR}\frac{1}{N\omega_N R^{N-1}}\int_{\partial B_R} u(x)\,d\Sigma \overset{(1)}{=}$

$$\frac{1-N}{N\omega_N R^N}\int_{\partial B_R} u(x)\,d\Sigma + \frac{1}{N\omega_N R^{N-1}}\frac{d}{dR}\int_{\partial B_R} u(x)\,d\Sigma$$

IN ORDER TO COMPUTE $\frac{d}{dR}\int_{\partial B_R} u(x)\,d\Sigma$ LET US WRITE $\int_{\partial B_R} u(x)\,d\Sigma = \int_{\partial B_1} u(Ry)\,R^{N-1}\,d\omega$

$y \in \partial B_1 \quad x = Ry \quad d\Sigma = R^{N-1}\,d\omega$

THEREFORE WE OBTAIN $\frac{d}{dR}\int_{\partial B_R} u(x)\,d\Sigma =$

$$= \frac{d}{dR}\, R^{N-1}\int_{\partial B_1} u(Ry)\,d\omega =$$

$$= (N-1)\,R^{N-2}\int_{\partial B_1} u(Ry)\,d\omega + R^{N-1}\int_{\partial B_1} \frac{\partial u}{\partial n}(Ry)$$

$$\frac{d}{dR}\int_{\partial B_R} u(x)\,d\Sigma = \frac{(N-1)}{R}\int_{\partial B_R} u(x)\,d\Sigma$$

$$+\int_{\partial B_R}\frac{\partial u}{\partial n}(x)\,d\Sigma = \frac{(N-1)}{R}\int_{\partial B_R} u(x)\,d\Sigma$$

NOW WE PLUG THIS INTO (1) AND OBTAIN

$$\frac{d}{dR}\,\frac{1}{N\omega_N R^{N-1}}\int_{\partial B_R} u(x)\,d\Sigma =$$

$$=\frac{1-N}{N\omega_N R^N}\int_{\partial B_R} u(x)\,d\Sigma$$

$$+\frac{1}{N\omega_N R^{N-1}}\,\frac{(N-1)}{R}\int_{\partial B_R} u(x)\,d\Sigma = 0$$

TO COMPLETE THE PROOF OF CLAIM (2), LET US CHECK THAT

$$\lim_{R\to 0^+}\frac{1}{N\omega_N R^{N-1}}\int_{\partial B(x_0,R)} u(x)\,d\Sigma = u(x_0)$$

IT IS A CONSEQUENCE OF CONTINUITY: LET $\varepsilon\in(0,+\infty)$. THERE EXISTS $\delta\in(0,+\infty)$ SUCH THAT $u(x_0)-\varepsilon < u(x) < u(x_0)+\varepsilon$ FOR EVERY $x\in B(x_0,\delta)$. IF WE TAKE $R<\delta$ WE HAVE

$$\frac{1}{N\omega_N R^{N-1}}\int_{\partial B_R}(u(x_0)-\varepsilon)\,d\Sigma < \frac{1}{N\omega_N R^{N-1}}\int_{\partial B(x_0,R)} u(x)\,d\Sigma$$

$$< \frac{1}{N\omega_N R^{N-1}}\int_{\partial B_R}(u(x_0)+\varepsilon)\,d\Sigma = u(x_0)+\varepsilon$$

IN SHORT

$$u(x_0)-\varepsilon < \frac{1}{N\omega_N R^{N-1}}\int_{\partial B(x_0,R)} u(x)\,d\Sigma < u(x_0)+\varepsilon$$

AND THE CONCLUSION FOLLOWS.

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REMARK CLAIM (1) IS EQUIVALENT TO CLAIM (2). EXERCISE PROVE CLAIM (1).

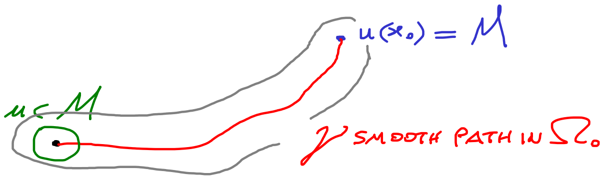
HINT: $$\int_{B(x_0,R)} u(x)\,dx = \int_0^R\left(\int_{\partial B_r} u(x)\,d\Sigma\right)dr$$

EXERCISE DERIVE CLAIM (2) FROM CLAIM (1).

LET US PROVE THE MAXIMUM PRINCIPLE

SUPPOSE THERE EXISTS $x_0\in\Omega$ SUCH THAT $u(x_0)=\max_{x\in\Omega} u(x) =: M$. WE PROVE THAT $u$ IS CONSTANT IN THE CONNECTED COMPONENT $\Omega_0$ OF $\Omega$ CONTAINING $x_0$, I.E., $u(x)=u(x_0)$ FOR ALL $x\in\Omega_0$.

WE ARGUE BY CONTRADICTION. SUPPOSE THERE EXISTS $x_1\in\Omega_0$ SUCH THAT $u(x_1)<u(x_0)$.




$$U_\gamma=\{x : \mathrm{dist}(x,\gamma)<\varrho\}\subset\Omega_0$$

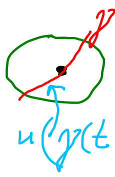
$$\gamma:[0,1]\to\Omega_0$$

$\gamma(0)=x_1$ $\quad\exists t\in(0,1]$

$\gamma(1)=x_0$ $\quad u(\gamma(t))=M$

$t_0$ THE SMALLEST OF SUCH $t$'s

$u(\gamma(t_0))=M$




$t<t_0$ $\quad u(\gamma(t))<M$

CONTRADICTS GAUSS' THEOREM:

$$M=u(\gamma(t_0))=\frac{1}{|B|}\int_B u(x)\,dx < \frac{1}{|B|}\int_B M\,dx = M$$

STRONG MAXIMUM PRINCIPLE:

IF $u(x_0) = M = \max_{\Omega} u$ THEN $u \equiv M$ IN $\Omega_0$

(THE CONNECTED COMPONENT OF $\Omega$ CONTAINING $x_0$)

WEAK MAXIMUM PRINCIPLE:

$$\sup_{\Omega} u = \sup_{\partial\Omega} u$$

MAXIMUM PRINCIPLE À LA BERESTYCKI:

IF $u \geq 0$ ON $\partial\Omega$ THEN $u \geq 0$ IN $\Omega$

WEAK MINIMUM PRINCIPLE:

$$\inf_{\Omega} u = \inf_{\partial\Omega} u$$

IMMEDIATE CONSEQUENCE OF THE PRECEDING THEOREM: COMPARISON PRINCIPLE

SUPPOSE $u, v \in C^0(\bar{\Omega})$, $\Omega$ BOUNDED OPEN, ARE HARMONIC IN $\Omega$.

IF $u \geq v$ ON $\partial\Omega$ THEN $u \geq v$ IN $\bar{\Omega}$

PROOF. THE DIFFERENCE $u - v$ IS HARMONIC AND THEREFORE $\min_{\bar{\Omega}}(u-v) \geq \min_{\partial\Omega}(u-v) \geq 0$.

REMARK THE CONCLUSION STILL HOLDS IF IN PLACE OF $\Delta u = \Delta v = 0$ WE ASSUME $\Delta u = \Delta v$.

THE CONVERSE OF THE THEOREM OF THE MEAN (KOEBE'S THEOREM): IF $u$ IS CONTINUOUS IN AN OPEN SET $\Omega$ AND SATISFIES THE PROPERTY OF THE MEAN (1) OR (2) THEN $u \in C^2(\Omega)$ AND $\Delta u = 0$ IN $\Omega$ (O. D. KELLOGG, FOUNDATIONS OF POTENTIAL THEORY)

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THE MAXIMUM PRINCIPLE STILL HOLDS UNDER THE MORE GENERAL ASSUMPTION THAT $\Delta u \geq 0$

THIS IS IMMEDIATE IF $\Delta u > 0$ BECAUSE AT ANY INTERIOR MAXIMUM $x_0$ WE OBVIOUSLY HAVE $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \leq 0$. MORE GENERALLY, IF $\Delta u \geq 0$, WE HAVE $\int_{\partial\Omega'} \frac{\partial u}{\partial n}\, d\Sigma \geq 0$

FOR EVERY $\bar{\Omega}' \subset \Omega$. FURTHERMORE, WE HAVE

$$\frac{d}{dR}\int_{\partial B_R} u(x)\, d\Sigma \geq \frac{(N-1)}{R}\int_{\partial B_R} u(x)\, d\Sigma$$

AND THEREFORE $\frac{d}{dR}\frac{1}{N\omega_N R^{N-1}}\int_{\partial B_R} u(x)\, d\Sigma \geq$

$$\frac{1-N}{N\omega_N R^N}\int_{\partial B_R} u(x)\, d\Sigma + \frac{1}{N\omega_N R^{N-1}}\frac{(N-1)}{R}\int_{\partial B_R} u(x)\, d\Sigma = 0.$$

AS BEFORE, WE STILL HAVE

$$u(x_0) \xleftarrow[0 \leftarrow R]{} \frac{1}{N\omega_N R^{N-1}}\int_{\partial B_R} u(x)\, d\Sigma$$

HENCE $u(x_0) \leq \frac{1}{N\omega_N R^{N-1}}\int_{\partial B_R} u(x)\, d\Sigma$. HENCE

THE SAME ARGUMENT AS BEFORE LEADS TO

$$M = u(\varphi(t_0)) \leq \frac{1}{|B|}\int_B u(x)\, dx < \frac{1}{|B|}\int_B M\, dx = M$$

CONCLUSION: IF $u(x_0) = M = \max_{\Omega} u$ THEN $u \equiv M$ IN THE COMPONENT $\Omega_0 \ni x_0$. THE STRONG MAXIMUM PRINCIPLE HOLDS FOR SUBHARMONIC FUNCTIONS (THOSE SATISFYING $\Delta u \geq 0$).

$V \equiv 0$ IS HARMONIC; IF $\Delta u \geq 0$ AND $u = V$ ON $\partial\Omega$ THEN $u \leq V$ IN $\Omega$

EBERHARD HOPF (1927) GENERALIZED TO LINEAR ELLIPTIC INEQUALITIES: $u \in C^2(\Omega)$,

$$\sum_{i,j=1}^{N} a^{ij}(x)\, u_{ij}(x) + \sum_{i=1}^{N} b^{i}(x)\, u_i(x) \geq 0 \text{ in } \Omega$$

SHORTENED AS $a^{ij}u_{ij} + b^i u_i \geq 0$, WHERE

$u_i(x) = \dfrac{\partial u}{\partial x_i}$, $u_{ij}(x) = \dfrac{\partial u}{\partial x_i \partial x_j}$, $a^{ij}(x) = a^{ji}(x)$ FOR ALL $x \in \Omega$. ASSUME

$$\lambda(x)|\xi|^2 \leq a^{ij}(x)\,\xi^i \xi^j \leq \Lambda(x)|\xi|^2$$

WITH $\lambda(x) > 0$ IN $\Omega$ (ELLIPTICITY).

WE ALSO NEED $\sup_{\Omega} \dfrac{\Lambda(x)}{\lambda(x)} < +\infty$ UNIFORM ELLIPTICITY

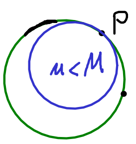
AND $\sup_{x \in \Omega} \dfrac{\max\{|b^1(x)|, \dots, |b^N(x)|\}}{\lambda(x)} < +\infty$

I.E. $\sup_{x \in \Omega} \dfrac{|b^i(x)|}{\lambda(x)} < +\infty$ FOR EACH $i$.

CLAIM: IF $u(x_0) = M = \sup_{\Omega} u$ THEN $u \equiv M$ IN $\Omega_0 \ni x_0$. PROOF. SUPPOSE $\exists x_1 \in \Omega_0$ SUCH THAT $u(x_1) < M$. TAKE A SMOOTH PATH $\gamma : [0,1] \to \Omega_0$ WITH $\gamma(0) = x_1$, $\gamma(1) = x_0$ AND A NEIGHBORHOOD $U_\rho = \{ x \in \mathbb{R}^N : \text{dist}(x, \gamma) < \rho \} \subset \Omega_0$ FOR $\rho$ POSITIVE AND SMALL.


TAKE $B(x_1, r)$ SUCH THAT $u < M$ IN THERE WITH $r < \frac{\rho}{2}$. THERE EXISTS $t \in [0,1)$ SUCH THAT $u < M$ IN $B(\gamma(t), r)$ AND THERE EXISTS $P \in \partial B(\gamma(t), r)$ WITH $u(P) = M$

WITHOUT LOSS OF GENERALITY, THE POINT $P$ IS UNIQUE:




IF NECESSARY, I TAKE AN INTERNALLY TANGENT, SMALLER BALL.

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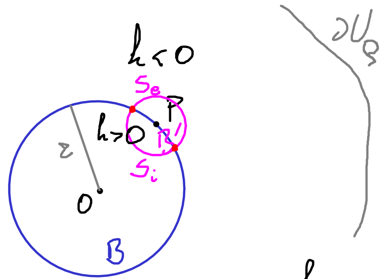
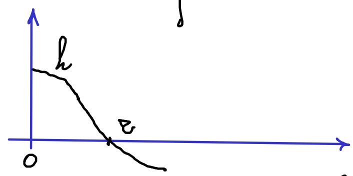



TAKE $B' = B(P, r')$ WITH $r' = \frac{r}{2}$ AND CONSIDER THE RADIAL FUNCTION $h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2}$

WHERE $\alpha$ IS CHOSEN SUCH THAT $a^{ij}(x)\, h_{ij} + b^i(x)\, h_i > 0$ IN $B(P, r') = B'$

$h\big|_{\partial B} = 0$.




BY COMPUTATION, WE FIND $h_i(x) = -2\alpha x_i e^{-\alpha|x|^2}$

$h_{ij}(x) = \left(-2\alpha\delta^{ij} + 4\alpha^2 x_i x_j\right) e^{-\alpha|x|^2}$ AND

THEREFORE $e^{\alpha|x|^2}\left(a^{ij}(x)\, h_{ij} + b^i h_i\right) =$

$= -2\alpha \operatorname{Tr}(a^{ij}(x)) + 4\alpha^2 a^{ij}(x)\, x_i x_j - 2\alpha\, b^i(x)\, x_i$

$= a^{ij}(x)\, x_i x_j \left( -2\alpha \dfrac{\operatorname{Tr}(a^{ij}(x))}{a^{ij} x_i x_j} + 4\alpha^2 - 2\alpha \dfrac{b^i x_i}{a^{ij} x_i x_j} \right)$

BUT $\dfrac{\operatorname{Tr}(a^{ij}(x))}{a^{ij} x_i x_j} \leq \dfrac{N\Lambda(x)}{\lambda(x)|x|^2}$ IS BOUNDED IN $B' \not\ni 0$

AND $\dfrac{|b^i x_i|}{a^{ij} x_i x_j} \leq \dfrac{|b^i x_i|}{\lambda(x)|x|^2}$ IS ALSO BOUNDED

AND THEREFORE $a^{ij}(x)\, h_{ij} + b^i(x)\, h_i > 0$ IN $B'$ FOR $\alpha$ LARGE. HENCE $v = u + \varepsilon h$ SATISFIES:

$a^{ij}(x)\, v_{ij} + b^i(x)\, v_i > 0$ IN $B'$ FOR ALL $\varepsilon > 0$.

THE BOUNDARY OF $B'$ IS MADE UP OF: $\partial B' \cap \overline{B} = S_i = \overline{S}_i$ AND $\partial B' \setminus S_i = S_e$. ON $S_e$ WE HAVE $u \leq M$, $h < 0$ HENCE $v < M$.

FURTHERMORE, $\max_{\overline{S}_i} u = \mu < M$ HENCE IF $\varepsilon$ IS SMALL ENOUGH I GET $v < M$ ON $\overline{S}_i$ AS WELL.

HENCE $\max_{\partial B'} v < M$ BUT $v(P) = u(P) + \varepsilon h(P) = u(P) = M$. THEREFORE

$\max_{\overline{B'}} v \geq M > \max_{\partial B'} v$.

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# THE BOUNDARY-POINT LEMMA (1952)

LEMMA: LET $u \in C^0(\bar{B}) \cap C^2(B)$ SATISFY $a^{ij}(x)\, u_{ij}(x) + b^i(x)\, u_i(x) \geq 0$ IN $B$, WITH $a^{ij} = a^{ji}$, $\lambda(x)\, |\xi|^2 \leq a^{ij}(x)\, \xi^i \xi^j \leq \Lambda(x)\, |\xi|^2$, $\lambda(x) > 0$ IN $B$, $\Lambda(x)/\lambda(x)$ BOUNDED, $|b^i(x)|/\lambda(x)$ BOUNDED. THE MAXIMUM IS ATTAINED AT SOME $P \in \partial B$. LET $n$ BE THE OUTER NORMAL TO $\partial B$ AT $P$ AND SUPPOSE, FOR SIMPLICITY, THAT THERE EXISTS

$$\frac{\partial u}{\partial n}(P) = \lim_{t \to 0^-} \frac{u(P + tn) - u(P)}{t}.$$

THEN EITHER

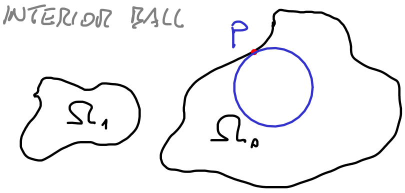
$$\frac{\partial u}{\partial n}(P) > 0 \quad \text{OR} \quad u = \text{CONST. IN } B.$$

REMARK: SINCE $u(P + tn) \leq u(P) = \max_{\bar{B}} u$ AND $t < 0$ WE OBVIOUSLY HAVE $\frac{\partial u}{\partial n}(P) \geq 0$.

REMARK: IN CASE THE LIMIT ABOVE DOES NOT EXIST, WE CAN SAY THAT EITHER $u = $ CONST. IN $B$ OR

$$\liminf_{t \to 0^-} \frac{u(P + tn) - u(P)}{t} > 0.$$

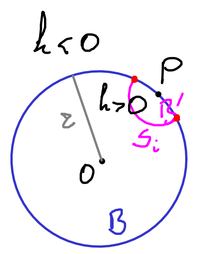
CONSEQUENCE: IF THE BALL $B$ IS REPLACED BY ANY DOMAIN $\Omega_0$ AND $\max_{\bar{\Omega}} u = u(P)$ FOR SOME $P \in \partial\Omega_0$, THEN THE CONCLUSION STILL FOLLOWS PROVIDED THERE EXISTS $B \subset \Omega_0$ WITH $P \in \partial B$ (INTERIOR BALL CONDITION)




EXERCISE: PROVE THAT A DOMAIN OF CLASS $C^1$ DOES NOT SATISFY THE INTERIOR BALL CONDITION AT EACH BOUNDARY POINT, IN GENERAL.

IN THIS RESPECT, SEE THE BOOK BY FRAENKEL.

PROOF OF THE LEMMA. ARGUING AS BEFORE, I.E. SHRINKING THE BALL IF NECESSARY A LITTLE BIT, WE MAY ASSUME THAT EITHER $u = $ CONST. OR THERE IS A UNIQUE $P \in \partial B$ WHERE $u(P) = M = \max_{\bar{B}} u$.




WE TAKE $B' = B(P, r')$ WITH $r' < r$ AS BEFORE AND IN THE SET $G = B \cap B'$ WE CONSIDER $v = u + \varepsilon h$ WITH $h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2}$ AS BEFORE. WE KNOW THAT FOR $\alpha$ LARGE AND $\varepsilon$ SMALL THE FUNCTION $v$ SATISFIES $a^{ij}(x)\, v_{ij}(x) + b^i(x)\, v_i(x) > 0$ IN $G$ AND $v < M$ ON $S_i = \overline{B \cap \partial B'}$. FURTHERMORE $v = u \leq M$ ON $\partial B$, AND $v\big|_{\partial B}(x) = M$ IF AND ONLY IF $x = P$. BY THE (STRAIGHTFORWARD) MAXIMUM PRINCIPLE WE HAVE $v(x) < M$ IN $G$.

THEREFORE BY THE DEFINITION OF DERIVATIVE WE OBVIOUSLY HAVE $\frac{\partial v}{\partial n}(P) \geq 0$. HENCE

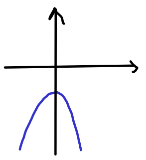
$$\frac{\partial u}{\partial n}(P) = \frac{\partial(v - \varepsilon h)}{\partial n}(P) \geq -\varepsilon \frac{\partial h}{\partial n}(P) =$$

$$= -\varepsilon \frac{dh}{dr}(r) > 0 \quad \text{AS CLAIMED.}$$

WE GO NOW BACK TO THE FIRST PAPER (1927)

LET US CONSIDER $a^{ij}(x)u_{ij}(x) + b^i(x)u_i(x) + c(x)u(x) \geq 0$. EVEN THOUGH WE ASSUME $|c(x)|/\lambda(x)$ BOUNDED, THE CONCLUSION FAILS, IN GENERAL.

COUNTEREXAMPLE. TAKE $u(x) = -1-x^2$ FOR $x \in \mathbb{R}$



WE HAVE $u'(x) = -2x$ AND $u''(x) = -2$

THEREFORE $u(x)$ SATISFIES

$$u''(x) = \frac{2\,u(x)}{1+x^2} \quad \text{I.E.}$$

$$u''(x) + c(x)\,u(x) = 0 \quad \text{WHERE } c(x) = \frac{-2}{1+x^2}.$$

NEVERTHELESS, WE HAVE:

THEOREM. IF $c(x) \leq 0$ IN $\Omega_0$ AND THERE EXISTS $x_0 \in \Omega_0$ SUCH THAT $u(x_0) = \max_{\Omega} u > 0$ THEN $u = \text{CONST.}$ IN SOME $B(x_0, r)$.

PROOF OF THE THEOREM. SINCE $u(x_0) > 0$ THERE IS $B(x_0, r)$ WHERE $u > 0$ AND THEREFORE

$$a^{ij}(x)u_{ij}(x) + b^i(x)u_i(x) \geq -c(x)u(x) \geq 0$$

AND THE CONCLUSION FOLLOWS.

COROLLARY: $u = \text{CONST.}$ IN $\Omega_0$ (CONNECTED).

PROOF. $\Omega_M := \{ x \in \Omega_0 : u(x) = M \}$

IS OPEN IN $\Omega_0$ BY THE THEOREM; BUT IT IS OBVIOUSLY CLOSED, HENCE $\Omega_M = \Omega_0$.



THEOREM. REGARDLESS OF THE SIGN OF $c(x)$, IF THE RATIO $\dfrac{c^-(x)}{\lambda(x)}$ IS BOUNDED, WHERE

$$c^-(x) = \max\{-c(x), 0\} = \begin{cases} 0, & c(x) \geq 0 \\ -c(x), & c(x) < 0 \end{cases}$$

AND THERE IS $x_0 \in \Omega_0$ SUCH THAT $u(x_0) = \max_{\overline{\Omega}_0} u = 0$ THEN $u \equiv 0$ IN SOME $B(x_0, r)$.

COROLLARY: IN FACT, $u \equiv 0$ IN $\Omega_0$ (CONNECTED).

REMARK IF $c(x) \geq 0$ THE ASSUMPTION HOLDS AND IN FACT $0 \leq a^{ij}u_{ij} + b^i u_i + c(x)u \leq a^{ij}u_{ij} + b^i u_i$ AND THE CLAIM FOLLOWS

FOR THE PROOF, WE DEFINE $V(x) = e^{-\alpha x_1} u(x)$ SO THAT $u(x) = e^{\alpha x_1} V(x)$ AND THEREFORE WE DERIVE AN INEQUALITY FOR $V$ WITHOUT ZERO-ORDER TERM. WE GET

$$u_1 = e^{\alpha x_1}\left(\alpha V(x) + V_1\right)$$

$$u_k = e^{\alpha x_1} V_k \quad \text{FOR } k = 2, \dots, N$$

$$u_{11} = e^{\alpha x_1}\left(\alpha^2 V(x) + 2\alpha V_1 + V_{11}(x)\right)$$

$$u_{1k} = e^{\alpha x_1}\left(\alpha V_k(x) + V_{1k}\right) \quad \text{FOR } k \geq 2$$

$$u_{hk} = e^{\alpha x_1} V_{hk} \quad \text{FOR } h, k \geq 2$$

THEREFORE $0 \leq e^{-\alpha x_1}\left(a^{ij}u_{ij} + b^i u_i + c(x)u\right) = a^{ij}V_{ij} + 2\alpha a^{11}V_1 + 2\alpha a^{1k}V_k + b^i V_i + \left(\alpha^2 a^{11} + \alpha b^1 + c(x)\right)V(x)$

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IF I TAKE $\alpha$ LARGE, I GET

$$\left(\alpha^2 a^{11} + \alpha b^1 + c(x)\right) \geq 0$$

INDEED

$$\left(\alpha^2 a^{11} + \alpha b^1 + c(x)\right) =$$

$$\lambda(x)\left(\frac{a^{11}(x)}{\lambda(x)}\alpha^2 + \frac{b^1}{\lambda(x)}\alpha + \frac{c}{\lambda}\right)$$

AND $\inf \frac{c}{\lambda} > -\infty$

THEREFORE I REDUCE TO $a^{ij} v_{ij} + \tilde{b}^i v_i + \tilde{c} v \geq 0$ WITH $\tilde{c} > 0$

IN PARTICULAR, FROM $a^{ij} \xi_i \xi_j \leq \Lambda(x)|\xi|^2$ TAKING $\xi = e_1 = (1, 0, \ldots, 0)$ WE GET $a^{11} \leq \Lambda(x)$ AND $\Lambda(x)/\lambda(x)$ BDD. BY ASSUMPTION.

HERE $\tilde{b}^i = b^i + 2\alpha a^{1i}$ AND $\frac{\tilde{b}^i(x)}{\lambda(x)}$ IS BOUNDED.

CORRESPONDING VERSIONS OF THE BOUNDARY POINT LEMMA HOLD:

IF $a^{ij} u_{ij} + b^i u_i + c(x) u(x) \geq 0$ IN A BALL $B$ AND $u(P) = \max_{\overline{B}} u = 0$

THEN IF $\frac{c^-(x)}{\lambda(x)}$ IS BOUNDED IN $B$

WE GET THAT EITHER $\frac{\partial u}{\partial n}(P) > 0$

OR $u \equiv 0$ IN $\overline{B}$.

Friday, November 17, 2023

## SERRIN'S OVERDETERMINED PROBLEM AND THE MOVING PLANE METHOD

LET $\Omega$ BE A BOUNDED DOMAIN OF CLASS $C^2$ AND $f \in C^1(\mathbb{R})$. SUPPOSE THERE EXISTS A POSITIVE $u \in C^2(\overline{\Omega})$ SATISFYING

$$\begin{cases} \Delta u = f(u) \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \qquad -\frac{\partial u}{\partial n} = \|Du\| \\ \frac{\partial u}{\partial n} = \text{CONST. ON } \partial\Omega \end{cases}$$

THEN $\Omega$ IS A BALL.

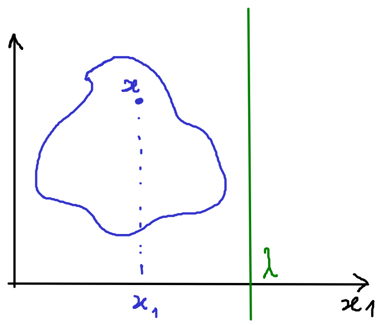
PROOF: FOR EVERY $\xi \in \partial B(0,1)$ THERE EXISTS A PLANE ORTHOGONAL TO $\xi$ SUCH THAT $\Omega$ IS SYMMETRIC WITH RESPECT TO IT.

REMARK: SERRIN'S PROBLEM IS INVARIANT UNDER TRANSLATIONS, I.E. THE BALL $\Omega$ CAN BE CENTERED AT ANY POINT.

REMARK: IF WE REPLACE $\frac{\partial u}{\partial n} = \text{CONST.}$ WITH $-\frac{\partial u}{\partial n} = c|x|$ AND WE KNOW THAT $\Omega \ni 0$ THEN $\Omega = B(0,R)$

WITHOUT LOSS OF GENERALITY LET US TAKE $\xi = e_1 = (1,0,\dots,0)$ AND CONSIDER THE PLANE $T_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\}$.

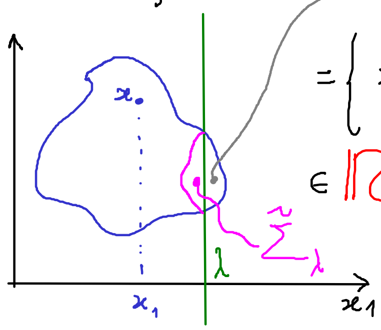
SINCE $\Omega$ IS BOUNDED THERE IS $\lambda > \max_{x \in \overline{\Omega}} x_1$




THE INFIMUM $\lambda_0$ OF SUCH $\lambda$ MAKES $T_{\lambda_0} \cap \overline{\Omega} \neq \emptyset$ AND $T_{\lambda_0} \cap \Omega = \emptyset$

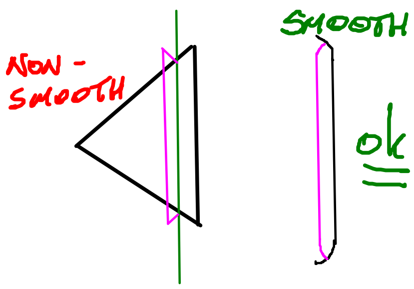
SINCE $\partial\Omega$ BELONGS TO $C^2$, THERE IS $\varepsilon_0 > 0$ SUCH THAT FOR $\lambda_0 - \varepsilon_0 < \lambda < \lambda_0$ THE CAP $\Sigma_\lambda = \{x \in \Omega : x_1 > \lambda\}$ IS REFLECTED INTO $\tilde{\Sigma}_\lambda$

$$= \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : (2\lambda - x_1, x_2, \dots, x_N) \in \Sigma_\lambda\}$$




$\tilde{\Sigma}_\lambda \subset \Omega$

COUNTEREXAMPLE:



**REMARK:** WEINBERGER'S PROOF REQUIRES LESS REGULARITY BUT HOLDS ONLY FOR $\Delta u = -1$: REMARK ON THE PRECEDING PAPER OF SERRIN.

TO PROCEED WITH THE PROOF WE INTRODUCE

$$w^\lambda(x) = u(x^\lambda) - u(x)$$

FOR $x = (x_1, \dots, x_N) \in \Sigma_\lambda$ WITH

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_N) \in \tilde{\Sigma}_\lambda \subset \Omega$$

WE FIND $\Delta w^\lambda = \Delta u(x^\lambda) - \Delta u(x)$

$$= f(u(x^\lambda)) - f(u(x)) =$$

$$= \frac{f(u(x^\lambda)) - f(u(x))}{w^\lambda} w^\lambda, \quad w^\lambda \neq 0$$

AND IN GENERAL $\Delta w^\lambda = c(x) w^\lambda$

$$\text{WHERE } c(x) = \begin{cases} \dfrac{f(u(x^\lambda)) - f(u(x))}{w^\lambda}, & w^\lambda \neq 0 \\ 0, & w^\lambda = 0 \end{cases}$$

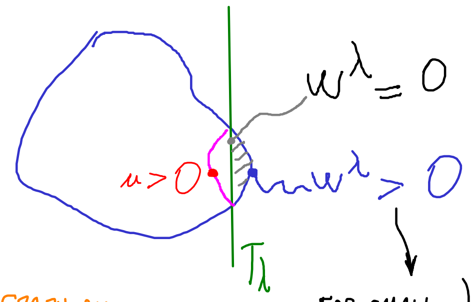
IS BOUNDED IN $\Omega$, THE MAXIMUM PRINCIPLE (MINIMUM, IN FACT) HOLDS PROVIDED THAT $w^\lambda \geq 0$ IN $\Sigma_\lambda$.

IF $\varepsilon_0$ IS SUFFICIENTLY SMALL, THIS IS ACHIEVED FROM $-\frac{\partial u}{\partial n} = \text{CONST.} > 0$

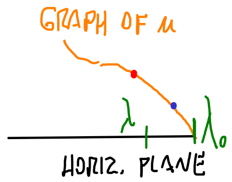
ALTERNATIVELY, WE MAY USE THE MAXIMUM PRINCIPLE IN DOMAINS OF SMALL MEASURE



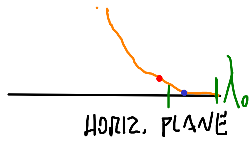
**INDEED, LET US CHECK THE BDRY COND.**

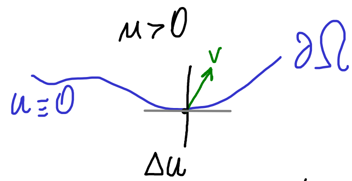



FOR SMALL $\lambda_0 - \lambda$ IT FOLLOWS FROM $-\frac{\partial u}{\partial n} > 0$




IN THE CASE WHEN $\frac{\partial u}{\partial n} = 0$ WE USE THE EQUATION $\Delta u = f(u)$ AT $\partial\Omega$ TO SEE THAT $\frac{\partial^2 u}{\partial n^2} > 0$




SUPP. $\frac{\partial u}{\partial n} = 0$

THEN $\nabla u = 0$ AT $\partial\Omega$

HENCE $D^2 u \geq 0$

$$(v) D^2 u (v) = \frac{\partial^2 u}{\partial v^2} \geq 0 \quad \forall v$$

(details on pp. 12-13)

Friday, November 24, 2023

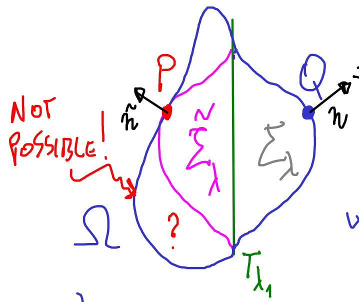
## THE MOVING PROCEDURE

WE SAW THAT $w^\lambda > 0$ IN $\Sigma_\lambda$ FOR $\lambda_0 - \lambda$ SMALL: THE SET

$$\{\lambda < \lambda_0 : w^\lambda > 0 \text{ IN } \Sigma_\lambda\} \neq \emptyset$$

LET $\lambda_1$ BE THE INFIMUM OF SUCH $\lambda$

TWO CASES MAY OCCUR: CASE 1




$\exists P \in \partial\Omega \cap \partial\tilde{\Sigma}_\lambda$

CHECK $w^\lambda$ AT $Q$

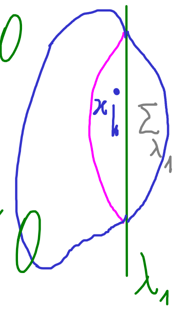
$$w^\lambda(Q) = u(P) - u(Q) = 0$$

$\frac{\partial w^\lambda}{\partial n}(Q) = 0$ BECAUSE $\frac{\partial u}{\partial n}$ = CONST. AT $\partial\Omega$

BY HOPF'S LEMMA IT FOLLOWS $w^\lambda \equiv 0$ IN $\Sigma_\lambda$ THEREFORE THE SET $\Omega \cap \{x_1 < \lambda_1\}$ COINCIDES WITH $\tilde{\Sigma}_\lambda$ HENCE $\Omega$ IS SYMMETRIC WITH RESPECT TO $T_{\lambda_1}$

REMARK AS LONG AS NO CONTACT POINT LIKE P EXISTS, WE CAN DIMINISH $\lambda$ UNLESS

$w^{\lambda_1} = 0$ ON $T_{\lambda_1}$ HENCE $\frac{\partial w^{\lambda_1}}{\partial n} \neq 0$




FOR EVERY $\lambda_k = \lambda_1 - \frac{1}{k}$

$$\exists x_k \in \Sigma_{\lambda_k} : w^{\lambda_k}(x_k) < 0$$

IMPOSSIBLE NEAR $T_{\lambda_1}$

$x_k \to x_\infty \in T_{\lambda_1}$

$w^{\lambda_1}(x_\infty) = 0$

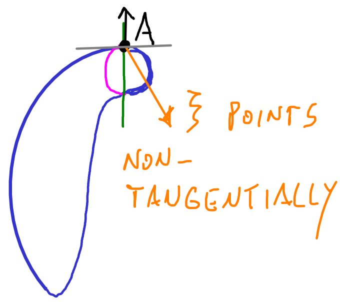
CASE 2: $\partial\Omega$ IS ORTHOGONAL TO $T_{\lambda_1}$

SERRIN'S CORNER LEMMA:

$w^{\lambda_1} > 0$ IN $\Sigma_{\lambda_1}$

$w^{\lambda_1}(A) = 0$

$\Delta w^{\lambda_1} = c(x)\, w^{\lambda_1}$




CLAIM: EITHER

$$\frac{\partial w^{\lambda_1}}{\partial \xi} > 0 \quad \text{OR} \quad \frac{\partial^2 w^{\lambda_1}}{\partial \xi^2} > 0$$

HOWEVER, WE SEE THAT $\nabla w^{\lambda_1}(A) = \bar{0}$ AND $D^2 w^{\lambda_1}(A) = 0$: CONTRADICTION.

$\nabla w^{\lambda_1}(A) = \bar{0}$: $w^{\lambda_1} \equiv 0$ ON $T_{\lambda_1}$

HENCE $\frac{\partial w^{\lambda_1}}{\partial n} = 0$ AND $\frac{\partial w^{\lambda_1}}{\partial \tau} = 0$

FOR EVERY $\tau$ TANGENT TO $\partial\Omega$ AT $A$.

SIMILARLY, $\frac{\partial^2 w^{\lambda_1}}{\partial n^2} = 0$ FOR $\frac{\partial^2 w^{\lambda_1}}{\partial \tau^2}$

REMEMBER THAT $w^{\lambda_1} = u(x^{\lambda_1}) - u(x)$

AND WE OBTAIN ZERO.

CONSIDER, FOR SIMPLICITY, THE CASE $N=2$. WE HAVE SEEN THAT

$$D^2 w^{\lambda_1}(A) = \begin{pmatrix} 0 & ? \\ ? & 0 \end{pmatrix}$$

TO CHECK THAT $\dfrac{\partial^2 w^{\lambda_1}}{\partial x \partial y}(A) = 0$

WE MAY DIFFERENTIATE THE IDENTITY $\left(\dfrac{\partial u}{\partial n}\right)^2 = 0$. IN SERRIN'S PAPER THE $\partial\Omega$ IS REPRESENTED AS THE GRAPH OF SOME $\varphi$. IN OUR CASE, WE MAY WRITE $y = \varphi(x)$ AND WE KNOW THAT $u(x, \varphi(x)) = 0$

THE NORMAL $n$ IS $\dfrac{(-\varphi'(x), 1)}{\sqrt{1 + (\varphi'(x))^2}}$

THE NORMAL DERIVATIVE IS $\dfrac{\partial u}{\partial n} = n \cdot \nabla u$

$$= \frac{-\varphi'(x)\, u_x + u_y}{\sqrt{1 + (\varphi'(x))^2}} = \text{CONST.}$$

IF WE DIFFERENTIATE THIS WE GET $u_{xy}(A) = 0$ AND THIS IMPLIES

$w^{\lambda_1}_{xy}(A) = 0$ AND THE CLAIM FOLLOWS.



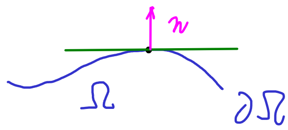
REMARK: IF $u \in C^2(\overline{\Omega})$ IS A SOLUTION OF

$$\begin{cases} \Delta u = f(u), \quad u > 0 \text{ IN } \Omega \\ u = 0 \text{ ON } \partial\Omega \\ \dfrac{\partial u}{\partial n} = 0 \text{ ON } \partial\Omega \end{cases} \qquad \text{WHERE } f \in C^1(\mathbb{R})$$

AND $\Omega$ IS A BDD. $C^2$-DOMAIN, THEN $\dfrac{\partial^2 u}{\partial n^2} > 0$ ON $\partial\Omega$.

PROOF: ① $\nabla u \equiv 0$ AT $\partial\Omega$: INDEED

$u = 0$ ON $\partial\Omega$

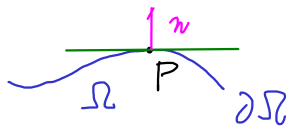



HENCE $\dfrac{\partial u}{\partial \tau} = 0$

THUS $\dfrac{\partial u}{\partial n} = 0$ IMPLIES $\nabla u = 0$

② $N = 2$, $\partial\Omega$: GRAPH OF $\varphi(x)$

TAKE $P \in \partial\Omega$ AND ASSUME THAT $\varphi'(x_p) = 0$




FROM ① WE KNOW

$u_x(x, \varphi(x)) = 0$

AND THEREFORE $u_{xx}(P) + u_{xy}(P)\, \varphi'(x_p) = 0$

HENCE $u_{xx}(P) = 0$.

FOR $N \geq 2$ SIMILARLY $\dfrac{\partial^2 u}{\partial \tau^2}(P) = 0$.

③ SUPPOSE BY CONTRAD. $\dfrac{\partial^2 u}{\partial n^2}(P) = 0$

THEN $\Delta u(P) = 0 = f(u(P)) = f(0)$

Friday, December 1, 2023

④ $\Delta u = f(u) = f(u) - f(0) = f'(\xi) \cdot u$

LINEAR! $\begin{cases} \Delta u = c(x) \cdot u & \text{IN } \Omega \\ u > 0 & \text{IN } \Omega \end{cases}$

$u\big|_{\partial\Omega} = 0 = \min_{\bar{\Omega}} u$ HENCE BY

HOPF LEMMA WE SHOULD HAVE $\frac{\partial u}{\partial n} < 0$

WHICH IS NOT THE CASE.

CONCLUSION: $\frac{\partial^2 u}{\partial n^2} > 0$

$\boxed{\text{CONVEXITY}}$

IS $u$ A CONVEX FUNCTION ?

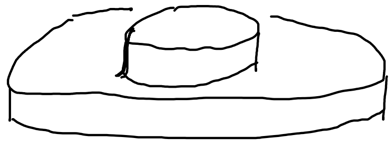
DOES $u$ HAVE CONVEX LEVEL SETS ?
(QUASICONCAVITY / QUASICONVEXITY)

EXAMPLE: HEAT

$$\begin{cases} u_t = \Delta u & \text{IN } \mathbb{R}^N \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{IN } \mathbb{R}^N \end{cases}$$

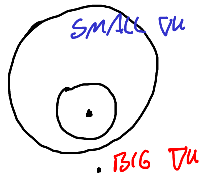
QUASICONCAVITY IS NOT PRESERVED!

$u_0(x)$

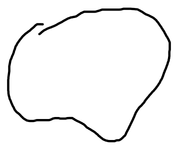


INITIAL TEMPERATURE

FROM THE TOP:
$u$ QUASI CONCAVE




FOR $t > 0$ SMALL:
NON QUASICONCAVE !



(ISHIGE - SALANI
2008)

## ELLIPTIC EQUATIONS

① IF $\Omega$ IS CONVEX BOUNDED
THE FIRST DIRICHLET EIGENFUNCTION
WILL NOT BE CONVEX !

$$\begin{cases} -\Delta u = \lambda_1 u & \text{in } \Omega, \quad \lambda_1 \text{ FIRST EIGENVALUE} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

CONSIDER $\Omega = B(0, r) \subset \mathbb{R}^2$,
$u$ RADIAL (BESSEL FUNCTION)

SEE WATSON: A TREATISE ON THE THEORY OF BESSEL FUNCTIONS.

POLAR COORDINATES:

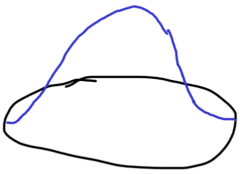
$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

AT $\partial\Omega = \partial B$ WHAT HAPPENS ?

$$-\lambda_1 u = 0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

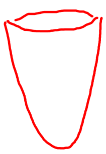
$u > 0$ in $\Omega$, $\frac{\partial u}{\partial r} < 0$ on $\partial\Omega$

HENCE $\frac{\partial^2 u}{\partial r^2} > 0$ :



BUT: $-\log u$ IS CONVEX !

$u > 0$ IS LOG-CONCAVE
FOR ANY CONVEX $\Omega$ ! $-\log u$



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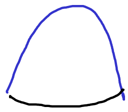
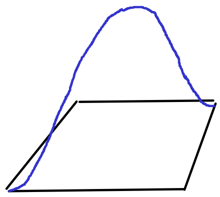
② TORSION PROBLEM $\begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = 0 & \text{at } \partial\Omega \end{cases}$

IF $\Omega$ IS CONVEX, WILL $u$ BE CONCAVE ?

IF $\Omega = B(0, R)$ THEN $u(x,y) = \frac{R^2 - x^2 - y^2}{4}$



BUT IF $\Omega$ = SQUARE THEN $\nabla u$ VANISHES AT THE CORNERS:



NOT CONCAVE !

HOWEVER, $v(x) = -\sqrt{u(x)}$ IS CONCAVE !

« $u(x)$ IS $\frac{1}{2}$ - CONCAVE » AND
THEREFORE QUASICONCAVE FOR $\Omega$ CONVEX !

Friday, December 1, 2023

REMARK: EIGENVALUES HAVE BEEN INVESTIGATED FOR THE FINSLER LAPLACIAN.
IS THE FIRST EIGENFUNCTION IN A CONVEX DOMAIN log-CONCAVE?

TECHNIQUE (GRECO-PORRU)

BASE: THE CONCAVITY FUNCTION ASSOCIATED TO $V(x)$ IS

$$C(x,y) = 2V(z) - V(x) - V(y)$$

WHERE $z = \frac{x+y}{2}$

$C(x,y) \le 0$ IN $\Omega \times \Omega$ IFF $V(x)$ IS CONVEX. $V(x) = -\log u(x)$

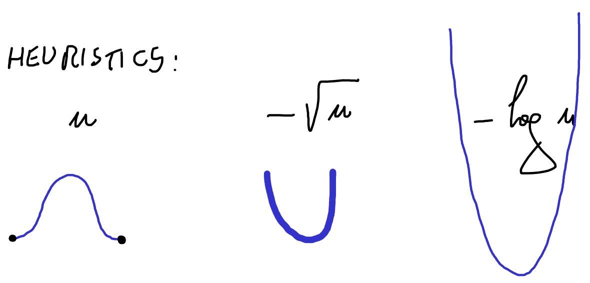
OR $V(x) = -\sqrt{u(x)}, \dots$

1. CHECK THAT $C(x,y) \le 0$ AT

$$\partial(\Omega \times \Omega) \subset (\partial\Omega \times \overline{\Omega}) \cup (\overline{\Omega} \times \partial\Omega)$$

USING $u\big|_{\partial\Omega} = 0$, $u > 0$ IN $\Omega$, $\frac{\partial u}{\partial n} < 0$

HEURISTICS:




MORE PRECISELY, $\limsup_{(x,y)\to\partial(\Omega\times\Omega)} C(x,y) \le 0$

2. CONCAVITY MAXIMUM PRINCIPLE

KNOW THAT $\Delta V = f(V, \nabla V)$

WANT TO FIND $LC \ge 0$

PIONEERING PAPERS (KAWOHL, KENNINGTON, KOREVAAR) ARGUE AT $(x_0, y_0) \in \Omega \times \Omega$ WHERE $C(x_0,y_0) \ge C(x,y)$ AND GET A CONTRADICTION.

EASY: $f(V,p)$ CONCAVE IN $V$ FOR ALL $p$

MORE GENERAL: $f(V,p)$ HARMONIC CONCAVE, I.E. POSITIVE AND $\frac{1}{f(V,p)}$ CONVEX IN $V$

DETAILS: START FROM $C(x,y) =$

$= 2V(z) - V(x) - V(y)$ COMPUTE

$$\frac{\partial C}{\partial x_i} = V_i(z) - V_i(x) \qquad \frac{\partial C}{\partial y_i} = V_i(z) - V_i(y)$$

$$\frac{\partial^2 C}{\partial x_i^2} = \frac{1}{2} V_{ii}(z) - V_{ii}(x)$$

$$\frac{\partial^2 C}{\partial y_i^2} = \frac{1}{2} V_{ii}(z) - V_{ii}(y); \qquad \frac{\partial^2 C}{\partial x_i \partial y_i} = \frac{1}{2} V_{ii}(z)$$

$$LC = \sum_{i=1}^{N} \left( \frac{\partial^2 C}{\partial x_i^2} + 2\frac{\partial^2 C}{\partial x_i \partial y_i} + \frac{\partial^2 C}{\partial y_i^2} \right)$$

$$= 2\Delta V(z) - \Delta V(x) - \Delta V(y)$$

$$= 2f(V(z), \nabla V(z)) - f(V(x), \nabla V(x)) - f(V(y), \nabla V(y))$$

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CONSIDER $\Delta V = f(V)$ FOR SIMPLICITY:

$$LC = 2f(V(z)) - f(V(x)) - f(V(y))$$

$$= 2f\left(\frac{V(x)+V(y)}{2}\right) - f(V(x)) - f(V(y))$$

$$+ 2f(V(z)) - 2f\left(\frac{V(x)+V(y)}{2}\right) \geq$$

$$\geq 2f(V(z)) - 2f\left(\frac{V(x)+V(y)}{2}\right) =$$

$$= f'(\xi)\, C(x,y)$$

THEREFORE $LC \geq f'(\xi)\, C(x,y)$

THE M.P. HOLDS IF $f' \geq 0$

$\boxed{\text{CONNECTION } u - V}$

SUPP. $\begin{cases} \Delta u = -1 & \text{IN } \Omega \text{ CONVEX} \\ u|_{\partial\Omega} = 0 \end{cases}$

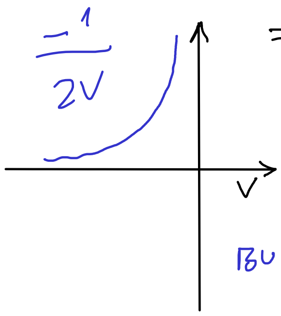
$V = -\sqrt{u}$, $u = V^2$, $u_i = 2VV_i$

$u_{ii} = 2V_i^2 + 2VV_{ii}$ THEREFORE

$$-1 = \Delta u = 2|\nabla V|^2 + 2V\Delta V$$

REWRITTEN $\Delta V = \dfrac{-1-2|\nabla V|^2}{2V}$

$$= \frac{-1}{2V} - \frac{|\nabla V|^2}{V}$$




POSITIVE
NOT CONCAVE

BUT $\dfrac{2V}{-1-2|\nabla V|^2}$ CONVEX !

IF, INSTEAD, $-\Delta u = \lambda_1 u$ AND $V = -\log u$

THEN $u = e^{-V}$, $u_i = -e^{-V} V_i$

$$u_{ii} = e^{-V} V_i^2 - e^{-V} V_{ii}$$

$$\Delta u = -\lambda_1 e^{-V} = e^{-V}|\nabla V|^2 - e^{-V}\Delta V$$

HENCE $\Delta V = \lambda_1 + |\nabla V|^2$ CONST. IN V

$$\frac{\partial f}{\partial V} \equiv 0, \quad f(V,p) = \lambda_1 + |p|^2.$$

REMARK: IF $C(x,y)$ BECOMES EXTREMAL AT $(x_0,y_0)$ THEN $\nabla_x C(x_0,y_0) = \nabla_y C(x_0,y_0) = 0$

THEREFORE $\nabla V(z) = \nabla V(x) = \nabla V(y)$ AND WE MAY WRITE

$$LC = f(v(z), \nabla v(z)) - f(v(x), \nabla v(x)) + f(v(z), \nabla v(z)) - f(v(y), \nabla v(y)) =$$

$$= f(v(z), \nabla v(z)) - f(v(x), \nabla v(z)) + f(v(z), \nabla v(z)) - f(v(y), \nabla v(z)) + f(v(x), \nabla v(z)) - f(v(x), \nabla v(x)) + f(v(y), \nabla v(z)) - f(v(y), \nabla v(y))$$

$$= f(v(z), \nabla v(z)) - f(v(x), \nabla v(z)) + f(v(z), \nabla v(z)) - f(v(y), \nabla v(z)) + \sum_{i=1}^{N} \frac{\partial f}{\partial p_i}(v(x), p) \frac{\partial C}{\partial x_i}(x,y) + \sum_{i=1}^{N} \frac{\partial f}{\partial p_i}(v(y), \tilde{p}) \frac{\partial C}{\partial y_i}(x,y)$$

AND, BY CONCAVITY, $LC \geq b^i C_{x_i} + \tilde{b}^i C_{y_i} + f'(\xi) C$

MENTION OF A MICROSCOPIC TECHNIQUE:

$D^2 V$ HAS CONSTANT RANK

MANAGE TO PROVE THAT $D^2 V \geq 0$

Thursday, December 7, 2023

# SYMMETRY VIA MORSE THEORY

IN SHORT, THE MORSE INDEX OF A POINT $x_0$ IN $\mathbb{R}^N$ WHERE $\nabla f = 0$ IS THE NUMBER OF NEGATIVE EIGENVALUES OF $D^2 f(x_0)$

MORE GENERALLY, THE INDEX OF THE SOLUTION $u$ OF $\Delta u = f(u)$ IS DEFINED BY MEANS OF THE VARIATIONAL FORMULATION: $u$ IS A CRITICAL POINT OF THE FUNCTIONAL

$$\mathcal{J}[v] = \int_\Omega \left( \frac{1}{2} |\nabla v|^2 + F(v) \right) dx$$

WHERE $F(v) = \int_0^v f(t)\,dt$. LET US TAKE $\eta \in C_c^1(\Omega) \subset H_0^1(\Omega)$ AND COMPUTE

$$\left( \frac{d^2}{dt^2} \mathcal{J}(u + t\eta) \right)_{t=0}.$$

NOTE THAT

$$\mathcal{J}(u+t\eta) = \int_\Omega \left( \frac{1}{2} |\nabla u + t \nabla \eta|^2 + F(u+t\eta) \right) dx$$

HENCE $\frac{d}{dt} \mathcal{J}(u+t\eta) = \int_\Omega \left( \nabla u \cdot \nabla \eta + t |\nabla \eta|^2 + f(u+t\eta)\eta \right) dx$ AND THEREFORE

$$\left( \frac{d^2}{dt^2} \mathcal{J}(u+t\eta) \right)_{t=0} = \int_\Omega \left( |\nabla \eta|^2 + f'(u)\eta^2 \right) dx$$

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DEFINE $Q[\eta] = \int_\Omega \left( |\nabla \eta|^2 + f'(u)\, \eta^2 \right) dx$

$= \delta^2 \mathcal{J}$

THE MORSE INDEX OF $u$ IS THE MAXIMAL DIMENSION OF A LINEAR SPACE $V$ OF DIRECTIONS $\eta$ IN WHICH $Q[\eta]$ IS NEGATIVE DEFINITE.

REMARK: IF $f' \geq 0$ THEN $m(u) = 0$

IN FACT $F$ IS CONVEX AND THEREFORE

$\mathcal{J}[v] = \int_\Omega \left( \frac{1}{2} |\nabla v|^2 + F(v) \right) dx$ IS ALSO

CONVEX AND ANY CRITICAL POINT $u$ IS AN ABSOLUTE MINIMIZER

REMARK: IF $f' \geq 0$ THE DIRICHLET PBM.

$$\begin{cases} \Delta u = f(u) & \text{IN } \Omega \text{ BOUNDED} \\ u(x) = g(x) & \text{ON } \partial\Omega \end{cases}$$

HAS AT MOST ONE SOLUTION: IN FACT IF $v$ IS ANOTHER SOL.

THEN WE DEFINE $w = u - v$ AND GET

$\Delta w = \Delta u - \Delta v = f(u) - f(v)$

$= f'(\xi)\, w$

AS WELL AS $w|_{\partial\Omega} = 0$ ON $\partial\Omega$ AND

THEREFORE $w \equiv 0$ IN $\Omega$ BY THE M. P.

GIVEN A SOLUTION $u$ OF $\Delta u = f(u)$, LET US CONSIDER THE EIGENVALUES OF THE LINEAR OPERATOR $v \mapsto \Delta v - f'(u)\, v$:

$$\begin{cases} \Delta v - f'(u)\, v = -\lambda v & \text{IN } \Omega \text{ BOUNDED} \\ v = 0 & \text{ON } \partial\Omega \end{cases}$$

EIGENVECTORS $v \not\equiv 0$ ARE EXTREMAL POINTS OF THE RAYLEIGH QUOTIENT

$$R[\eta] = \frac{\int_\Omega \left( |\nabla \eta|^2 + f'(u)\, \eta^2 \right) dx}{\int_\Omega \eta^2\, dx} = \frac{Q[\eta]}{\int_\Omega \eta^2\, dx}$$

AND EIGENVALUES ARE THE VALUES $R[v]$.

THUS, $m(u)$ EQUALS THE NUMBER OF NEGATIVE EIGENVALUES OF $\Delta v - f'(u)\, v$.

EXAMPLE LET US CONSIDER $f(u) = -\lambda_1 u$

WHERE $\lambda_1 = \min_{\eta \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \eta|^2\, dx}{\int_\Omega \eta^2\, dx} > 0$.

THERE EXISTS $\varphi_1 > 0$ SUCH THAT $-\Delta \varphi_1 = \lambda_1 \varphi_1$ IN $\Omega$, $\varphi_1 = 0$ ON $\partial\Omega$ (THE FIRST EIGEN-FUNCTION). OF COURSE, EVERY $t\varphi_1$ WILL DO THE JOB. NO OTHER SOLUTIONS EXIST.

WE DEFINE $u = \varphi_1$ SO THAT $\Delta u = -\lambda_1 u$

$= f(u)$

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CONSTRUCT $\mathcal{J}[v]$ ASSOCIATED TO $\Delta u = -\lambda_1 u$:

$$\mathcal{J}[v] = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 - \lambda_1 v^2\right) dx$$

WHAT IS $Q[\eta]$?

$$Q[\eta] = \int_\Omega \left(|\nabla \eta|^2 - \lambda_1 \eta^2\right) dx$$

THINK ABOUT $f(x,y) = x^2 - y^2$ AT $(0,0)$:

$D^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ AND

$$(x, y)\, D^2 f(0,0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 - 2y^2$$

REMEMBER: $0 < \lambda_1 \leq \dfrac{\int_\Omega |\nabla \eta|^2 dx}{\int_\Omega \eta^2 dx}$

HENCE $\int_\Omega |\nabla\eta|^2 dx \geq \lambda_1 \int_\Omega \eta^2 dx$

AND THEREFORE $2\mathcal{J}[\eta] = Q[\eta] \geq 0$ FOR ALL $\eta$

EXERCISE: LET $\lambda_2 > \lambda_1$ BE THE SECOND EIGENVALUE OF $-\Delta u = \lambda u$ IN $\Omega$ BOUNDED, $u|_{\partial\Omega} = 0$. LET $\varphi_2$ BE A CORRESPONDING EIGENFUNCTION. DEFINE $\mathcal{J}[v] = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 - \lambda_2 v^2\right) dx$. $-\Delta\varphi_2 = \lambda_2 \varphi_2$ COMPUTE $m(\varphi_2)$

Friday, December 15, 2023

PART 1: $m(\varphi_2) > 0$

CONSTRUCT $\eta$ SUCH THAT $Q_2[\eta] =$

$$= \int_\Omega \left(|\nabla\eta|^2 - \lambda_2 \eta^2\right) dx < 0$$

TAKE $\eta = t\varphi_1$ AND GET $Q_2[t\varphi_1] =$

$$= t^2 \int_\Omega \left(|\nabla\varphi_1|^2 - \lambda_2 \varphi_1^2\right) dx$$

$(\lambda_1 < \lambda_2)$

$$< t^2 \int_\Omega \left(|\nabla\varphi_1|^2 - \lambda_1 \varphi_1^2\right) dx =$$

$$= 2t^2 \mathcal{J}_1[\varphi_1] = 0$$

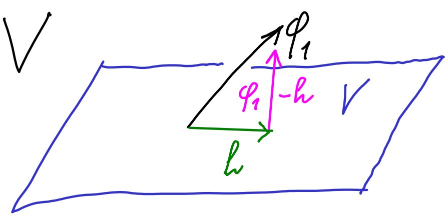
HENCE $Q_2[t\varphi_1] < 0$.

PART 2: IF I TAKE A VECTOR SPACE $V$ OF DIRECTIONS $\eta$ WITH $\dim V \geq 2$ I FIND $0 \neq \eta_0 \in V$ S.T. $Q_2[\eta_0] \geq 0$

$\Rightarrow$ MAXIMAL DIM OF $V$ S.T. $Q_2|_V < 0$ IS 1

LET $\dim V \geq 2$, WRITE $h = \pi_V[\varphi_1]$ SO THAT $\varphi_1 = h + (\varphi_1 - h)$ AND

$\varphi_1 - h \perp V$



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CASE 1 $h=0, \quad \varphi_1 \perp V$

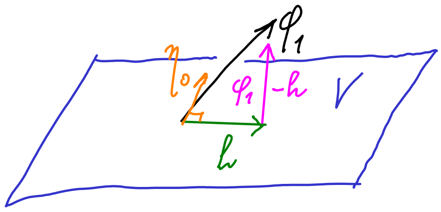
TAKE ANY $0\neq \eta_0 \in V$ AND RECALL THAT

$$\lambda_2 = \min_{0\neq\eta\perp\varphi_1} \frac{\int_\Omega |\nabla\eta|^2}{\int_\Omega \eta^2} \le \frac{\int_\Omega|\nabla\eta_0|^2}{\int_\Omega \eta_0^2}$$

IMPLIES $Q_2[\eta_0]\ge 0$ AS CLAIMED.

CASE 2 $h\neq 0$

$\varphi_1 = h + (\varphi_1 - h), \quad (\varphi_1-h)\perp V$




CHOOSE $h\perp\eta_0\in V$ AND OBTAIN

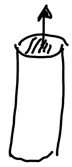
$$\eta_0\cdot\varphi_1 = \eta_0\cdot\big(h+(\varphi_1-h)\big)$$
$$= \eta_0\cdot h + \eta_0\cdot(\varphi_1-h) = 0$$

HENCE $Q_2[\eta_0]\ge 0$ AS BEFORE

CONCLUSION: $m(\varphi_2)=1$.

## A RECENT RESULT FOR A MIXED PROBLEM

TO KEEP IT SIMPLE:



$\Omega = \omega\times(0,h)$

$$\begin{cases}\Delta u = f(u) \text{ in } \Omega \\ u_N = 0 \text{ on } \partial\Omega\end{cases} \rightarrow \begin{cases}\frac{\partial u}{\partial \nu} = 0 \text{ on } \omega\times\{0\} \text{ AND } \omega\times\{h\} \\ u(x) = g(x_1,\dots,x_{N-1}) \\ \text{on } \partial\omega\times(0,h)\end{cases}$$

CLAIM 1: IF $m(u)=0$ THEN $\frac{\partial u}{\partial x_N}\equiv 0$

CLAIM 2: IF $m(u)=1$ THEN EITHER $\frac{\partial u}{\partial x_N}\equiv 0$ OR $u$ IS STRICTLY MONOTONE IN $x_N$ AND $\frac{\partial u}{\partial x_N}\neq 0$ IN $\Omega$ (SEE GLADIALI - GRECO)

CONJECTURE: IF $m(u)=2$ THEN EITHER $u_N\equiv 0$, OR $u_N>0$, OR $u_N<0$ IN $\Omega$ OR $u(x_1,\dots,x_{N-1},x_N) =$

$= u(x_1,\dots,x_{N-1},h-x_N)$