Le soluzioni dell'esame di Geometria 1 - 08/09/2023

Esercizio 1

a) Sappiamo che
$$f(1,2,0) = (1,1,0)$$
$$f(-1,0,1) = (1,2,1)$$
$$f(0,1,1) = (0,0,0)$$

Si noti che $\mathcal{B} = \{\underset{v_1}{(1,2,0)}, \underset{v_2}{(-1,0,1)}, \underset{v_3}{(0,1,1)}\}$ è una base di $\mathbb{R}^3$. Infatti:

$$\det\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1 \cdot \det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = -1+2 \neq 0$$

Quindi
$$M_{\mathcal{B}\mathcal{B}_c}(f) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underset{\mathcal{B}_c}{\mathbb{R}^3} \xrightarrow{id} \underset{\mathcal{B}}{\mathbb{R}^3} \xrightarrow{f} \underset{\mathcal{B}_c}{\mathbb{R}^3}$$

Troviamo la matrice di passaggio da $\mathcal{B}_c :=$ base canonica di $\mathbb{R}^3$ a $\mathcal{B}$.

- $(1,0,0) = \alpha(1,2,0) + \beta(-1,0,1) + \gamma(0,1,1)$
$$= (\alpha-\beta, 2\alpha+\gamma, \beta+\gamma) \Rightarrow \begin{cases} \alpha - \beta = 1 \\ 2\alpha + \gamma = 0 \rightarrow \gamma = -2\alpha \\ \beta + \gamma = 0 \end{cases}$$
$$\Rightarrow \begin{cases} \alpha - \beta = 1 \\ -2\alpha + \beta = 0 \end{cases} \Rightarrow -\alpha = 1 \Rightarrow \gamma = 2 \Rightarrow \beta = -2$$

- $(0,1,0) = \alpha(1,2,0) + \beta(-1,0,1) + \gamma(0,1,1)$

$$= (\alpha - \beta,\ 2\alpha + \gamma,\ \beta + \gamma) \Rightarrow \begin{cases} \alpha - \beta = 0 \rightarrow \alpha = \beta \\ 2\alpha \quad + \gamma = 1 \\ \quad \beta + \gamma = 0 \end{cases} \Rightarrow$$

$$\begin{cases} 2\beta + \gamma = 1 \\ \beta + \gamma = 0 \end{cases} \Rightarrow \beta = 1$$

$$\Rightarrow \alpha = 1 \Rightarrow \gamma = -\beta = -1.$$

- $(0,0,1) = \alpha(1,2,0) + \beta(-1,0,1) + \gamma(0,1,1) = (\alpha - \beta,\ 2\alpha + \gamma,\ \beta + \gamma)$

$$\Rightarrow \begin{cases} \alpha - \beta = 0 \longrightarrow \alpha = \beta \\ 2\alpha + \gamma = 0 \\ \quad \beta + \gamma = 1 \end{cases} \Rightarrow \begin{cases} 2\beta + \gamma = 0 \\ \beta + \gamma = 1 \end{cases} \Rightarrow \beta = -1$$

$$\Rightarrow \gamma = 2,\ \alpha = -1$$

Allora

$$M_{B_c B} = \begin{pmatrix} -1 & 1 & -1 \\ -2 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$

$$\begin{tikzcd} \underset{B_c}{\mathbb{R}^3} \arrow[r, "id"] \arrow[rr, bend right, "f = f \circ id"'] & \underset{B}{\mathbb{R}^3} \arrow[r, "f"] & \underset{B_c}{\mathbb{R}^3} \end{tikzcd}$$

Di conseguenza

$$M_{B_c B_c}(f) = M_{B B_c}(f) \cdot M_{B_c B}(id) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & -1 \\ -2 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 & -2 \\ -5 & 3 & -3 \\ -2 & 1 & -1 \end{pmatrix}$$

b) $(3,4,1) \in \mathrm{Im}(f) = L(f(v_1), f(v_2), f(v_3)) \Leftrightarrow$

$(3,4,1) \in L((1,1,0), (1,2,1), (0,0,0)) = L((1,1,0), (1,2,1))$

$\Leftrightarrow \exists \alpha, \beta \in \mathbb{R} : (3,4,1) = \alpha(1,1,0) + \beta(1,2,1)$

$= (\alpha+\beta, \alpha+2\beta, \beta)$

$$\Leftrightarrow \begin{cases} \alpha+\beta = 3 \\ \alpha+2\beta = 4 \\ \beta = 1 \end{cases} \Leftrightarrow \begin{cases} \alpha+1 = 3 \\ \alpha+2 = 4 \\ \beta = 1 \end{cases} \Leftrightarrow \begin{cases} \alpha = 2 \\ \alpha = 2 \\ \beta = 1 \end{cases}$$

La risposta è "sì", $(3,4,1) \in \mathrm{Im}(f)$.

c) $v = x_1 v_1 + x_2 v_2 + x_3 v_3 \in \ker(f) \Leftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 + 2x_2 = 0 \\ x_2 = 0 \end{cases} \quad x_3 = t, \text{ da cui}$$

$\ker(f) = \{(0,0,t) : t \in \mathbb{R}\} = L(0,0,1)$.

Una base di $\ker(f)$ è $\{(0,0,1)\}$.

d)

$$M_{\mathcal{B}_c \mathcal{B}'}(g) = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\begin{tikzcd} \underset{\mathcal{B}}{\mathbb{R}^3} \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \underset{\mathcal{B}_c}{\mathbb{R}^3} \arrow[r, "g"] & \underset{\mathcal{B}'}{\mathbb{R}^2} \end{tikzcd}$$

$\mathcal{B}' = \{(1,1), (0,1)\}$

$$M_{\mathcal{B}\mathcal{B}'}(g \circ f) = M_{\mathcal{B}_c \mathcal{B}'}(g) \cdot M_{\mathcal{B}\mathcal{B}_c}(f)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$

$$(1,1,1) = x_1 v_1 + x_2 v_2 + x_3 v_3$$
$$= x_1(1,2,0) + x_2(-1,0,1) + x_3(0,1,1)$$
$$= (x_1 - x_2,\ 2x_1 + x_3,\ x_2 + x_3) \iff$$

$$\begin{cases} x_1 - x_2 = 1 \\ 2x_1 + x_3 = 1 \\ x_2 + x_3 = 1 \end{cases} \rightarrow \begin{matrix} x_1 - x_2 = 1 \\ 2x_1 - x_2 = 0 \end{matrix} \rightarrow -x_1 = 1 \rightarrow x_2 = -2$$

e $x_3 = 1 - 2x_1 = 3$

Allora $(1,1,1) = -1 \cdot v_1 - 2v_2 + 3v_3$ e quindi $(g \circ f)(1,1,1) =$

$= y_1(1,1) + y_2(0,1) = (y_1, y_1 + y_2) =$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}$$

$= (-5, -13).$

Esercizio 2

Affinché l'insieme delle soluzioni S sia un sottospazio occorre e basta che il sistema sia omogeneo $\Leftrightarrow$ $h=1$.

Sappiamo, dal Teorema di Rouché-Capelli, che

$$\dim(S) = n - rg(A)$$
$$= 3 - rg\begin{pmatrix} k+2 & k & 1 \\ 0 & 1 & -1 \\ k+2 & 2 & -1 \\ 2k & -3 & 3 \end{pmatrix}$$

Quindi $\dim(S)=1 \Leftrightarrow rg\begin{pmatrix} k+2 & k & 1 \\ 0 & \boxed{1} & \boxed{-1} \\ k+2 & \boxed{2} & \boxed{-1} \\ 2k & -3 & 3 \end{pmatrix} = 2$

Dato che vi è un minore di ordine 2 con il det $\neq 0$, precisamente

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix},$$

affinché $rg(A)=2$ occorre e basta che tutti gli orlati del suddetto minore abbiano det $=0$. Ebbene,

$$\det\begin{pmatrix} k+2 & k & 1 \\ 0 & 1 & -1 \\ k+2 & 2 & -1 \end{pmatrix} \overset{R_1 \to R_1 - R_3}{=} \det\begin{pmatrix} 0 & k-2 & 2 \\ 0 & 1 & -1 \\ k+2 & 2 & -1 \end{pmatrix} =$$
$$= (k+2)\det\begin{pmatrix} k-2 & 2 \\ 1 & -1 \end{pmatrix} = (k+2)(-k+2-2)$$
$$= -k(k+2).$$

$$\det\begin{pmatrix} 0 & 1 & -1 \\ k+2 & 2 & -1 \\ 2k & -3 & 3 \end{pmatrix} \underset{\substack{\uparrow \\ C_2 \to C_2 + C_3}}{=} \det\begin{pmatrix} 0 & 0 & -1 \\ k+2 & 1 & -1 \\ 2k & 0 & 3 \end{pmatrix} =$$

$$= -\det\begin{pmatrix} k+2 & 1 \\ 2k & 0 \end{pmatrix} = 2k$$

I due minori hanno <u>entrambi</u> $\det = 0 \iff k = 0$.

<u>Conclusione</u>: S è sottospazio vettoriale di $\mathbb{R}^3$ di dimensione 2 $\iff h = 1$ e $k = 0$.

ESERCIZIO 3

a) $\forall A, B \in M_2(\mathbb{R})\quad \forall a, b \in \mathbb{R}$

$$f(aA+bB) = (aA+bB)\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$
$$= a\left(A\cdot\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\right) + b\left(B\cdot\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\right)$$
$$= a\,f(A) + b\,f(B)$$

b) Troviamo la matrice associata a f rispetto alla base

$$\mathcal{B} = \left\{ \underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}}_{e_3}, \underbrace{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}}_{e_4} \right\}$$

di $M_2(\mathbb{R})$.

$$f(e_1) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix} = 1\,e_1 + 2e_2 + 0\cdot e_3 + 0\cdot e_4$$
$$f(e_2) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 0\cdot e_1 + 1\cdot e_2 + 0\cdot e_3 + 0\cdot e_4$$
$$f(e_3) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 2\end{pmatrix} = 0\cdot e_1 + 0\cdot e_2 + 1\cdot e_3 + 2\cdot e_4$$
$$f(e_4) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = 0\cdot e_1 + 0\cdot e_2 + 0\cdot e_3 + 1\cdot e_4$$

$$\Rightarrow\quad M_{\mathcal{B}}(f) = \begin{pmatrix}1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 2 & 1\end{pmatrix}$$

Troviamo gli autovalori.

$$P(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 2 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^4$$

Vi è un unico autovalore, $\lambda = 1$, con molteplicità algebrica 4.

Troviamo $V(1)$.

$$v = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 \in V(1) \iff$$

$$f(v) = 1 \cdot v \iff \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 1 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\iff \begin{pmatrix} x_1 \\ 2x_1 + x_2 \\ x_3 \\ 2x_3 + x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \iff \begin{cases} x_1 = x_1 \\ 2x_1 + x_2 = x_2 \\ x_3 = x_3 \\ 2x_3 + x_4 = x_4 \end{cases} \iff$$

$$\begin{cases} 2x_1 = 0 \\ 2x_3 = 0 \end{cases} \iff v = 0 \cdot e_1 + s \cdot e_2 + 0 \cdot e_3 + t \cdot e_4, \quad s, t \in \mathbb{R}$$

Quindi

$$V(1) = \{ s e_2 + t e_4 : s, t \in \mathbb{R} \} = L(e_2, e_4)$$

c) $m_g(1) = \dim V(1) = 2 \neq m_a(1) = 4 \implies$

f NON è diagonalizzabile