Morse Theory and Nonlinear Differential Equations

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Introduction

This paper is aimed to provide an introduction about Morse theory that will be used in the study of Critical points theory and of Non linear elliptic partial differential equations. During the course ”Equazioni alle derivate parziali non lineari” there were treated the principal abstract results of the critical points theory and these results were applied in the study of PDEs; known by variational methods . Here, we will deal with Morse theory as an alternative method to the Variational methods.Starting by introducing critical groups and Morse disequations, we end up applying Morse theory on the nonlinear elliptic PDE in which we demonstrate the existence of non trivial solution.

Chapter 1
Preliminaries

Compactness-type Condition

Definition 1.1

Let (X,||.||) be a Banach space and $\phi \in C^1$ (X, $\mathbb{R}$) .
(a) ϕ satisfies the Cerami condition at the level c $\in R$ ((C)$_{\mathrm{c}}-$ condition) if every sequence $\{x_n\}_n\geq_1 \subset$ X such that

$$\phi(x_n) \to c \ in \ \mathbb{R} \ and \ (1 + ||x_n||)\phi'(x_n) \to 0 \ in \ X^*$$

admits a strongly convergent subsequence. We say that ϕ satisfies Cerami condition((C)$-$condition) if it satisfies the (C)$_{\mathrm{c}}$ $-$condition at every level c $\in \mathbb{R}$.

Let X be a Banach space and $\phi \in C^1(X, \mathbb{R})$. Let

$$K_\phi = \{x \in X : \phi'(x) = 0\} \ and \ K_\phi^C = \{x \in K_\phi : \phi(x) \in C\}$$

denote the critical set of ϕ and the critical set of ϕ with the critical values in C $\subset \mathbb{R}$, respectively.
• Suppose that H is a Hilbert space with the inner product (.,.). We can define the gradient ∇ of $\phi \in C^1(H, \mathbb{R})$, denoted by $\nabla\phi$, via the relation

$$(\nabla\phi(x), y) = \langle \phi'(x), y\rangle \ for \ all \ x, y \in H.$$

Introduction To Morse Theory

Morse theory has been widely used in the study of multiplicity of solutions for semilinear elliptic boundary value problems, arising in the calculus of variations. It is standard that the solutions of such a differential problem can be seen as critical points of a suitable smooth energy functional ϕ, defined on a Hilbert space H. The basic idea of Morse theory is that the number of solutions of the differential problem can be estimated by investigating the variations of the topological structures of the level sets of ϕ . Therefore it becomes crucial to describe locally the behavior of the energy functional near its critical points. A way to investigate such a behavior is to evaluate the critical groups at the isolated critical points.What follows represents a presentation of basic results and techniques of Morse theory that are useful for studying the multiplicity of solutions of nonlinear elliptic boundary value problems with variational structures

Critical Groups

Let us introduce the notion of critical groups, which describe the local behavior of a C^1-function on a Banach space X (or, more generally, on a Banach manifold).Critical groups help to distinguish between different types of critical points and are extremely useful in producing multiple critical points for a functional. (For the construction of Critical groups see [1, 6.2])

Let $H_k(.,.)$ denotes the k-th singular homology group of a topological pair(see [1,6.1]) .
Definition 1.2 Let X be a Banach space and $\phi \in C^1(X,\mathbb{R})$, and x∈ X an isolated critical point of ϕ. The critical groups of ϕ at x are defined by

$$C_k(\phi,x) = H_K(\phi^c \cap U, \phi^c \cap U\backslash\{x\}) \; for \; all \; k \in \mathbb{N}_0$$

where c=$\phi(x)$ and U is a neighborhood of x such that $K_\phi \cap \phi^c \cap U = \{x\}.\phi^c = \{y \in X : \phi(y) \leq c\}$. Moreover we set by convention $C_k(\phi,x) = 0 \; if k \in \mathbb{Z}, k < 0$.

Remark 1.3
(a)The excision property of the singular homology implies that the preceding definition of critical groups is independent of the particular choice of the neighborhood U (i.e $C_k(\phi,x)$ is invariant with respect to U)
(b) The critical groups $C_k(\phi,x)$ depend only on the behavior of ϕ near x. In particular, they are also defined when ϕ is defined only in a neighborhood of x.

We will always compute critical groups on the field $\mathbb{R}$.
Example 1.4
(a) Let X be a Banach space, $\phi \in C^1(X,\mathbb{R})$, and x∈X a local minimizer of ϕ that is an isolated critical point. Then we can find a neighborhood U of x such that $K_\phi \cap U = \{x\} \; and c = \phi(x) < \phi(y) \; for \; all \; y \in U\backslash\{x\}$. Hence

$$C_K(\phi,x) = H_k(\{x\},\emptyset) = H_K(\{x\}) = \delta_{k,0}\mathbb{R} \; for \; all \; k \in \mathbb{N}_0$$

with$\delta_{k,0}$ being the Kronecker $\delta-$symbol.

(b) Let$\phi \in C^1(X,\mathbb{R})$ and x be a local maximizer of ϕ that is an isolated critical point. Then we can find $\rho > 0$ small such that $K_\phi \cap \overline{B_\rho(x)} = \{x\} \; and \; c = \phi(x) > \phi(y) \; for \; all \; y \in \overline{B_\rho(x)}\backslash\{x\}$. Thus,

$$C_k(\phi,x) = H_k(\overline{B_\rho(x)}, \overline{B_\rho(x)}\backslash\{x\}) \; for \; all \; k \in \mathbb{N}_0.$$

If dimX is infinte, then both $\overline{B_\rho(x)} \; and \, \overline{B_\rho(x)}\backslash\{x\}$ are contractible,then we get $C_k(\phi,x) = 0 \; for \; all \; k \in \mathbb{N}_0$. Now let's assume that m:=dimX< $+\infty$. Then $\overline{B_\rho(x)}\backslash\{x\}$ is homotopy equivalent to the sphere S^{m-1}, so that $C_k(\phi,x) = H_{k-1}(\overline{B_m}, S^{m-1}) = \delta_{k,m}\mathbb{R} \; for \; all \; k \in \mathbb{N}_0$.
In this way, in all cases , we have shown

$$C_k(\phi,x) = \begin{cases} \mathbb{R} & \text{if } k = dimX \\ 0 & \text{otherwise,} \end{cases} \quad for \; all \; k \in \mathbb{N}_0.$$

• Here, we consider a Hilbert space H with inner product $(.,.)_H$. Let U⊂ H be an open set $\phi \in C^2(U,\mathbb{R})$. For each $x \in U, \phi''(x)$ can be seen as a symmetric bilinear form on H, and there is a unique L_x ∈L(H) such that

$$(L_x(y),z)_H = \phi''(x)(y,z) \; for \; all \; y,z \in H.$$

In particular, L_x is self-adjoint, so we have the orthogonal decomposition $H = kerL_x \oplus imL_x$. We can identify $L_x \; with \; \phi''(x)$.

Definition 1.5 Let $\phi \in C^2(U,\mathbb{R})$ be as above and x$\in$ U be a critical point of ϕ.

(a) The **Morse index** of x is defined as the supremum of the dimensions of the vector subspaces of H on which $\phi''(x)$ is negative definite.
(b) We say that x $\in K_\phi$is **nondegenerate** if $\phi''(x)$ isnondegenerate (i.e L_x is invertible) .
Remark By the inverse function theorem, a nondegenerate critical point is always isolated .

The critical groups of a nondegenerate critical point depend only on its Morse index.
Theorem 1.6 If H is a Hilbert space, U $\subset$ H is open, $\phi \in C^2(U,\mathbb{R})$ and x$\in K_\phi$ is non degenerate with Morse index m(possibly $+\infty$), then for k $\in \mathbb{N}_0$ we have

$$C_k(\phi,x) = \begin{cases} \mathbb{R} & \text{if } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

The critical groups at infinity are useful tools for dealing with Morse relations.They are defined for functional whose sets of critical values is bounded below.

Definition 1.7 Let $\phi \in C^1(X,\mathbb{R})$ be a map such that inf$\phi(K_\phi) > -\infty$ and satisfying the (C)-condition. The critical groups of ϕ at infinty are defined by

$$C_k(\phi,\infty) = H_k(X,\phi^a) \; for \; all \; k \in \mathbb{N}_0$$

for any a $< inf\phi(K_\phi)$.

Proposition 1.8 Let $\phi \in C^1(X,\mathbb{R})$ satisfy the (C)-condition and inf$\phi(K_\phi) > -\infty$.
(a) If a$< inf\phi(K_\phi) \leq sup\phi(K_\phi) < b$,then $C_k(\phi,\infty) = H_k(\phi^b,\phi^a)$ for all k$\in \mathbb{N}_0$.
(b) If $K_\phi = \emptyset$,then $C_k(\phi,\infty) = 0$ for all k$\geq$ 0.
(c) If $K_\phi = \{x_0\}$, then $C_k(\phi,\infty) = C_k(\phi,x_0)$ for all k $\geq$ 0.

Theorem 1.9 Let $\phi \in C^1(X,\mathbb{R})$ satisfy the (C)-condition and admit finitely many critical points.

(a) For all $k \in \mathbb{N}_0$ we have

$$\sum_{x\in K_\phi} dimC_k(\phi,x) \geq dimC_k(\phi,\infty).$$

(b) Assume that $C_k(\phi,x)$ has a finite dimension for all $k \in \mathbb{N}_0$, all x$\in K_\phi$, and vanishes for all $k \in \mathbb{N}_0$ large. Then there exists a polynomial `Q`(t) with non negative integer coefficients such that

$$\sum_{x\in K_\phi}(\sum_{k\in\mathbb{N}_0} dimC_k(\phi,x)t^k) = \sum_{k\in\mathbb{N}_0} dimC_k(\phi,\infty)t^k + (1+t)\mathtt{Q}(t).$$

Proposition 1.10 Let $\phi \in C^1(X,\mathbb{R})$ satisfy the (C)-condition and inf$\phi(K_\phi) > -\infty$.Assume that X=Y$\oplus$V with dimY $< +\infty$, $\phi|_V$ bounded below,and $\phi|_Y$ anticoercive [i.e $\phi(y) \to -\infty \; as \; ||y|| \to +\infty, y \in Y$]. Then $C_{dimY}(\phi,\infty) \neq 0$

(see [1,p.160])

Chapter2
Application

The Morse theory can be refined to localize critical points and solutions, for instance , of non-linear elliptic boundary value problems.

Let $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary $\partial\Omega$, we consider the following semi-linear elliptic Dirichlet problem:

$$(D)\begin{cases} -\Delta u(x) = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

driven by the laplacian $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ and involving a Carathéodory function f.

• Let us recall that :
The space $H_0^1(\Omega)$ is a Hilbert separable space and on which is defined a scalar product as

$$\langle u, v\rangle = \int_\Omega \nabla u.\nabla v dx$$

$H^{-1}(\Omega)$ is its dual space.
Let $\{\lambda_n\}_{n\geq 1}$ be the non decreasing sequence of eigenvalues of the negative Laplacian under Dirichlet boundary conditions($-\Delta^D, for\ short$), repeated according to their multiplicities(which is finite) and $\{\hat{u}_n\}_{n\geq 1}$ be an orthogonal basis of $H_0^1(\Omega)$ made of corresponding eigenfunctions.

We state the precise hypotheses on f and its primitive F(x,s)=$\int_0^s f(x,t)dt$:

H(f) (i) $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Caratheodory function with f(x,0)=0 a.e in Ω ,$f(x,.) \in C^1(\mathbb{R})\ for\ all x \in \Omega$, and there is $c > 0$ such that

$$|f(x,s)| \leq c(1+|s|) for\ a.a\ x \in \Omega, all s \in \mathbb{R};$$

(ii) There exist $\delta > 0$ and $\tau \in (1,2)$ such that for all $s \in [-\delta,\delta]\backslash\{0\}$

$$\tau F(x,s) - f(x,s)s \geq 0\ for a.a\ x \in \Omega\ and\ essinf F(x,s) > 0;$$

(iii) There exists $\theta_0 < \lambda_{m+1}(m \geq 1)$ such that

$$\lambda_m \leq \liminf_{s\to\pm\infty} \frac{2F(x,s)}{|s|^2} \leq \limsup_{s\to\pm\infty} \frac{2F(x,s)}{|s|^2} \leq \theta_0\ uniformly\ for\ a.a\ x \in \Omega;$$

(iv) There exist $\beta_0 > 0$ and $\mu \in [1,2)$ such that

$$\beta_0 \leq \limsup_{s\to\pm\infty} \frac{2F(x,s) - f(x,s)s}{|s|^\mu}\ uniformly\ for\ a.a\ x \in \Omega.$$

Under the Hypothesis H(f)(i), we can define the energy functional $\phi : H_0^1(\Omega) \to \mathbb{R}$ for the problem (D),given by

$$\phi(u) = \frac{1}{2}||\nabla u||^2 - \int_\Omega F(x,u)dx\ for\ all\ u \in H_0^1(\Omega).$$

Evidently, $\phi \in C^2(H_0^1(\Omega), \mathbb{R})$ and

$$\phi'(u) = -\Delta u - N_f(u) \ for\ all\ u \in H_0^1(\Omega), \qquad (1)$$

where $N_f(u)(.) = f(., u(.)) \in L^2(\Omega)\ for\ all\ u \in H_0^1(\Omega).N_f$ is the Nemytskii operator.

proposition 2.1 If hypotheses H(f)(i),(iv) hold, then the functional ϕ satisfies the (C)-condition.

proof. Let $\{u_n\}_{n\geq 1} \subset H_0^1(\Omega)$ be a sequence such that

$$|\phi(u_n)| \leq M_1 \ for\ all\ n \geq 1, \qquad (1.1)$$

for some $M_1 > 0$, and

$$(1 + ||\nabla u_n||_2)\phi'(u_n) \to 0 \ in\ H^{-1}(\Omega)\ as\ n \to \infty. \qquad (1.2)$$

Let us prove that

$$\{u_n\}_{n\geq 1}\ is\ bounded\ in\ H_0^1(\Omega). \qquad (1.3)$$

From (1.2)and (1), then for all n$\geq$ 1, $all\ h \in H_0^1(\Omega)$, we have

$$|\langle \phi'(u_n), h\rangle| \leq |\langle -\Delta u_n, h\rangle - \int_\Omega f(x, u_n)h dx| \leq \frac{\epsilon_n ||\nabla h||_2}{1 + ||\nabla u_n||_2}, with\ \epsilon_n \to 0. \qquad (1.4)$$

Let us choose h=u_n in (1.4), we obtain

$$|\langle -\Delta u_n, u_n\rangle - \int_\Omega f(x, u_n)u_n dx| = |||\nabla u_n||_2^2 - \int f(x, u_n)u_n dx| \leq \frac{\epsilon_n ||\nabla u_n||_2}{1 + ||\nabla u_n||_2} \leq \epsilon_n\ for\ all\ n \geq 1. \quad (1.5)$$

On the other hand , (1.1) and the definition of ϕ yield

$$-||\nabla u_n||_2^2 + \int_\Omega 2F(x, u_n)dx \leq 2M_1 \ for\ all\ n \geq 1. \qquad (1.6)$$

Adding up (1.6) and (1.5), we will get

$$\int_\Omega (2F(x, u_n) - f(x, u_n)u_n)dx \leq M_2 \ for\ all\ n \geq 1, \qquad (1.7)$$

for some $M_2 > 0$.By H(f)(iv), we can find $\beta_1 \in (0, \beta_0)$ and $M_3 = M_3(\beta_1) > 0$ with

$$0 < \beta_1 |s|^\mu \leq 2F(x, s) - f(x, s)s \ for\ a.a\ x \in \Omega, all |s| \geq M_3. \qquad (1.8)$$

In view of the hypothesis H(f)(i), from (1.8) it follows that for all $|s| \geq M_3$,

$$2F(x, s) - f(x, s)s \geq \int_0^{|s|} f(x, t)dt - c(|s| + |s|^2) \geq -2\int_0^{|s|} c(1 + |t|)dt - c(|s| + |s|^2)$$

$$\geq -2c(|s| + \frac{|s|^2}{2}) - c(|s| + |s|^2) \geq -M_4 \ \ , with\ M_4 > 0$$

$$then\ we\ have\ \beta_1 |s|^\mu - M_4 \leq 2F(x, s) - f(x, s)s \ for\ a.a.\ x \in \Omega,\ all\ \boldsymbol{s} \in \mathbb{R}. \qquad (1.9)$$

Using (1.9)and(1.7), we infer that

$$\{u_n\}_{n\geq 1}\ is\ bounded\ in\ L^{\mu}(\Omega). \quad (1.10)$$

Since $1 \leq \mu < 2 < 2^*$ [by H(f)(iv)], fixing r $\in (2, 2^*)$, there is t $\in$[0,1) such that $\frac{1}{2} = \frac{1-t}{\mu} + \frac{t}{r}$. From the interpolation inequality (see [2]) we have

$$||u_n||_2 \leq ||u_n||_{\mu}^{1-t}||u_n||_r^t\ for\ all\ n \geq 1.$$

For that fact that exist a continuous and compact immersion from $H_0^1(\Omega)\ to\ L^{\nu}(\Omega)\ for\ all\ \nu \in [1, 2^*)$ and (1.10), this ensures that

$$||u_n||_2^2 \leq M_5||\nabla u_n||_2^{2t}\ for\ all\ n \geq 1, \qquad (1.11)$$

for some $M_5 > 0$. Returning to (1.5) and using (1.11) and hypothesis H(f)(i), we derive that

$$||\nabla u_n||_2^2 \leq c_1(1 + ||\nabla u_n||_2^{2t})\ for\ all\ n \geq 1,$$

with $c_1 > 0$. Since $t \in [0, 1)$, this proves (1.3).
Because of (1.3),along a relabeled subsequence , we have

$$u_n \rightharpoonup u\ in\ H_0^1(\Omega)\ and\ u_n \to u\ in\ L^2(\Omega)\ as\ n \to \infty. \quad (1.12)$$

In (1.4) we choose $h = u_n - u$ and pass to the limit as n $\to \infty$ through (1.12).Then we obtain $\lim\limits_{n\to+\infty}\langle -\Delta u_n, u_n - u\rangle = 0.(1.13)$

$$||u_n - u_n||^2 = \langle u_n - u, u_n - u\rangle = \langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle \to 0, through (1.12) and (1.13).$$

Then it follows that $u_n \to u$ in $H_0^1(\Omega)$, which completes the proof.

Theorem 2.2 Assume that H(f) holds. Then problem (D) admits at least one nontrivial solution $u_0 \in C_0^1(\overline{\Omega})$.
Proof. We know that the solutions of (D) coincide with the critical points of the $C^2 - functional\ \phi$ and of all them belong to $C_0^1(\overline{\Omega})$.
Thus, it suffices to show that ϕ admits at least one nontrivial critical point.To do this, we may assume that ϕ has only a finite number of critical points. We know that ϕ satisfies the (C)-condition, and we have by [theorem 11.42 ,[1]], 0 being an isolated critical point of ϕ by H_f(i)(f(x,0)=0 a.e in Ω)

$$C_k(\phi, 0) = 0\ for\ all\ k \geq 0. \quad (2.1)$$

As in [theorem 9.4 ,[1]], let us consider $\{\lambda_n\}_{n\geq 1}$and $\{\hat{u}_n\}_{n\geq 1}$ as they are defined before,and we consider the decomposition $H_0^1(\Omega) = H_m \oplus H_m^{\perp}$,where

$$H_m = span\{\hat{u}_n\ : 1 \leq n \leq m\}\ and\ H_m^{\perp} = \overline{span}\{\hat{u}_n : n \geq m + 1\}.$$

Claim 1: $\phi|_{H_m}$ is anticoercive, i.e $\phi(u) \to -\infty\ as\ ||\nabla u||_2 \to +\infty, u \in H_m$.
By (1.8) we see that

$$\frac{d}{ds}(\frac{F(x, s)}{s^2}) = \frac{\frac{d}{ds}F(x, s)s^2 - 2sF(x, s)}{s^4} = \frac{f(x, s)s - 2F(x, s)}{s^3} \leq -\beta_1 s^{\mu-3}$$

for a.a x $\in \Omega$, all s $\geq M_3$.Integrating, we have

$$\frac{F(x, s)}{s^2} - \frac{F(x, t)}{t^2} \leq \frac{\beta_1}{2 - \mu}(\frac{1}{s^{2-\mu}} - \frac{1}{t^{2-\mu}})$$

for a.a x $\in \Omega$, all $s \geq t \geq M_3$. Letting s$\to +\infty$, since $\mu < 2$ and using hypothesis H(f)(iii), we have that

$$\liminf_{s\to+\infty} \frac{F(x,s)}{s^2} \geq \frac{\lambda_m}{2}$$

$$then\ for\ all\ s >> M_3, \frac{F(x,s)}{s^2} > \frac{\lambda_m}{2},$$

$$and\ thus\ we\ obtain\ \frac{\lambda_m}{2} + \frac{f(x,t)}{t^2} \leq -\frac{\beta_1}{2-\mu}\frac{1}{t^{2-\mu}}$$

by repeating the same steps for the case of limsup and combing the results we finally obtain

$$\frac{\lambda_m}{2}|t|^2 - F(x,t) \leq -\frac{\beta_1}{2-\mu}|t|^\mu \ for\ a.a.\ x \in \Omega,\ all\ |t| \geq M_3.$$

Combinig it with H(f)(i), it follows that

$$\frac{\lambda_m}{2}|t|^2 - F(x,t) \leq \beta_2 - \frac{\beta_1}{2-\mu}|t|^\mu$$

for a.a. x $\in \Omega, all\ t \in \mathbb{R},\ for\ some \beta_2 > 0$. By [Proposition 9.9,[1]], each u $\in H_m$ satisfies $||\nabla u||_2^2 \leq \lambda_m ||u||_2^2$,whence

$$\phi(u) \leq \int_\Omega (\frac{\lambda_m}{2}|u|^2 - F(x,u))dx \leq -\frac{\beta_1}{2-\mu}||u||_\mu^\mu + \beta_2|\Omega|_N$$

for all u $\in H_m$. This yields that $\phi(u) \to -\infty\ as\ ||u||_\mu \to +\infty, u \in H_m$.Since all the norms are equivalent in H_m(which is finite dimensional), we conclude that Clam1 holds true.

Claim 2: $\phi|_{H_m^\perp}$ is bounded below.
By H(f)(i),(iii), we can find $\theta_1 \in (\theta_0, \lambda_{m+1})$ and $\theta_2 > 0$ such that

$$F(x,s) \leq \theta_2 + \frac{\theta_1}{2}|s|^2\ for\ a.a\ x \in \Omega,\ all\ s \in \mathbb{R}.$$

Since each u $\in H_m^\perp\ satisfies\ ||\nabla u||_2^2 \geq \lambda_{m+1}||u||_2^2$ (By proposition 9.9,[1]), we get

$$\phi(u) \geq \frac{\lambda_{m+1} - \theta_1}{2}||u||_2^2 - \theta_2|\Omega|_N\ for\ all\ u \in H_m^\perp.$$

Knowing that $\lambda_{m+1} > \theta_1$, this yields Claim 2.

Claim 1 and 2 alllow us to apply proposition (1.10), which yields $C_m(\phi, \infty) \neq 0$.
Then by Theorem (1.9) (a) implies that ϕ admits a critical point $u_0 \in H_0^1(\Omega)$ such that $C_m(\phi, u_0) \neq 0$.
In fact,let us suppose by contradiction that $K_\phi = \{0\}$, then by Morse Inequalities we have

$$dim C_k(\phi, 0) \geq dim C_k(\phi, \infty)\ for\ all\ k \geq 0$$

$$for\ k = m \implies 0 \geq dim C_m(\phi, \infty) \neq 0, contradiction$$

Comparing the fact that $C_m(\phi, u_0) \neq 0$ with (2.1), we deduce that u_0 is non trivial. Then the proof of this theorem is completed.

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