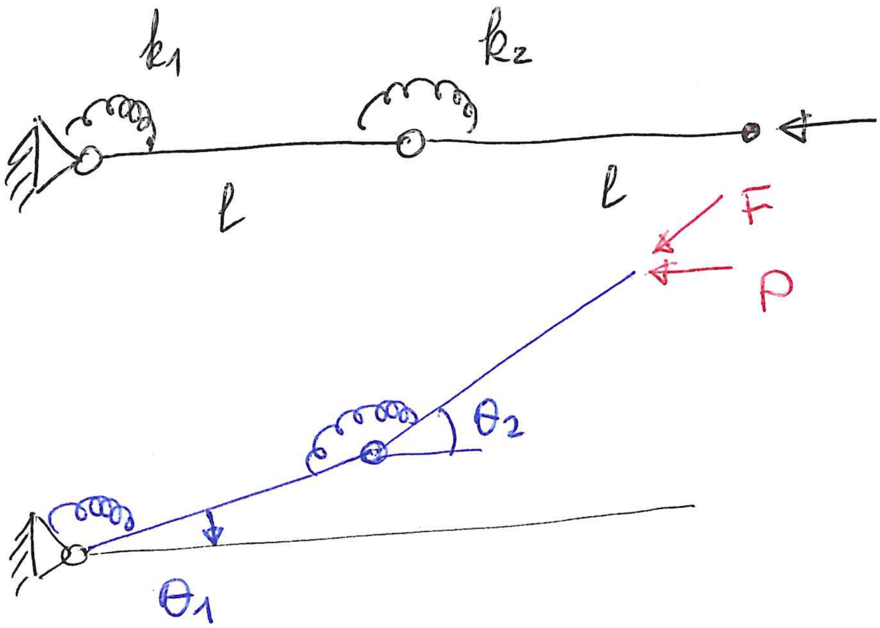
Ziegler column



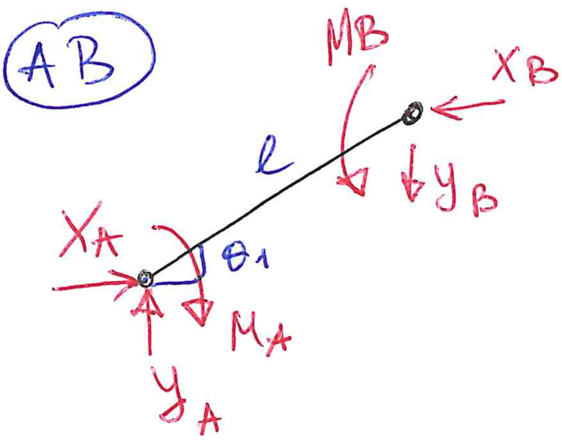
F is tangent

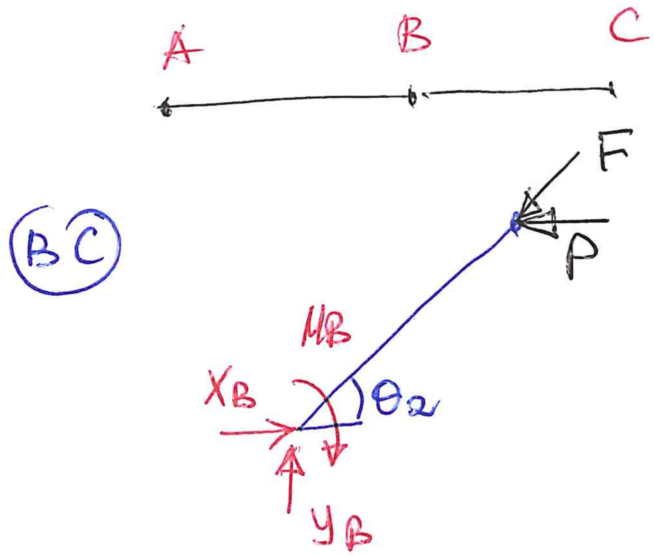
P is parallel

Linearized approach : $\theta_1 \approx 0, \quad \theta_2 \approx 0$

Static criterium

Equations of equilibrium





Unknowns are X_A, y_A, X_B, y_B :

AB

$X_A = X_B$

$Y_A = Y_B$

balance of forces

balance of moments

$$M_A - M_B - X_B \cdot l \sin\theta_1 + Y_B l \cos\theta_1 = 0$$

✓

BC balance of forces

$X_B = P + F\cos\theta_2$

$Y_B = F\sin\theta_2$

balance of moments

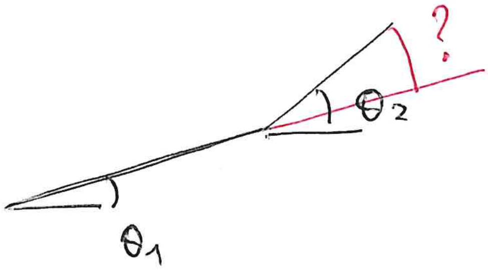
$$M_B - Pl\sin\theta_2 = 0$$

✓

M_A, M_B – ?

$M_A = k_1 \theta_1$

$M_B = k_2(\theta_2 - \theta_1)$



$$\begin{cases} M_A - M_B - X_B\, l \sin\theta_1 + Y_B\, l\cos\theta_1 = 0 \\ M_B - Pl\sin\theta_2 = 0 \end{cases}$$

$$\begin{cases} k_1\theta_1 - k_2(\theta_2-\theta_1) - (P+F\cos\theta_2)\,l\sin\theta_1 \\ \qquad\qquad + F\sin\theta_2\, l\cos\theta_1 = 0 \\ k_2(\theta_2-\theta_1) - Pl\sin\theta_2 = 0 \end{cases}$$

Linearization means that $\theta_1 \approx 0,\ \theta_2 \approx 0.$

because of the trivial solution is zero.

$\sin\theta_1 \to \theta_1 \qquad \cos\theta_{1,2} \to 1$

$\sin\theta_2 \to \theta_2$

$$\begin{cases} k_1\theta_1 - k_2(\theta_2-\theta_1) - (P+F)\,l\theta_1 + Fl\theta_2 = 0 \\ k_2(\theta_2-\theta_1) - Pl\theta_2 = 0 \end{cases}$$

1. Particular case $F = 0,\ P \neq 0$

$$\begin{cases} (k_1 + k_2)\theta_1 - k_2\theta_2 - Pl\,\theta_1 = 0 \\ -k_2\theta_1 + k_2\theta_2 - Pl\,\theta_2 = 0 \end{cases}$$

$$A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0 \Rightarrow \underline{\det A} - ?$$

$$A = \begin{pmatrix} k_1 + k_2 - Pl & -k_2 \\ -k_2 & k_2 - Pl \end{pmatrix}$$

Let us note that A is symmetric: $A = A^T$

$$\det A = (k_1 + k_2 - Pl)(k_2 - Pl) - k_2^2$$
$$= k_1 k_2 + \cancel{k_2^2} - Pl\,k_2 - Pl(k_1 + k_2) + (Pl)^2 - \cancel{k^2}$$

$$x = Pl: \quad \underline{x^2 - (k_1 + 2k_2)x + k_1 k_2 = 0}$$

$$x_{1,2} = \frac{k_1 + 2k_2 \pm \sqrt{(k_1 + 2k_2)^2 - 4k_1k_2}}{2}$$

$$x_{1,2} = \frac{k_1 + 2k_2 \pm \sqrt{k_1^2 + 4k_2^2}}{2}$$

$$P_{c_1} = \frac{k_1 + 2k_2 - \sqrt{k_1^2 + 4k_2^2}}{2\ell}$$

$$P_{c_2} = \frac{k_1 + 2k_2 + \sqrt{k_1^2 + 4k_2^2}}{2\ell}$$

$$k_1 = k_2 = k$$

$$P_{c_1} = \frac{1 + 2 - \cancel{k}\sqrt{1+4}}{2\ell}\, k$$

$$\boxed{\begin{aligned} P_{c_1} &= \frac{3-\sqrt{5}}{2\ell}\, k \\ P_{c_2} &= \frac{3+\sqrt{5}}{2\ell}\, k \end{aligned}}$$

$$\underline{P_{c_1} < P_{c_2}}$$

2. Particular case $F \neq 0,\ P = 0$

$$
\begin{cases}
(k_1 + k_2)\theta_1 - k_2\theta_2 - F\ell\theta_1 + F\ell\theta_2 = 0 \\
-k_2\theta_1 + k_2\theta_2 = 0
\end{cases}
$$

$$B \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0 \rightarrow \det \underline{\underline{B}} - ?$$

$$B = \begin{pmatrix} k_1 + k_2 - F\ell & -k_2 + F\ell \\ -k_2 & k_2 \end{pmatrix}$$

$$S = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$ – symmetric stiffness matrix (elastic)

But B is not symmetric : $B \neq B^T$

$\boxed{\text{Indication of non-conservative problems}}$

$$\det B = (k_1 + k_2 - Fl)k_2 - k_2(k_2 - Fl) = 0$$

$$-(-k_2)(-k_2 + Fl)$$

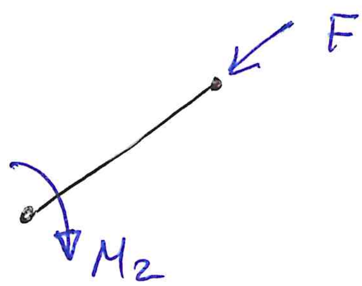
$$(k_1 + k_2)k_2 - \cancel{Flk_2} - k_2^2 + \cancel{Flk_2} = 0$$

$$= k_1 k_2 + k_2^2 - k^2 = k_1 k_2 \neq 0$$

$$\det B = k_1 k_2 \neq 0 \Rightarrow \theta_1 \text{ and } \theta_2 = 0$$

There is only trivial solution!

Another solution (BC)

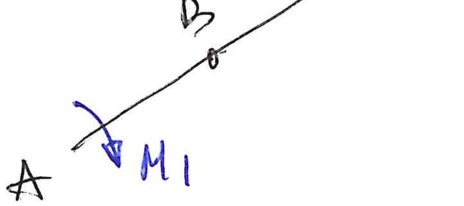


Equilibrium condition:

$$M_2 = 0$$

$$M_2 = k_2(\theta_2 - \theta_1) \Rightarrow \theta_2 = \theta_1$$

ABC



$$\Rightarrow M_1 = 0, \quad M_1 = k_1\theta_1$$

$$\Rightarrow \theta_1 = 0 \Rightarrow \theta_2 = 0$$

3. General case $F \neq 0$, $P \neq 0$

$$\left[\underbrace{S}_{\text{sym}} + \underbrace{\begin{pmatrix} -P\ell & 0 \\ 0 & -P\ell \end{pmatrix}}_{\text{sym}} + \underbrace{\begin{pmatrix} -F\ell & F\ell \\ 0 & 0 \end{pmatrix}}_{\text{not-sym.}} \right] \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0$$

$$C = \begin{pmatrix} k_1 + k_2 - P\ell - F\ell & -k_2 + F\ell \\ -k_2 & k_2 - P\ell \end{pmatrix}$$

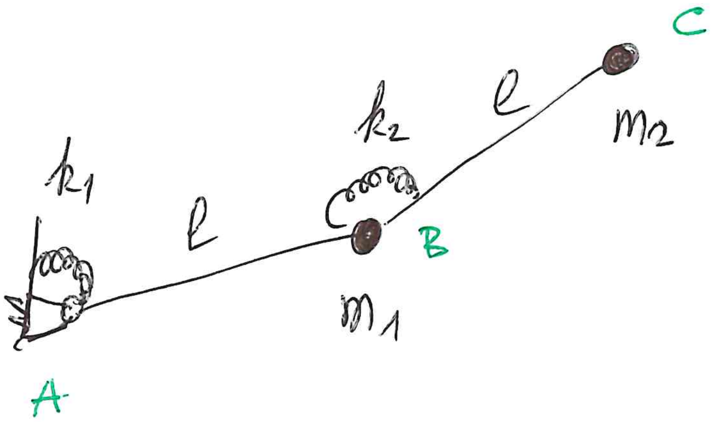
$$\det C = (k_1 + k_2 - P\ell - F\ell)(k_2 - P\ell) - $$
$$- k_2 (k_2 - F\ell) = 0$$

$$(k_1 + \cancel{k_2}) k_2 - P\ell k_2 - \cancel{F\ell k_2} - P\ell(k_1 + k_2) + (P\ell)^2$$
$$+ F\ell P\ell - \cancel{k_2^2} + \cancel{F\ell k_2} = 0$$

$$k_1 k_2 - P\ell(2k_2 + k_1) + (P\ell)^2 + F\ell P\ell = 0$$

$F = \mu P$ Danger!

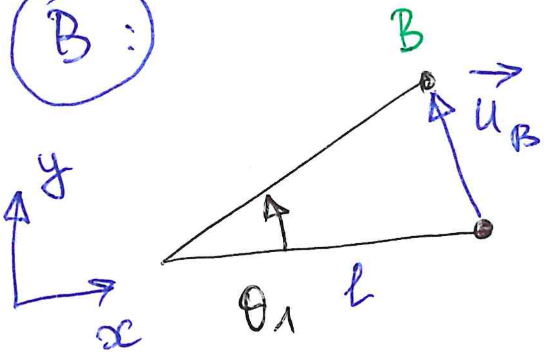
Dynamics.



Basic idea: to add forces of inertia to equations of statics.

displacements of B and C.

B :



$$\vec{u}_B = (u_B, v_B)$$

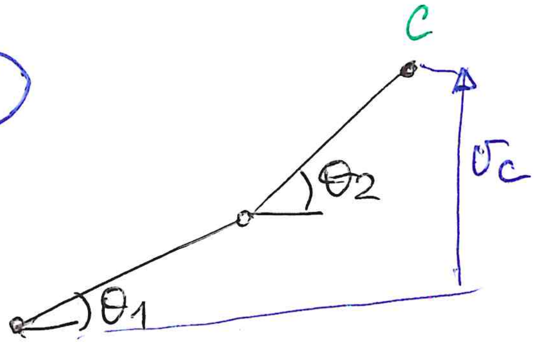
$$-u_B = \ell - \ell\cos\theta_1$$

$$v_B = \ell\sin\theta_1$$

θ_1 is small

$$v_B = \ell\theta_1 \;,\quad u_B \approx 0$$

C :



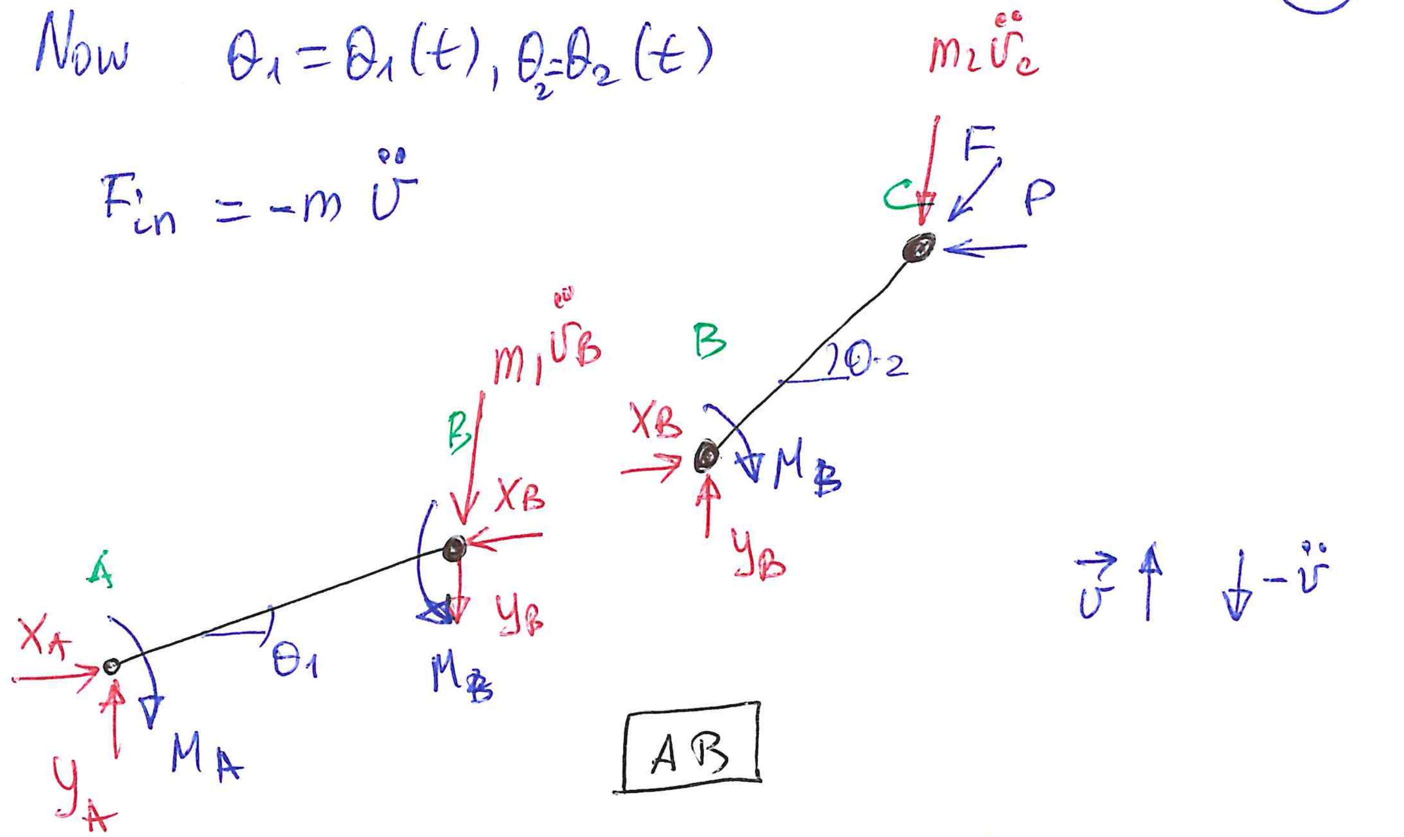
$$u_C \approx 0$$

$$v_C = \ell\sin\theta_1 + \ell\sin\theta_2$$

$$v_C = \ell\theta_1 + \ell\theta_2$$

Now $\theta_1 = \theta_1(t)$, $\theta_2 = \theta_2(t)$

$$F_{in} = -m\ddot{v}$$



$\boxed{AB}$

Forces: $X_A = X_B$, $\quad y_A = y_B + m_1\ddot{v}_B$

Balance of moments:

$$M_A - M_B - X_B l\sin\theta_1 + y_B l\cos\theta_1 + m_1\ddot{v}_B l\cos\theta_1 = 0$$

$\boxed{BC}$

$$X_B = F\cos\theta_2 + P, \quad y_B = F\sin\theta_2 + m_2\ddot{v}_c$$

$$M_B + m_2\ddot{v}_c l\cos\theta_2 - Pl\sin\theta_2 = 0$$

$$\sin\theta \to \theta, \quad \cos\theta \to 1$$

$$\begin{cases} M_A - M_B - (F \cancel{\cos} \theta_2 + P) l \theta_1 + (F\theta_2 + m_2 \ddot{v}_C) l \\ \qquad + m_1 \ddot{v}_B l = 0 \\ \\ M_B + m_2 \ddot{v}_C l - P l \theta_2 = 0 \end{cases}$$

$$k_1 \theta_1 - k_2(\theta_2 - \theta_1) - (P+F) l \theta_1 + F l \theta_2 + m_2 l^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_1 l^2 \ddot{\theta}_1 = 0$$

$v_B = l\theta_1$

$v_C = l\theta_1 + l\theta_2$

$M_A = k_1 \theta_1$

$M_B = k_2(\theta_2 - \theta_1)$

$$k_2(\theta_2 - \theta_1) - P l \theta_2 + m_2 l^2 (\ddot{\theta}_1 + \ddot{\theta}_2) = 0$$

$$\begin{cases} (m_2 + m_1) l^2 \ddot{\theta}_1 + (k_1 + k_2)\theta_1 - (P+F) l \theta_1 \\ \qquad + m_2 l^2 \ddot{\theta}_2 - k_2 \theta_2 + F l \theta_2 = 0 \\ \\ m_2 l^2 \ddot{\theta}_1 - k_2 \theta_1 + l^2 m_2 \ddot{\theta}_2 + k_2 \theta_2 - P l \theta_2 = 0 \end{cases}$$

$$\vec{q} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \qquad M\ddot{\vec{q}} + K_e \vec{q} + K_g \vec{q} = 0$$

$$K_g = K_g(F, P)$$

$$M = \begin{pmatrix} (m_1+m_2)\ell^2 & m_2\ell^2 \\ m_2\ell^2 & m_2\ell^2 \end{pmatrix}$$ – symmetric mass matrix

$$K_e = S = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$ – symmetric stiffness matrix

$$K_g = \begin{pmatrix} -(P+F)\ell & F\ell \\ 0 & -P\ell \end{pmatrix}$$ – non-symmetric due to F

For conservative force P K_g is symmetric

for non-conservative force F K_g is non-symm.

Simplifications: 1) $k_1 = k_2 = k$

2) $m_1 = \delta m, \quad m_2 = m$

$\delta = \dfrac{m_1}{m_2}$ is mass ratio

Matrix of mass

$$M = ml^2 \begin{pmatrix} 1+\delta & 1 \\ 1 & 1 \end{pmatrix}$$

Matrix of elastic stiffness

$$K_e = k \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Matrix of geometric stiffness

$$K_g = Pl \begin{pmatrix} -(1+\alpha) & \alpha \\ 0 & -1 \end{pmatrix}$$

Here $F = \alpha P$

$\vec{q} = \vec{Q} e^{\lambda t}$ – form of solution

So we get $\left[\lambda^2 M + K_e + K_g\right] \vec{Q} = 0$

$\vec{Q}$ is a constant vector. Or $\boxed{A \vec{Q} = 0}$

This matrix has the form

$$A = \lambda^2 m e^2 \begin{pmatrix} 1+\delta & 1 \\ 1 & 1 \end{pmatrix} + k \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$+ P\ell \begin{pmatrix} -1-\alpha & \alpha \\ 0 & -1 \end{pmatrix}$$

Let us divide A by k and introduce

$$\tilde{\lambda}^2 = \lambda^2 \frac{me^2}{k} \quad \text{and} \quad \mu = \frac{P\ell}{k}$$

Then A becomes

$$A = \tilde{\lambda}^2 \begin{pmatrix} 1+\delta & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} + \mu \begin{pmatrix} -1-\alpha & \alpha \\ 0 & -\alpha \end{pmatrix}$$

characteristic equation

$$\det A = 0$$

$$A = \begin{pmatrix} (1+\delta)\tilde{\lambda}^2 + 2 - \mu(1+\alpha) & \tilde{\lambda}^2 - 1 + \alpha\mu \\ \tilde{\lambda}^2 - 1 & \tilde{\lambda}^2 + 1 + \mu \end{pmatrix}$$

Distribution of mass

Let M be a total mass, i.e. $M = m_1 + m_2$

$m_1 = \delta m$, $m_2 = m$

$M = m_1 + m_2 = \delta m + m = (1+\delta) m$

$\Rightarrow \boxed{m = \frac{M}{1+\delta}}$

For example, $\delta = 9 \Rightarrow m_1 = 9m = \frac{9M}{10} = 0.9M$

$m_2 = 0.1M$

$\delta = 99$, $m_1 = \frac{99M}{1+99} = 0.99M$, $m_2 = 0.01M$

$\delta = \frac{1}{9}$ $m_1 = \frac{1}{9} \frac{M}{1+1/9} = \frac{1}{9} \frac{9}{10} M = 0.1M$

$m_2 = \frac{9}{10} M$.

$\Rightarrow \boxed{\delta \in (0, +\infty)}$

Let us find F_c as a function of δ.

Let P be zero for simplicity

Now $\mu = \frac{Fl}{k}$ and characteristic equation is

$$\det A = 0$$

with $A = \tilde{\lambda}^2 \begin{pmatrix} 1+\delta & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$$+ \mu \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

or $A = \begin{pmatrix} (1+\delta)\tilde{\lambda}^2 + 2 - \mu & \tilde{\lambda}^2 - 1 + \mu \\ \tilde{\lambda}^2 - 1 & \tilde{\lambda}^2 + 1 \end{pmatrix}$

$$\det A = \left[(1+\delta)\tilde{\lambda}^2 + 2 - \mu\right]\left[\tilde{\lambda}^2 + 1\right] - $$
$$- \left[\tilde{\lambda}^2 - 1\right]\left[\tilde{\lambda}^2 - 1 + \mu\right]$$

$$= \delta \tilde{\lambda}^4 + \tilde{\lambda}^2 \left[5 + \delta - 2\mu\right] + 1 = 0$$

$$\tilde{\lambda}_{1,2} = \frac{-(5+\delta-2\mu) \pm \sqrt{(5+\delta-2\mu)^2 - 4\delta}}{2\delta}$$

$$\Delta = (5+\delta-2\mu)^2 - 4\delta$$

We are looking for a value of μ such that

$$\Delta = 0$$

$$\Delta = (5+\delta-2\mu)^2 - 4\delta = 0$$

$$(5+\delta-2\mu)^2 = 4\delta \quad , \quad \underline{\delta > 0}$$

$$5+\delta-2\mu = 2\sqrt{\delta}$$

$$2\mu = \underline{5+\delta-2\sqrt{\delta}} \Rightarrow \underline{\mu_c = \frac{5}{2} + \frac{\delta}{2} - \sqrt{\delta}}$$

$$\mu = \frac{Fl}{k}$$

$$\boxed{F_c = \mu_c \frac{k}{l}}$$

$\delta = 9$; $\mu_c = 4$

$\delta = 100$; $\mu_c = 42.5$

$\delta \to \infty$, $\mu_c \to \infty$ and $F_c \to \infty$

$$\mu = \mu(\delta) = \frac{5}{2} + \frac{\delta}{2} - \sqrt{\delta}$$

$$F_c = \mu(\delta)\frac{k}{\ell}$$

$$\boxed{\begin{array}{c}\text{Ziegler case}: \delta = 2 \\ F_c = 2.09\,\frac{k}{\ell}\end{array}}$$

$$\mu(0) = \frac{5}{2} = 2.5$$

$$\mu(\infty) = \infty$$

Let us find minimum of μ

$$\mu' = \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{\delta}} = 0$$

$$\Rightarrow \delta = 1$$

$$\mu_{min} = \mu(1) = \frac{5}{2} + \frac{1}{2} - 1 = 2$$

Critical force depends on mass distribution