Università degli Studi di Cagliari
Facoltà di Ingegneria e Architettura
Dipartimento di Ingegneria Civile, Ambientale e Architettura
Magistrale Ingegneria Civile 2 anno **a.a. 2022/23**

Structural Stability and Limit Analysis of Structures

(Instabilità delle strutture e calcolo a rottura)

> Lezione 20

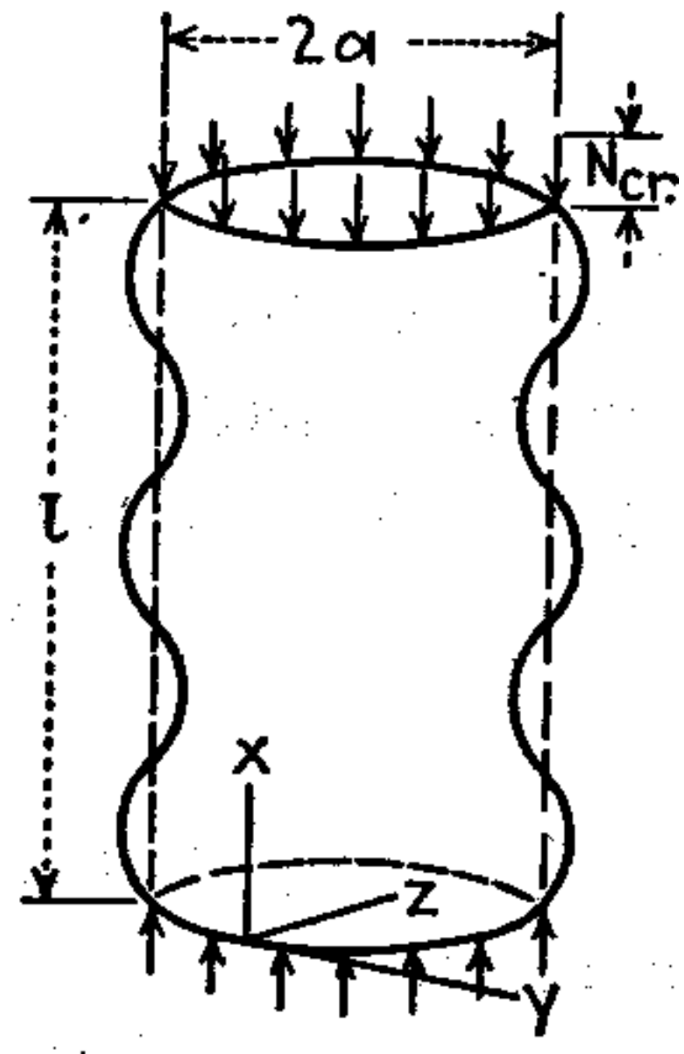
Instability of cylindrical shells

(Instabilità: le lastre curve (tubi))

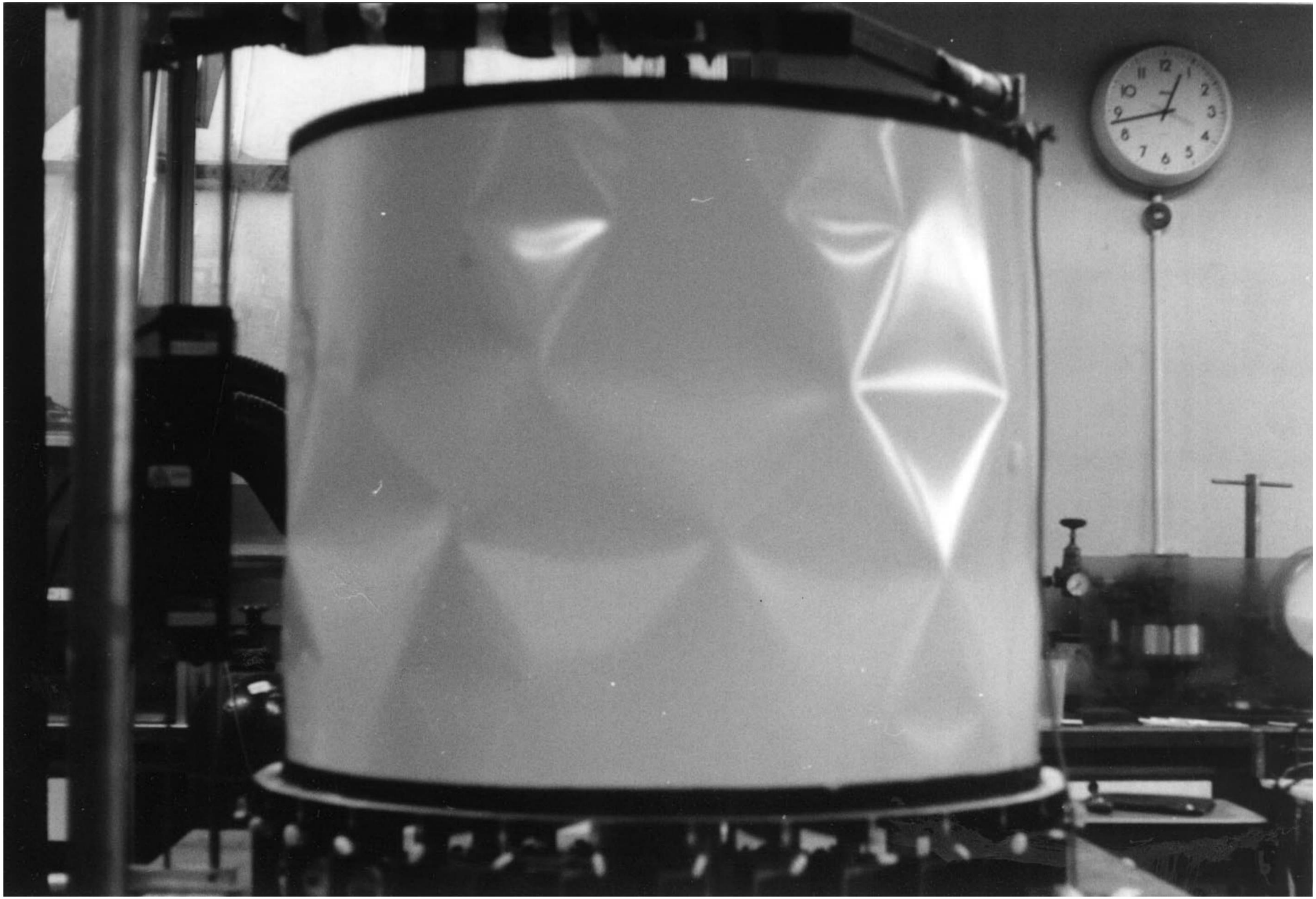
Victor Eremeev

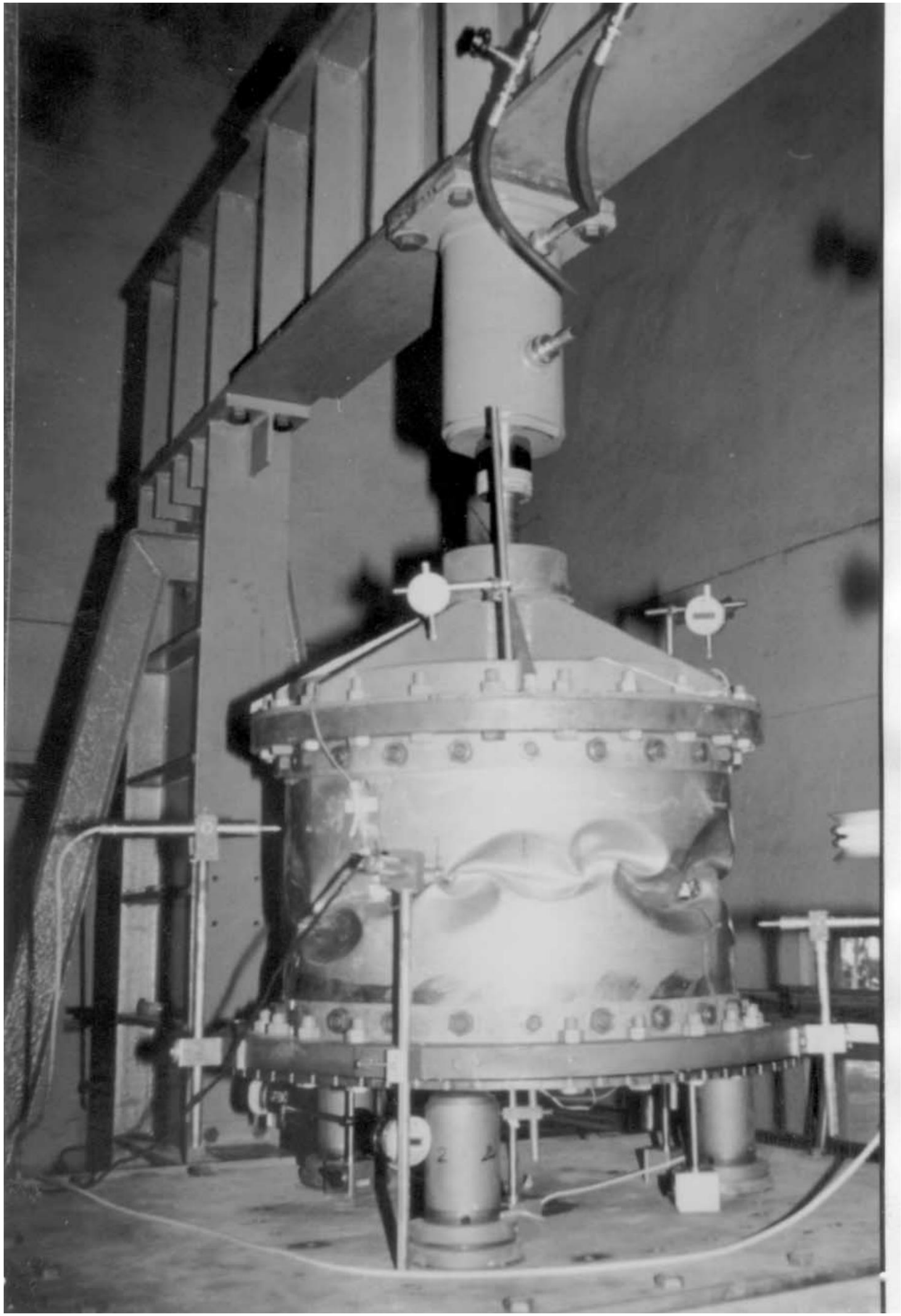
victor.eremeev@unica.it

Lastre cilindriche (tubi)



https://www.youtube.com/watch?v=XVJrcizcdwE

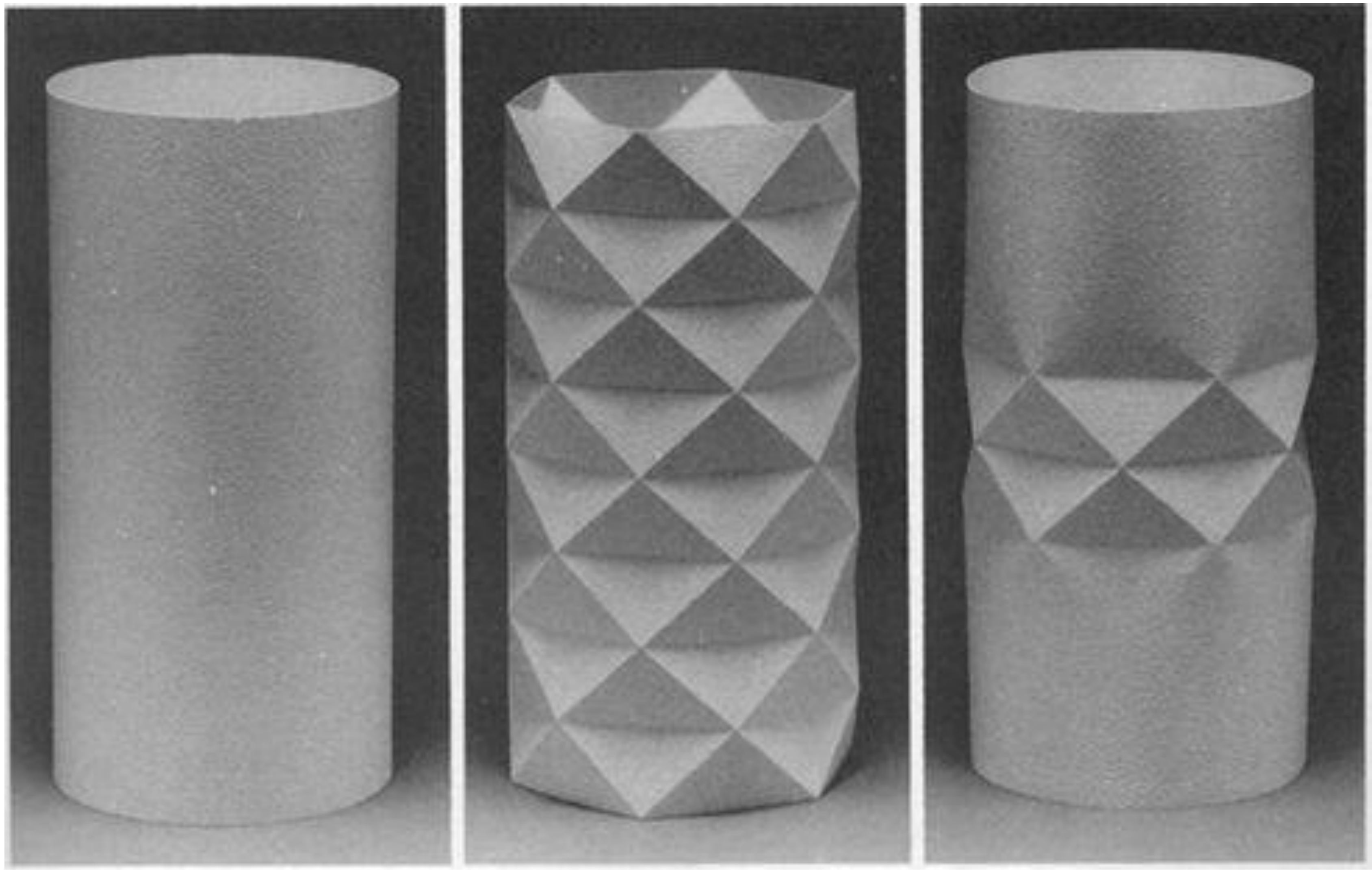


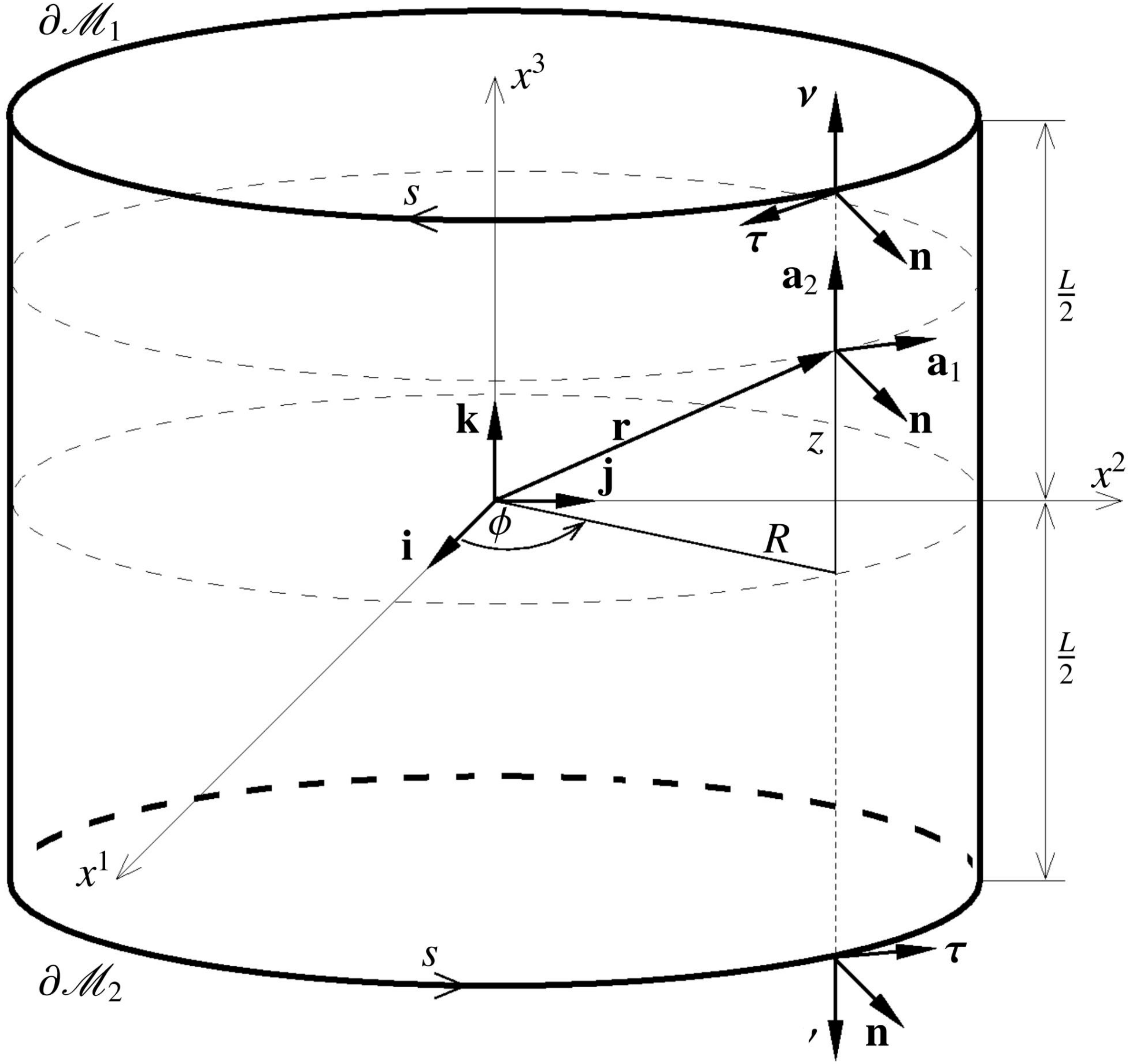


https://shellbuckling.com/

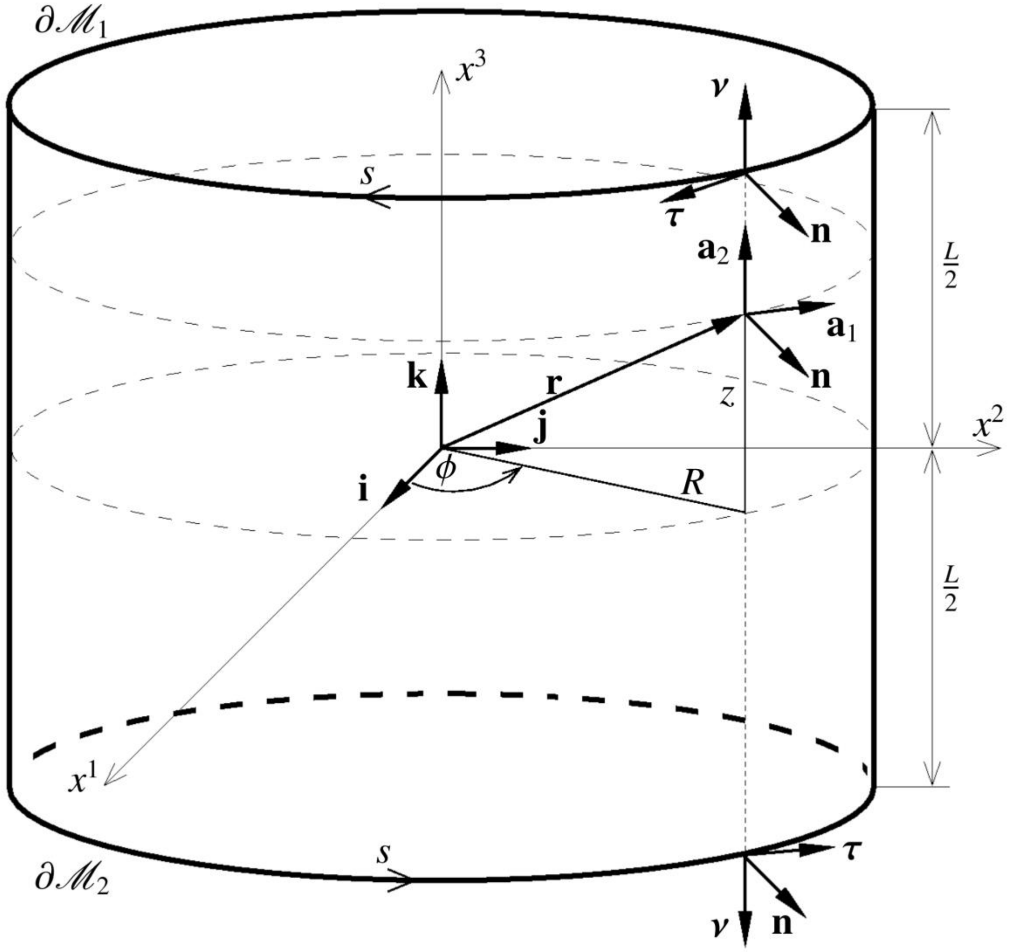
https://www.youtube.com/watch?v=y5VRww1Ypwk

https://orilab.art/natural/yoshimura





$$\mathbf{r} = R\cos\phi\,\mathbf{i} + R\sin\phi\,\mathbf{j} + z\,\mathbf{k}\ ,$$

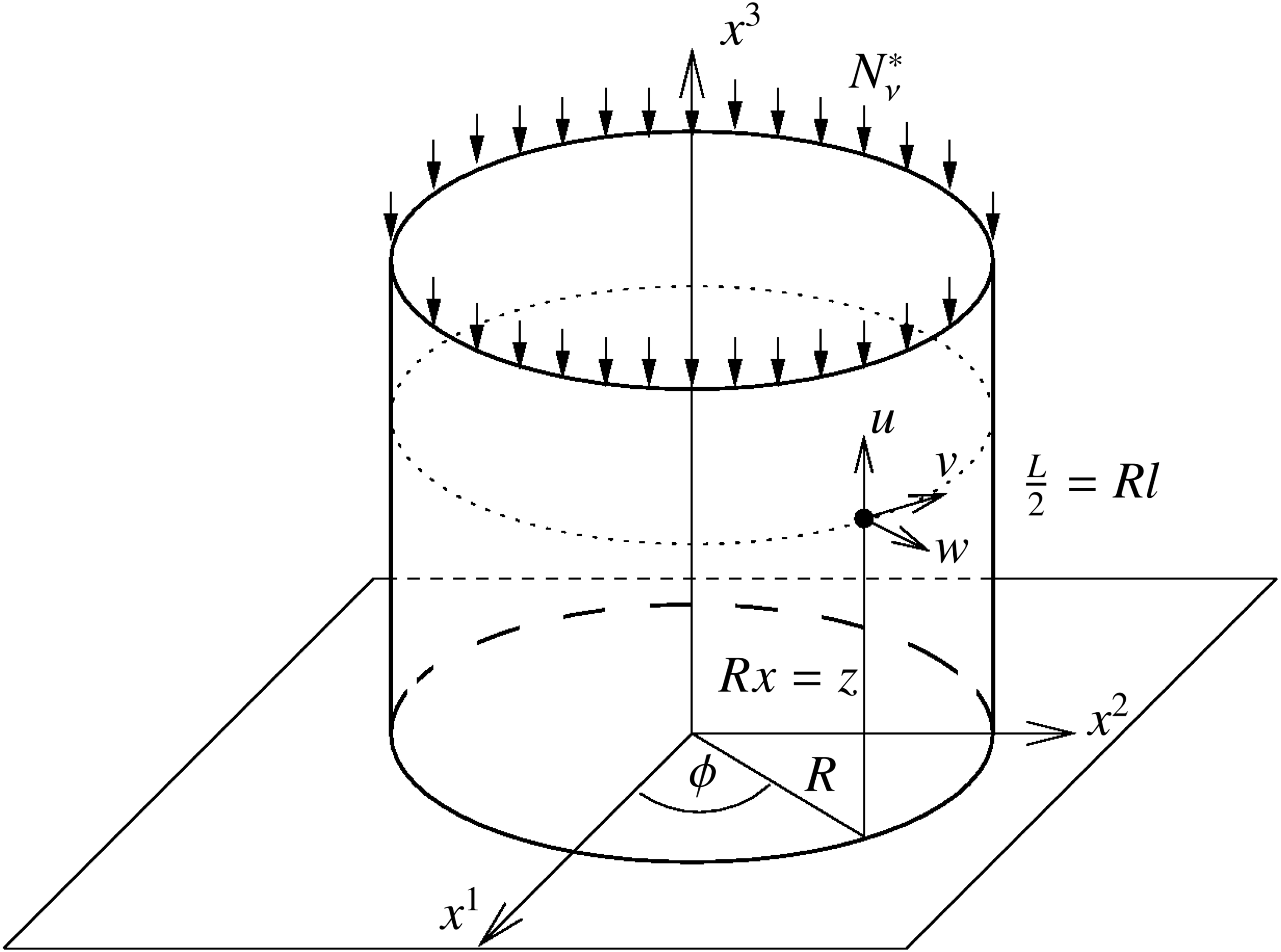


$$\mathbf{r} = R\cos\phi\,\mathbf{i} + R\sin\phi\,\mathbf{j} + z\,\mathbf{k}\ ,$$

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \phi} = -R\sin\phi\,\mathbf{i} + R\cos\phi\,\mathbf{j}\ , \qquad \mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}\ ,$$

$$\mathbf{a}^1 = \frac{\mathbf{a}_2\times\mathbf{n}}{\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{n})} = -\frac{1}{R}(\sin\phi\,\mathbf{i} - \cos\phi\,\mathbf{j})\ , \qquad \mathbf{a}^2 = \frac{\mathbf{n}\times\mathbf{a}_1}{\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{n})} = \mathbf{k}\ ,$$

$$\mathbf{n} = \frac{\mathbf{a}_1\times\mathbf{a}_2}{|\mathbf{a}_1\times\mathbf{a}_2|} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}\ .$$


x^3
N_ν^*
u
v
w
$\frac{L}{2} = Rl$
$Rx = z$
x^2
ϕ
R
x^1

Selected theoretical models used in stability analysis:

- Flügge (1932)
- Donnell (1933)
- Mushtari-Galimov (1957)
- Sanders (1963)
- Koiter (1967)
- Budiansky (1978)
- Stumpf (1984)
- Pietraszkiewicz (1984)

Membrane prebuckling state

Axial compressive force component $N_\nu^* = -\frac{2\rho}{\epsilon}$ generates membrane prebuckling state

$$u_0(\phi, x) = -2\epsilon\rho(1 + 3\epsilon\rho)x$$

$$v_0(\phi, x) = 0$$

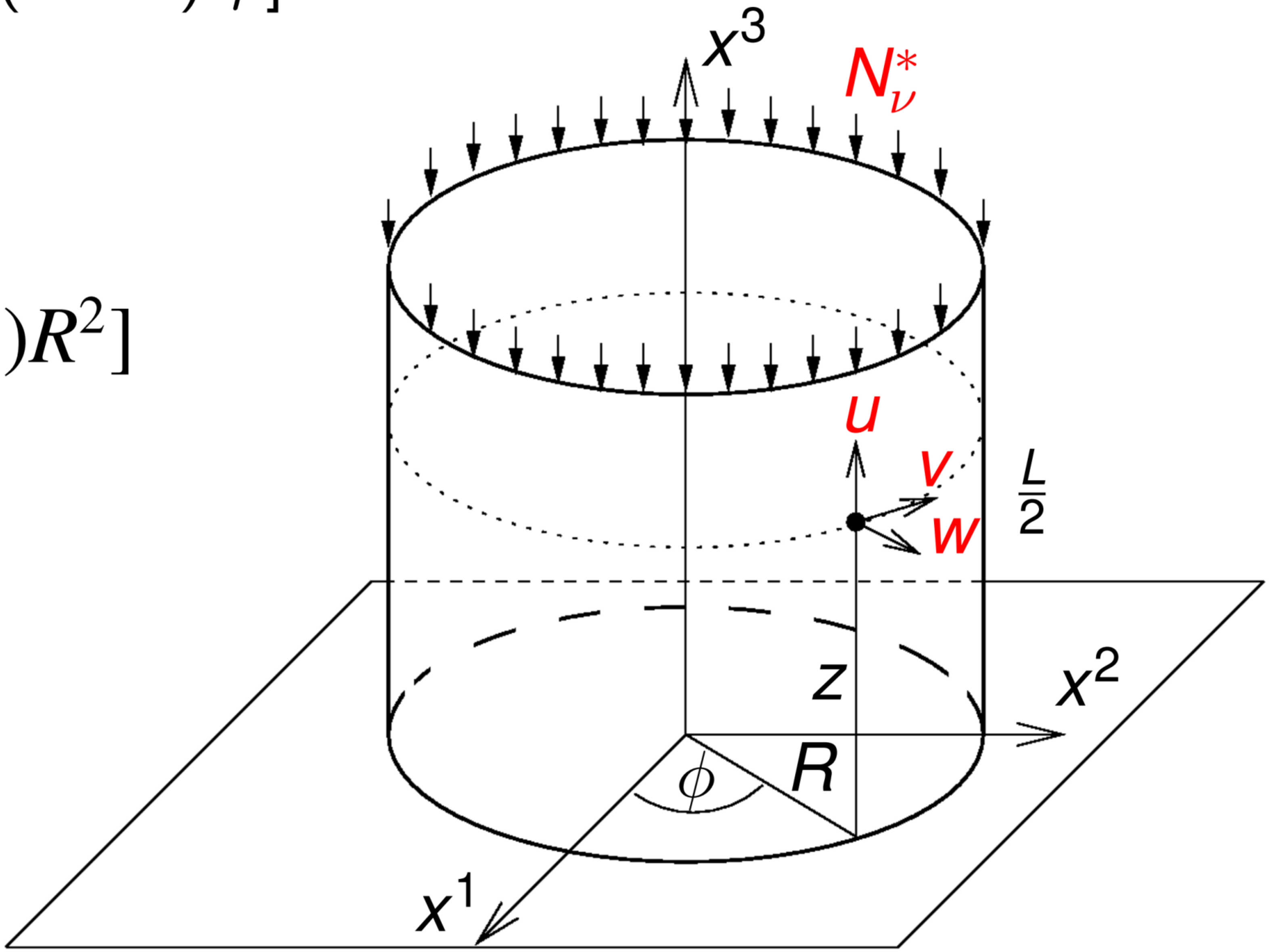
$$w_0(\phi, x) = 2\epsilon\nu\rho[1 + (2 - \nu)\epsilon\rho]$$

$$N_\nu^* = -\frac{2\rho}{\epsilon}$$

$$\epsilon^2 = h^2/[12(1 - \nu^2)R^2]$$

$$\rho = 1$$

$$\sigma_{cl} = 2\epsilon E h.$$



$$N_1^1 = \frac{D}{R^2} N_\phi(\phi, x) \ , \quad N_2^2 = \frac{D}{R^2} N_x(\phi, x) \ , \quad N_1^2 = R^2 N_2^1 = \frac{D}{R} N_{x\phi}(\phi, x) \ ,$$
$$M_1^1 = \frac{D}{R} M_\phi(\phi, x) \ , \quad M_2^2 = \frac{D}{R} M_x(\phi, x) \ , \quad M_1^2 = R^2 M_2^1 = D M_{x\phi}(\phi, x) \ ,$$

$$N_\phi = O(1) \ , \quad N_{x\phi} = O(1) \ , \quad N_x = -\frac{2\rho}{\epsilon} + O(1) \ ,$$
$$M_\phi = O(\epsilon) \ , \quad M_{x\phi} = O(\epsilon) \ , \quad M_x = O(\epsilon) \ ,$$

$$u_1 = u_0 + \mu u \ , \quad v_1 = v_0 + \mu v \ , \quad w_1 = w_0 + \mu w$$

Opoka, S. and Pietraszkiewicz, W., 2009. On refined analysis of bifurcation buckling for the axially compressed circular cylinder. *International Journal of Solids and Structures*, *46*(17), pp.3111-3123.

Bifurcation buckling: Linear stability equations and homogeneous work-conjugate boundary conditions

$$A_1 w^{\bullet\bullet\bullet} + A_2 w^{\prime\prime\bullet} + A_3 u^{\prime\bullet} + A_4 v^{\prime\prime} + A_5 v^{\bullet\bullet} + A_6 w^{\bullet} = 0$$

$$B_1 (w^{\prime\bullet\bullet} + \nu w^{\prime\prime\prime}) + B_2 u^{\prime\prime} + B_3 u^{\bullet\bullet} + B_4 [(1+\nu) v^{\prime\bullet} + 2\nu w^{\prime}] = 0$$

$$C_1 (w^{\prime\prime\prime\prime} + 2 w^{\prime\prime\bullet\bullet} + w^{\bullet\bullet\bullet\bullet}) + C_2 (u^{\prime\bullet\bullet} + \nu u^{\prime\prime\prime}) + C_3 v^{\prime\prime\bullet} + C_4 v^{\bullet\bullet\bullet} + C_5 w^{\prime\prime} + C_6 w^{\bullet\bullet} + C_7 u^{\prime} + C_8 v^{\bullet} + C_9 w = 0$$

$d_1 \equiv D_1 w^{\prime\prime} + D_2 u^{\prime} + D_3 (v^{\bullet} + w) = 0$ or $u = 0$

$d_2 \equiv E_1 w^{\prime\bullet} + E_2 u^{\bullet} + E_3 v^{\prime} = 0$ or $v = 0$

$d_3 \equiv F_1 [w^{\prime\prime\prime} + (2-\nu) w^{\prime\bullet\bullet}] + F_2 (u^{\bullet\bullet} + \nu u^{\prime\prime}) + F_3 v^{\prime\bullet} + F_4 w^{\prime} = 0$ or $w = 0$

$d_4 \equiv G_1 (w^{\prime\prime} + \nu w^{\bullet\bullet}) + G_2 v^{\bullet} + G_3 u^{\prime} + G_4 w = 0$ or $w^{\prime} = 0$

at $x = \pm l = \pm \frac{L}{2R}$, where $\frac{\partial()}{\partial x} = ()^{\prime}$, $\frac{\partial()}{\partial \phi} = ()^{\bullet}$

$$A_1 = 4\epsilon^2(1-\nu^2)[1-\epsilon(2-5\nu)\rho-\epsilon^2(7+2\nu)\rho^2]+O(\epsilon^5)\ ,$$
$$A_2 = 4\epsilon^2(1-\nu^2)[1-\epsilon(2-4\nu-\nu^2)\rho-\epsilon^2(6+\nu+2\nu^2)\rho^2]+O(\epsilon^5)\ ,$$
$$A_3 = -(1+\nu)[1-2\epsilon(2-3\nu)\rho-2\epsilon^2(4+6\nu-3\nu^2)\rho^2]+O(\epsilon^3)\ ,$$
$$A_4 = -(1-\nu)\{1-\epsilon(6-4\nu)\rho+2\epsilon^2[2(1-\nu^2)-(3+8\nu)\rho^2]\}+O(\epsilon^3)\ ,$$
$$A_5 = -2\{1-2\epsilon(1-4\nu)\rho+2\epsilon^2[2(1-\nu^2)-(3-8\nu^2)\rho^2]\}+O(\epsilon^3)\ ,$$
$$A_6 = -2\{1-2\epsilon(1-4\nu)\rho+2\epsilon^2[1-\nu^2-(3-8\nu^2)\rho^2]\}+O(\epsilon^3)\ ,$$
$$B_1 = 4\epsilon^3(1-\nu^2)\rho[\nu-\epsilon(1+2\nu)\rho]+O(\epsilon^5)\ ,$$
$$B_2 = -2[1-2\epsilon(5-\nu-\nu^2)\rho+2\epsilon^2(2-8\nu-3\nu^2+2\nu^3)]\rho^2]+O(\epsilon^3)\ ,$$
$$B_3 = -(1-\nu)[1-2\epsilon(4-\nu)\rho-2\epsilon^2\nu(6+\nu)\rho^2]+O(\epsilon^3)\ ,$$
$$B_4 = -1+\epsilon(6-4\nu)\rho+2\epsilon^2(3+8\nu)\rho^2+O(\epsilon^3)\ ,$$
$$C_1 = -2\epsilon^2(1-\nu^2)[1-\epsilon(6-4\nu)\rho-2\epsilon^2(3+8\nu)\rho^2]+O(\epsilon^5)\ ,$$
$$C_2 = 4\epsilon^3(1-\nu^2)\rho[\nu-\epsilon(1+4\nu-2\nu^2)\rho]+O(\epsilon^5)\ ,$$
$$C_3 = 4\epsilon^2(1-\nu^2)[1-\epsilon[6-4\nu-\nu^2]\rho-\epsilon^2(6+17\nu+6\nu^2)\rho^2]+O(\epsilon^5)\ ,$$
$$C_4 = 4\epsilon^2(1-\nu^2)[1-\epsilon(6-5\nu)\rho-\epsilon^2(7+22\nu)\rho^2]+O(\epsilon^5)\ ,$$
$$C_5 = -4\epsilon(1-\nu^2)[\rho-\epsilon(\nu+4(1-\nu)\rho^2)]+O(\epsilon^3)\ ,$$
$$C_6 = 4\epsilon^2(1-\nu^2)+O(\epsilon^3)\ ,$$
$$C_7 = -2\nu[1-2\epsilon(4-3\nu)\rho-6\epsilon^2(6-\nu)\nu\rho^2]+O(\epsilon^3)\ ,$$
$$C_8 = -2[1-2\epsilon(3-4\nu)\rho+2\epsilon^2(1-\nu^2-(3+16\nu-8\nu^2)\rho^2)]+O(\epsilon^3)\ ,$$
$$C_9 = -2[1-2\epsilon(3-4\nu)\rho+\epsilon^2(1-\nu^2-2(3+16\nu-8\nu^2)\rho^2)]+O(\epsilon^3)\ ,$$

$$D_1 = -4\epsilon^3\nu(1-\nu^2)\rho[\nu - \epsilon(1+2\nu)\rho] + O(\epsilon^5)\ ,$$
$$D_2 = 2[1 - 2\epsilon(5-\nu-\nu^2)\rho + 2\epsilon^2(2-8\nu-3\nu^2+2\nu^3)\rho^2] + O(\epsilon^3)\ ,$$
$$D_3 = 2\nu[1 - \epsilon(6-4\nu)\rho - 2\epsilon^2(3+8\nu)\rho^2] + O(\epsilon^3)\ ,$$
$$E_1 = 4\epsilon^2(1-\nu^2)[1-\nu-\epsilon(2-6\nu+3\nu^2)\rho - \epsilon^2(6-5\nu+2\nu^2)\rho^2] + O(\epsilon^5)\ ,$$
$$E_2 = -(1-\nu)[1 - 2\epsilon(2-3\nu)\rho - 2\epsilon^2(4+6\nu-3\nu^2)\rho^2] + O(\epsilon^3)\ ,$$
$$E_3 = -(1-\nu)[1 - \epsilon(6-4\nu)\rho + \epsilon^2(4-4\nu^2-6\rho^2-16\nu\rho^2)] + O(\epsilon^3)\ ,$$
$$F_1 = -2\epsilon^2(1-\nu^2)[1 - \epsilon(6-4\nu)\rho - 2\epsilon^2(3+8\nu)\rho^2] + O(\epsilon^5)\ ,$$
$$F_2 = 4\epsilon^3(1-\nu^2)\rho[\nu - \epsilon(1+4\nu-2\nu^2)\rho] + O(\epsilon^5)\ ,$$
$$F_3 = 4\epsilon^2(1-\nu^2)[1 - \epsilon(6-4\nu-\nu^2)\rho - \epsilon^2(6+17\nu+6\nu^2)\rho^2] + O(\epsilon^5)\ ,$$
$$F_4 = -2\epsilon(1-\nu^2)[2\rho - \epsilon(\nu + 8(1-\nu)\rho^2)] + O(\epsilon^3)\ ,$$
$$G_1 = -1 + 2\epsilon(1-\nu)\rho + 2\epsilon^2(3+\nu^2)\rho^2 + O(\epsilon^3)\ ,$$
$$G_2 = 2\nu[1 - \epsilon(3-2\nu)\rho - \epsilon^2(9+2\nu+2\nu^2)\rho^2] + O(\epsilon^3)\ ,$$
$$G_3 = 2\epsilon\nu\rho[1+2\nu+\epsilon(3+4\nu)\rho] + O(\epsilon^3)\ ,$$
$$G_4 = \nu[1 - 2\epsilon(2-\nu)\rho - 2\epsilon^2(6+2\nu+\nu^2)\rho^2] + O(\epsilon^3)\ .$$

Solution method

1. Solutions of the stability equations

$$\left.\begin{aligned} u(\phi, x) &= Ue^{px}\cos(n\phi) \\ v(\phi, x) &= Ve^{px}\sin(n\phi) \\ w(\phi, x) &= We^{px}\cos(n\phi) \end{aligned}\right\} \Rightarrow \begin{aligned} &V = V(\nu, \epsilon, n, \rho; p_j, U) \\ &W = W(\nu, \epsilon, n, \rho; p_j, U) \\ &\text{p}_j \text{ - roots of characteristic eq.} \end{aligned}$$

2. Substitution of solutions into boundary conditions

   Solutions exist if

$$Det[\ldots] = 0 \quad \Rightarrow \quad f(\nu, \epsilon, n, \rho, p_j, l) = 0 \quad \Rightarrow \quad \rho = \rho_{crit}$$

$$u(\phi, x) = \sum_j U_j e^{p_j x} \cos(n\phi) - \frac{1-\epsilon(6-8\nu)\rho+\epsilon^2[1-\nu^2-2(3+16\nu-8\nu^2)\rho^2]}{\nu[1-2\epsilon\rho(4-3\nu+3\epsilon(6-\nu)\nu\rho)]} S\,x + Z \,,$$

$$v(\phi, x) = \sum_j V(\nu, \epsilon, n, \rho; p_j, U_j) e^{p_j x} \sin(n\phi) \,,$$

$$w(\phi, x) = \sum_j W(\nu, \epsilon, n, \rho; p_j, U_j) e^{p_j x} \cos(n\phi) + S \,,$$

where non-zero A_k's are named S and Z, respectively.

---

axially compressed Euler column with simply-supported (clamped) boundaries

$$\rho = \frac{\pi^2}{16 l^2 \epsilon} \left(\rho = \frac{\pi^2}{4 l^2 \epsilon}\right),$$

▲ clamped column

■ simply-supported column

■ ▲

Nomenclature for boundary conditions

C–family and S–family				
C1(S1):	$u = 0$	$v = 0$	$w = 0$	$w' = 0 (d_4 = 0)$
C2(S2):	$d_1 = 0$	$v = 0$	$w = 0$	$w' = 0 (d_4 = 0)$
C3(S3):	$u = 0$	$d_2 = 0$	$w = 0$	$w' = 0 (d_4 = 0)$
C4(S4):	$d_1 = 0$	$d_2 = 0$	$w = 0$	$w' = 0 (d_4 = 0)$
C5(S5):	$u = 0$	$v = 0$	$d_3 = 0$	$w' = 0 (d_4 = 0)$
C6(S6):	$d_1 = 0$	$v = 0$	$d_3 = 0$	$w' = 0 (d_4 = 0)$
C7(S7):	$u = 0$	$d_2 = 0$	$d_3 = 0$	$w' = 0 (d_4 = 0)$
C8(S8):	$d_1 = 0$	$d_2 = 0$	$d_3 = 0$	$w' = 0 (d_4 = 0)$

Classical cases:
C1: clamped BC
S2: simply-supported BC

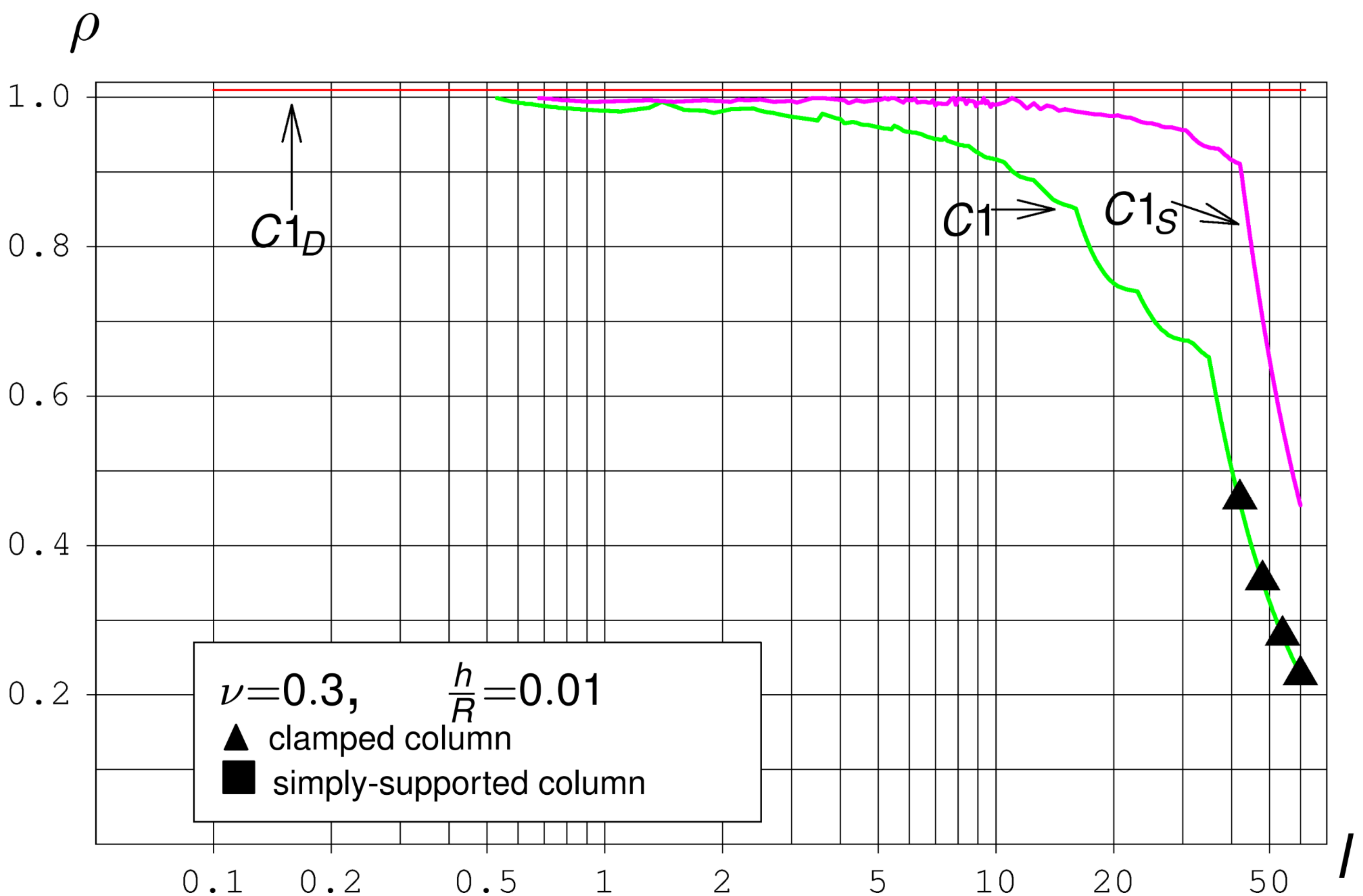
SIMPLIFIED VERSION (Timoshenko)

$$D \frac{d^4 w}{dx^4} + N_x \frac{d^2 w}{dx^2} + Eh \frac{w}{a^2} = 0$$

$$w = -A \sin \frac{m\pi x}{l}$$

Influence of different approximations in the derivation of the stability problem

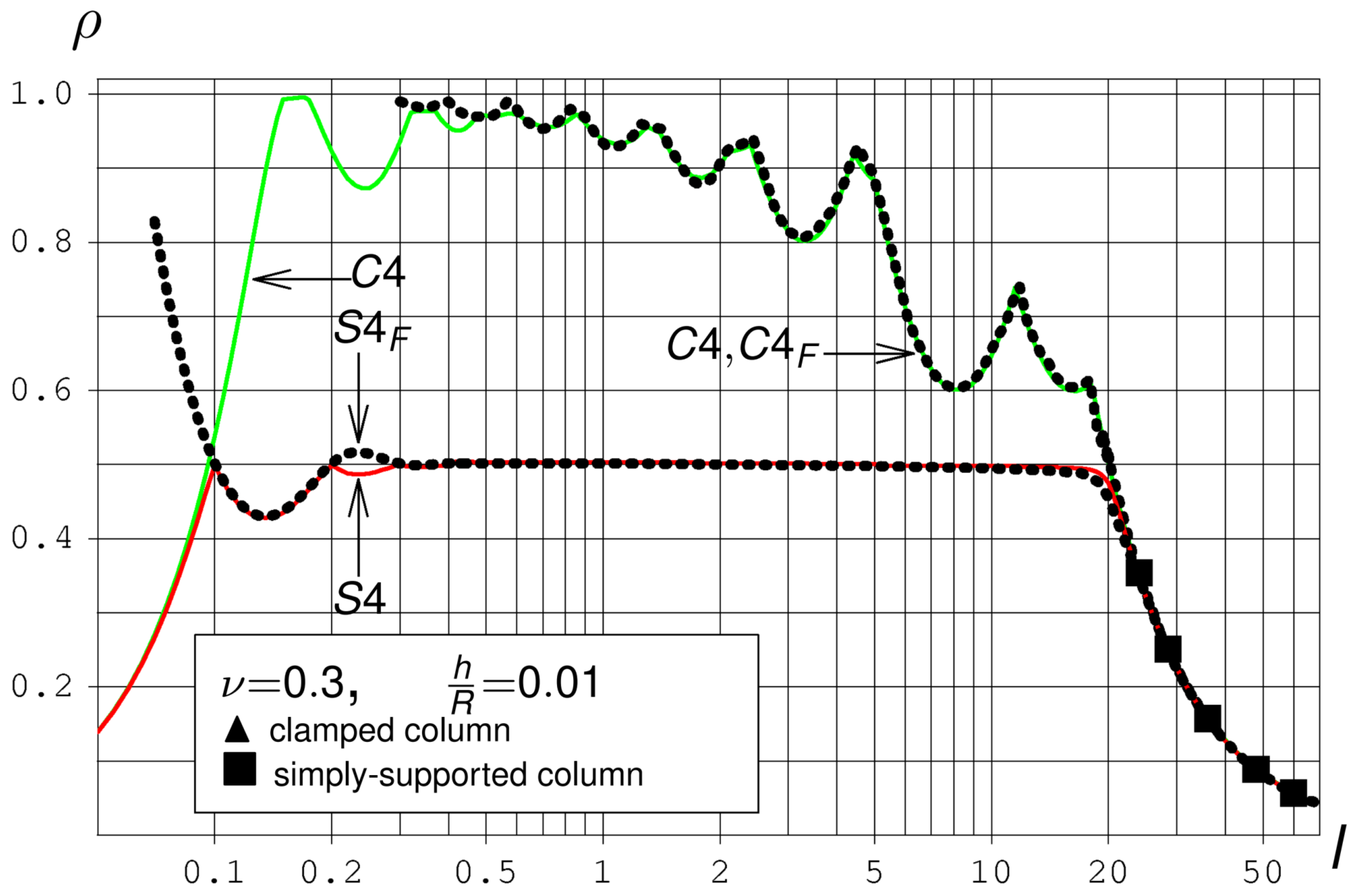
Even (energetically) small terms influence ρ_{crit}



Comparison with results obtained from Flügge equations

For all curves we noted good overall agreement
but short cylinders with bc S4 or C4 behave differently

$$l = \frac{L}{2R}$$



Similar result for S4 case: Simmonds and Danielson (1970)

$$l = \frac{L}{2R}$$

if $l \leq 0.1$ then the cylinder is regarded as short

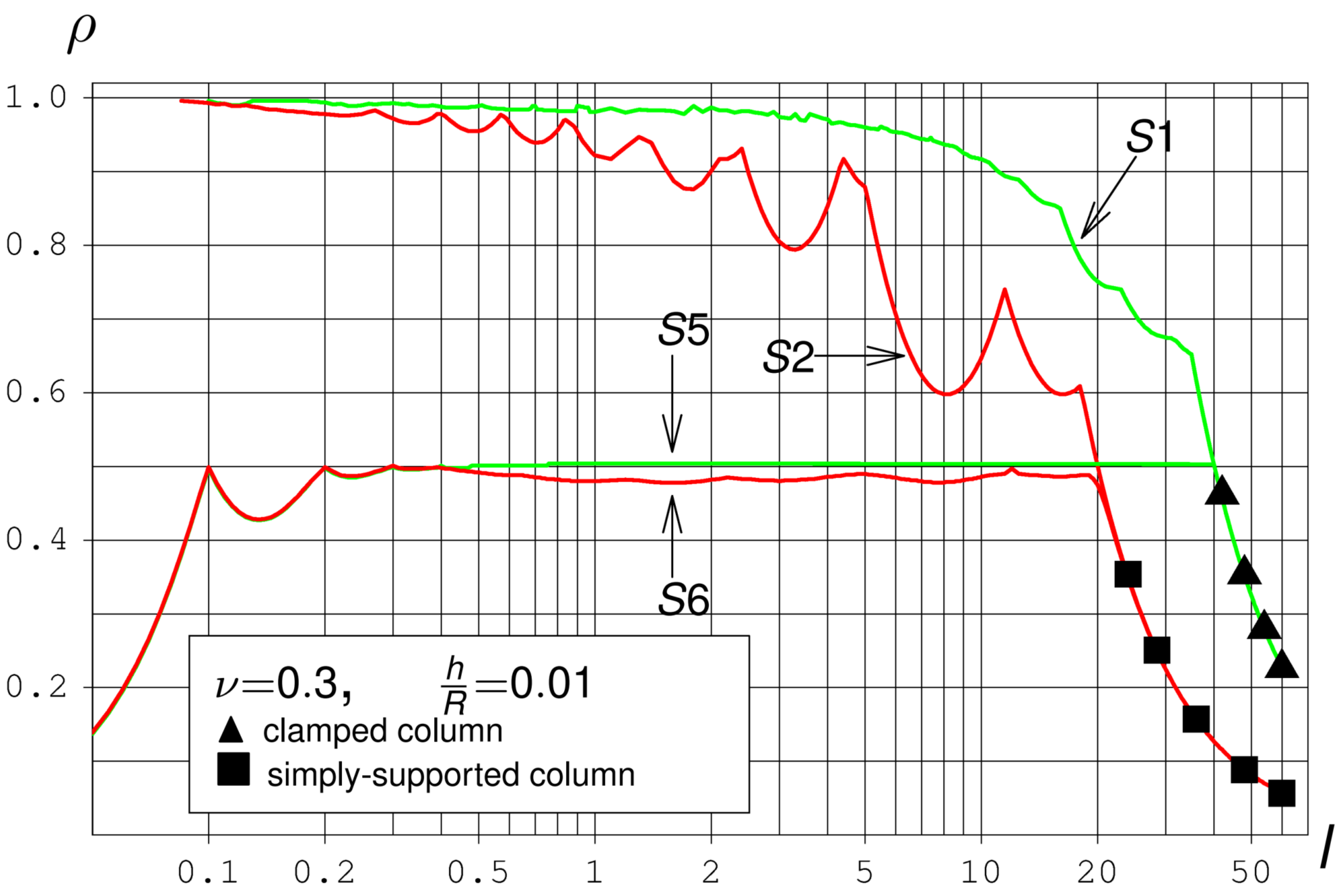
the practical cylinder lengths (PCL) cover the interval $l \in (0.1, 20)$

long cylinders $l \geq 20.$

experimental cylinder lengths (ECL) when $l \in (0.2, 5).$

Relaxation of geometric boundary conditions as a factor for decreasing the buckling load

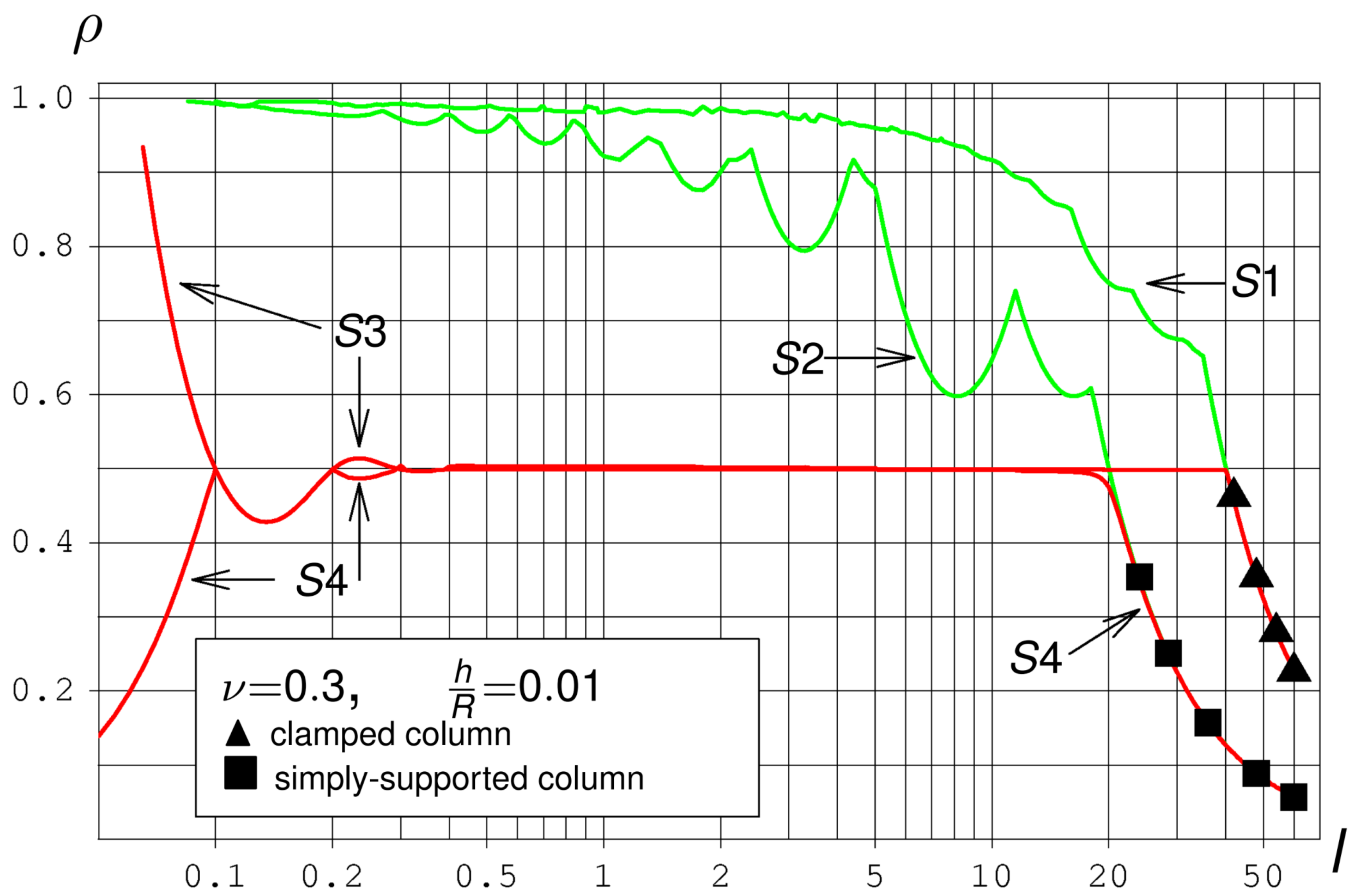
The exchange of $u = 0$ for $d_1 = 0$: $S1 \to S2$, $S5 \to S6$



Small influence in range of ECL

Relaxation of geometric boundary conditions as a factor for decreasing the buckling load

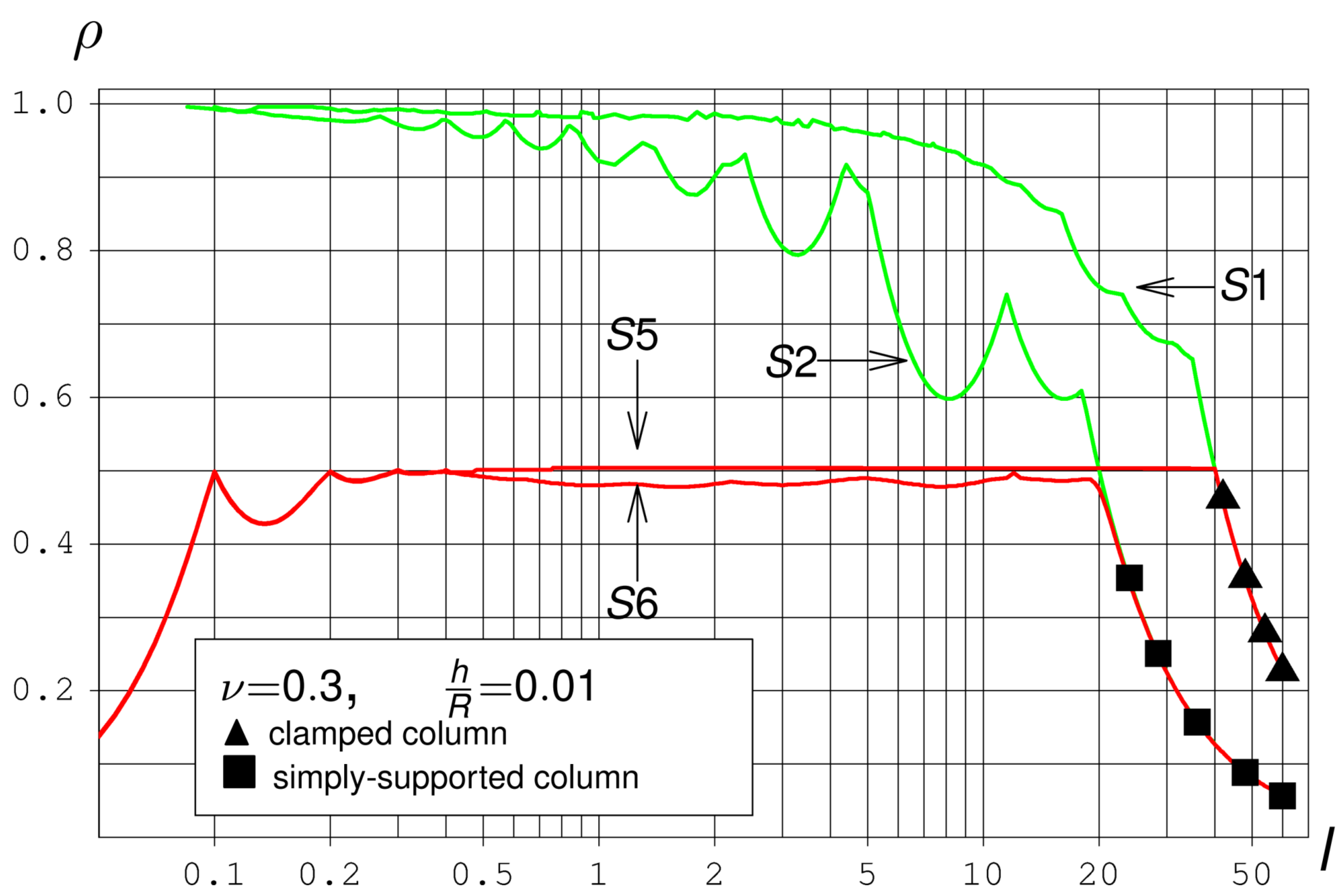
The exchange of $v = 0$ for $d_2 = 0$: $S1 \rightarrow S3$, $S2 \rightarrow S4$



Fall down to one half of the classical value Ohira (1961)

Relaxation of geometric boundary conditions as a factor for decreasing the buckling load

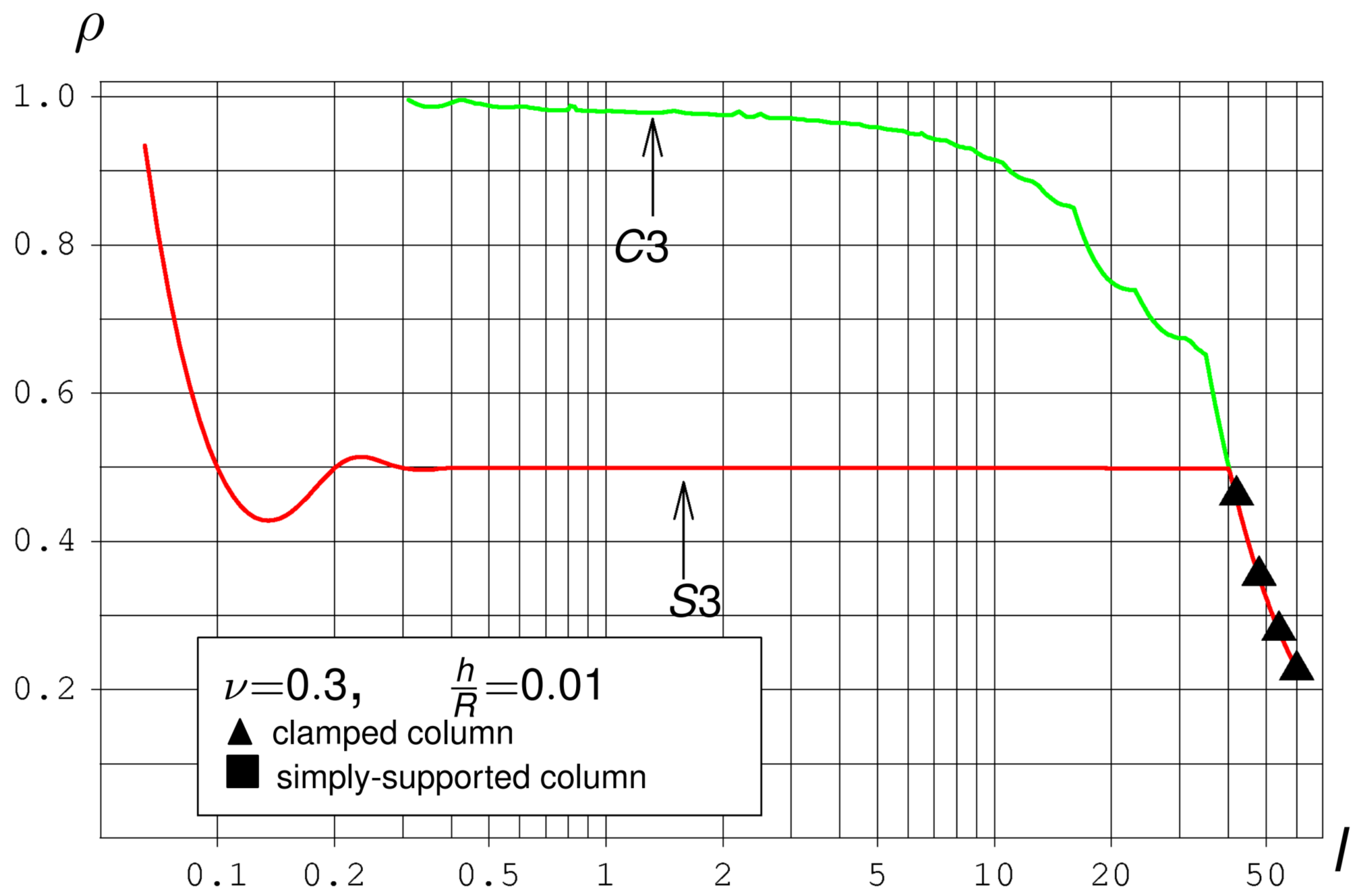
The exchange of $w = 0$ for $d_3 = 0$: $S1 \rightarrow S5$, $S2 \rightarrow S6$



Fall down to one half of the classical value in range of ECL

Relaxation of geometric boundary conditions as a factor for decreasing the buckling load

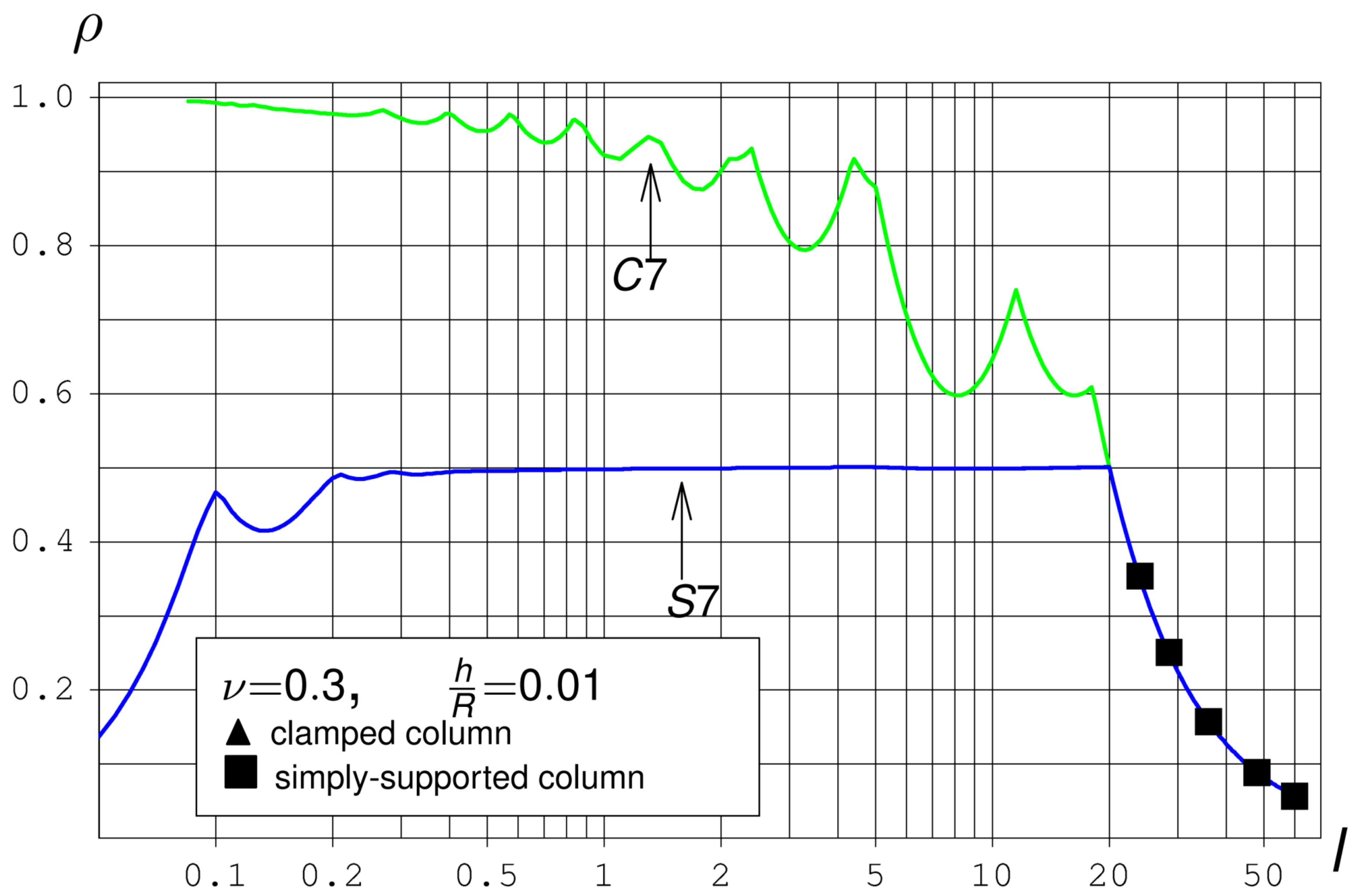
The exchange of $w' = 0$ for $d_4 = 0$: $C3 \rightarrow S3$



Fall down to one half of the classical value in range of ECL

Behaviour of long cylinder as Euler column

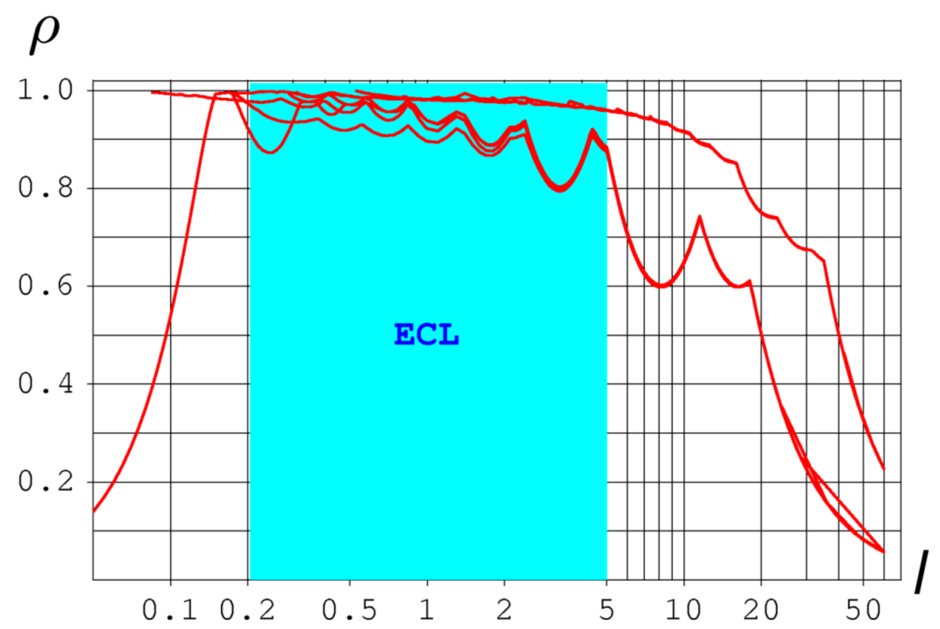
Long cylinder behaves as clamped Euler column
iff $u = v = 0$ or $u = w = 0$ (the constraint $u = 0$ is not sufficient)



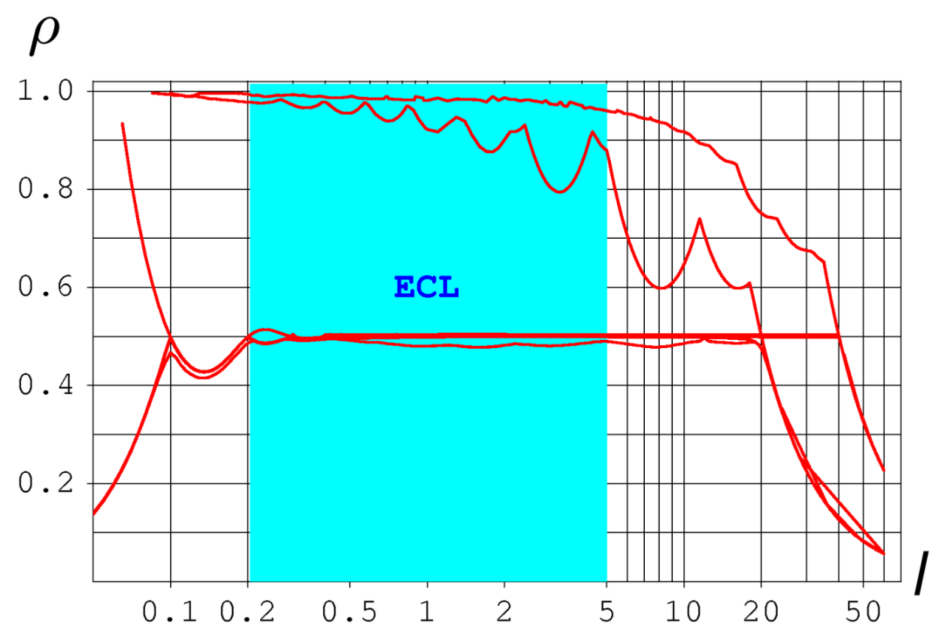
The influence of suppresion of rotations at the boundary

- Cylinders with clamped BCs

experimental cylinder lengths (ECL)



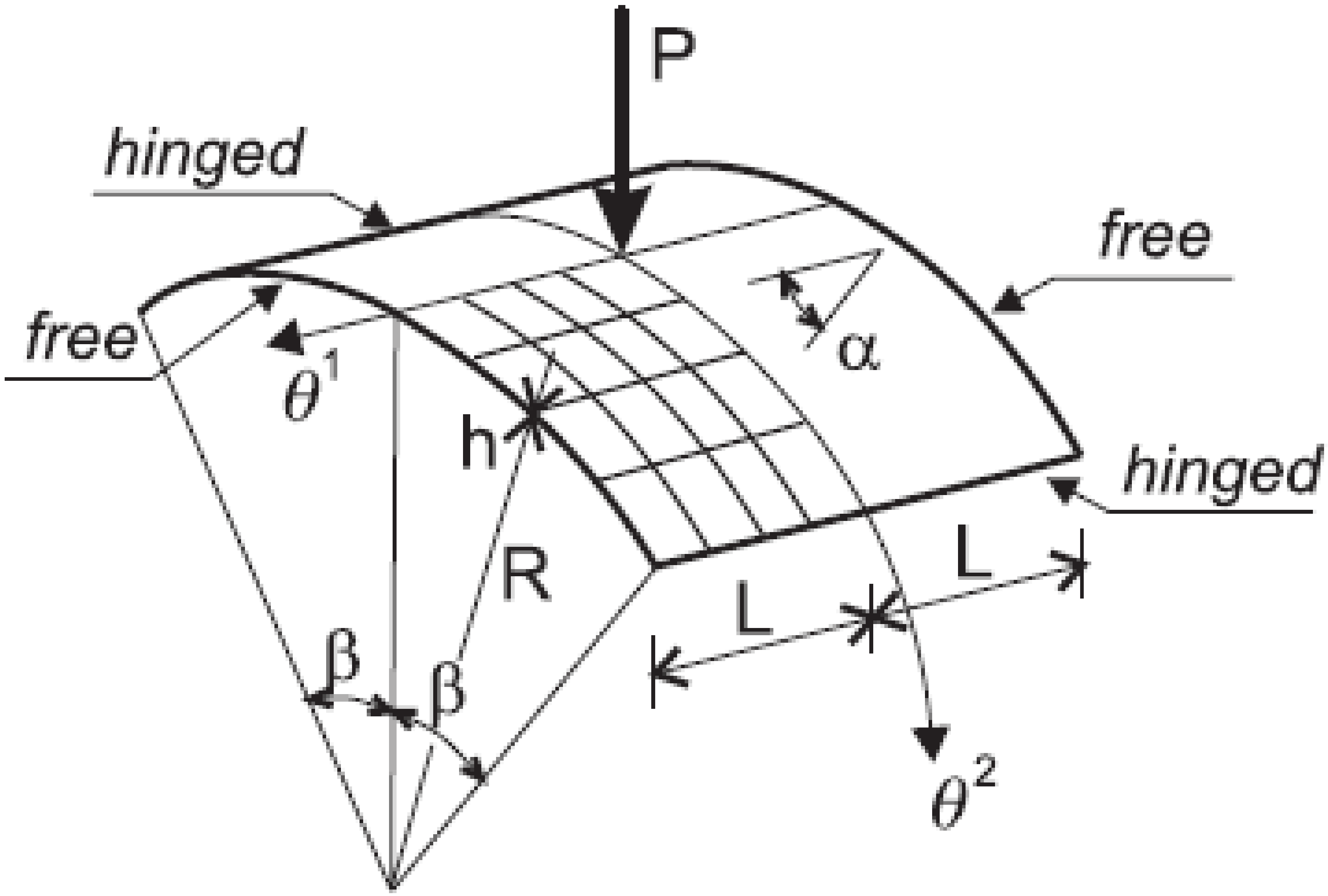
- Cylinders with simply-supported BCs



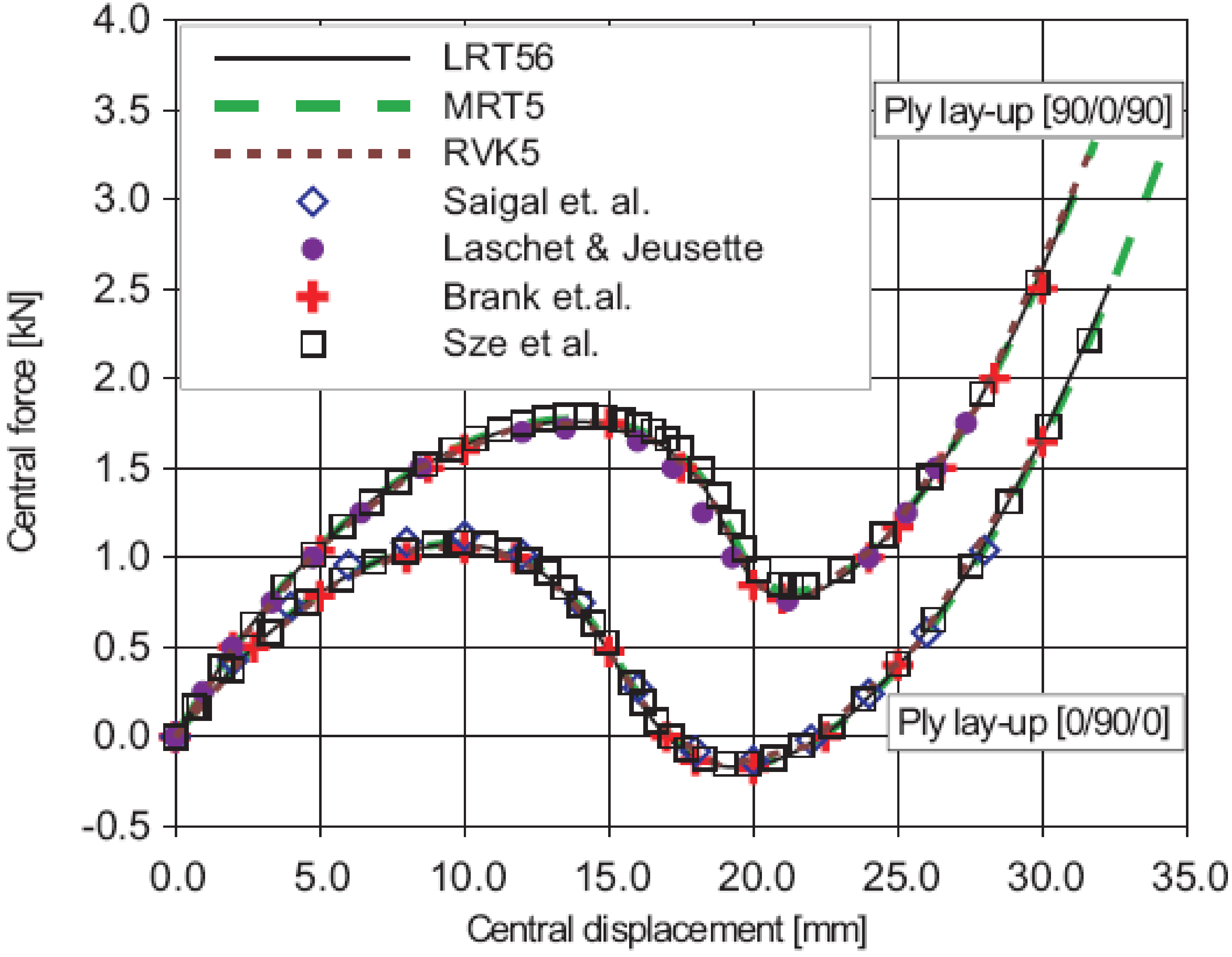
Finite elements analysis

Buckling e postbuckling

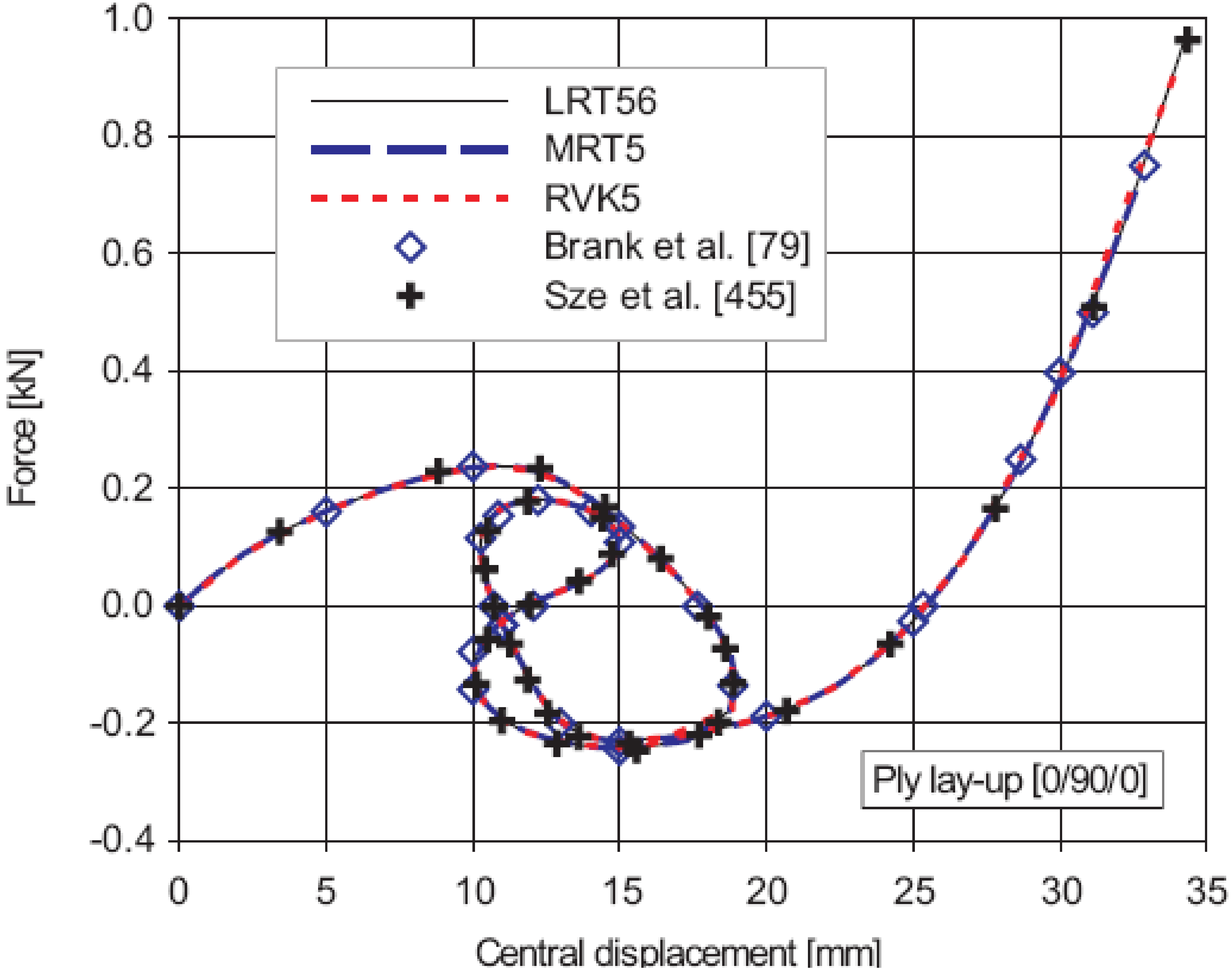
Hinged laminated cylindrical panels under point load



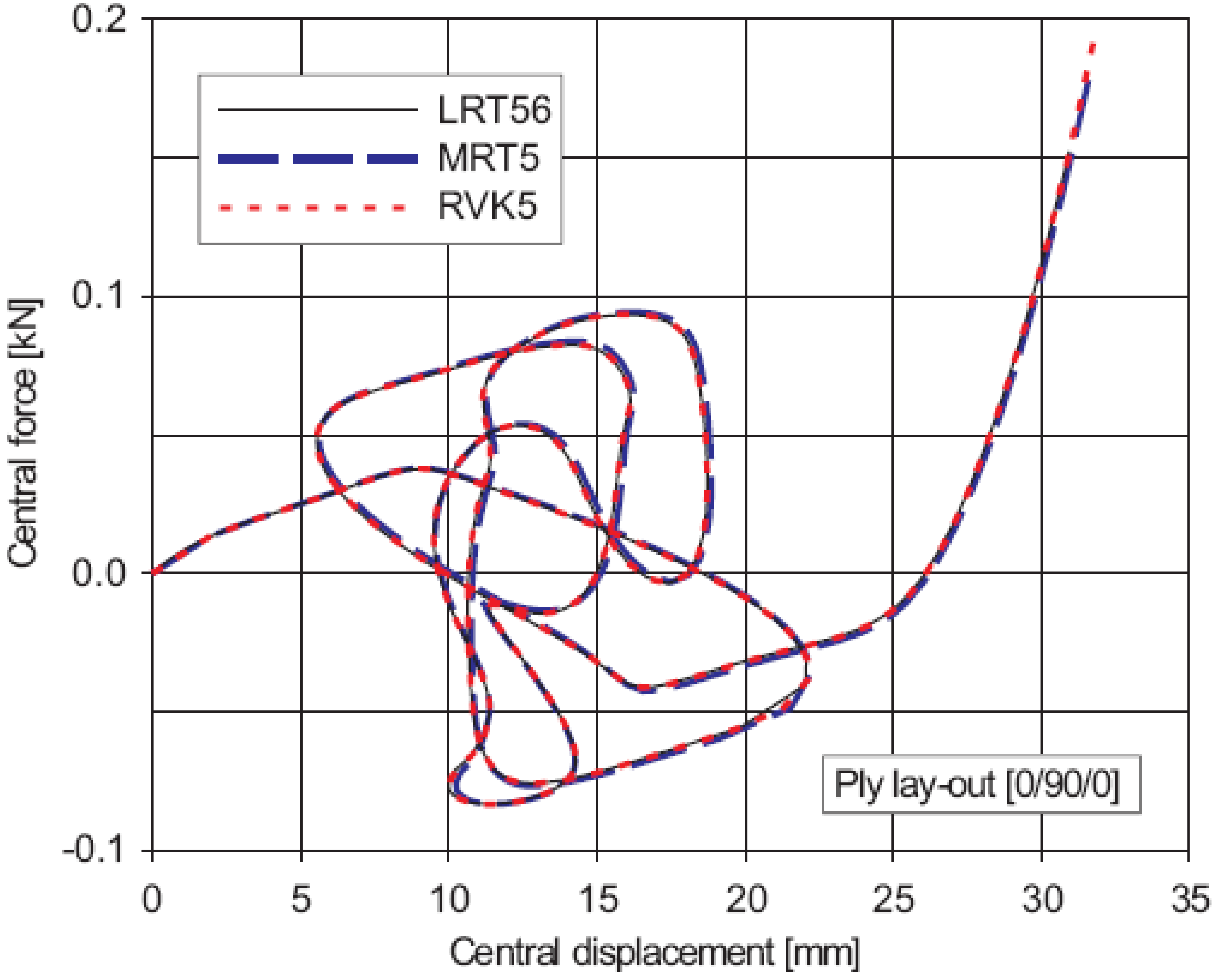
Central deflection for the cross ply laminate panel A (h=12.6 mm)



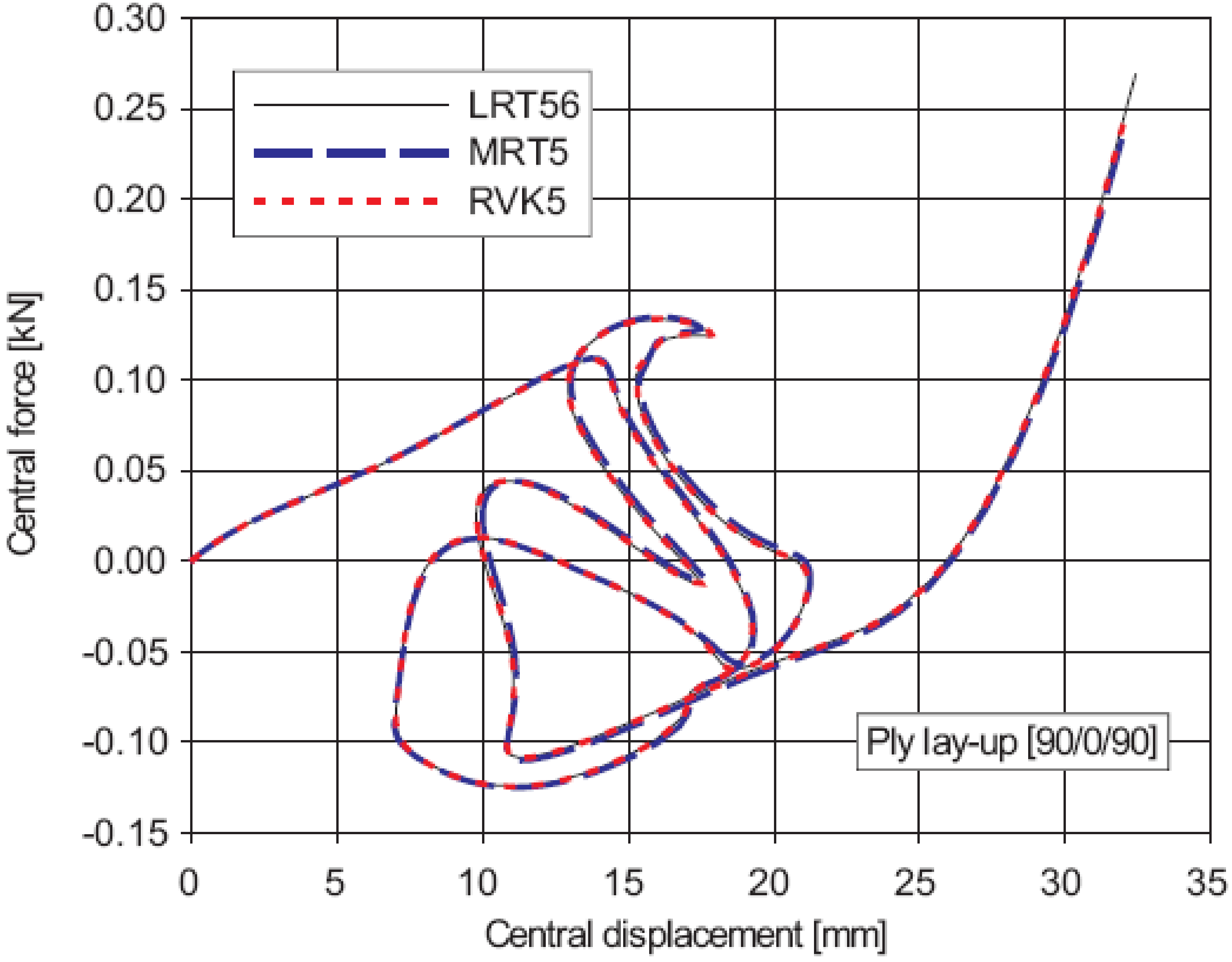
Central deflection for the cross ply [0/90/0] laminate panel B (h=6.3 mm)



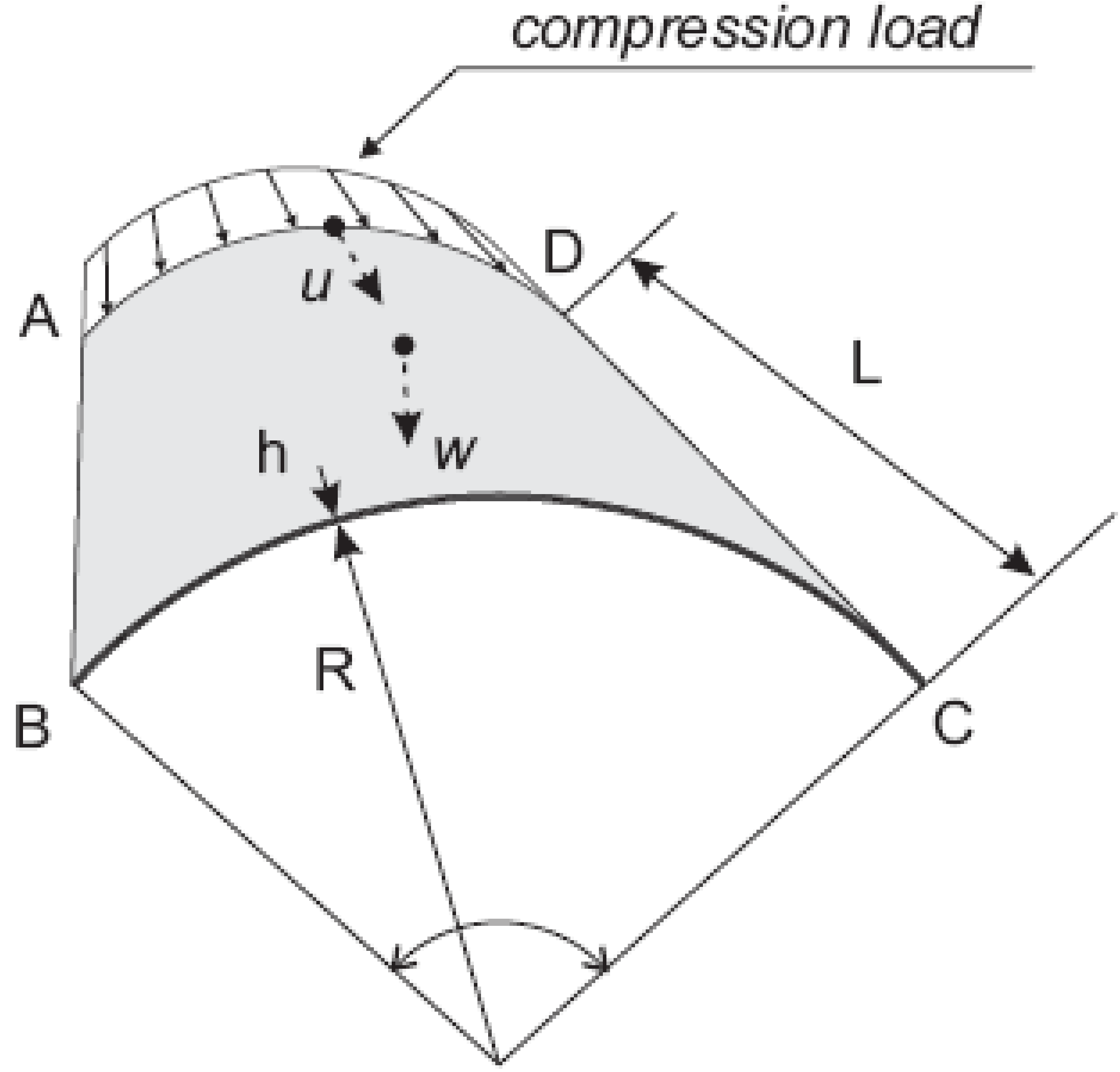
Central deflection for the laminate panel C (h=3.15 mm)

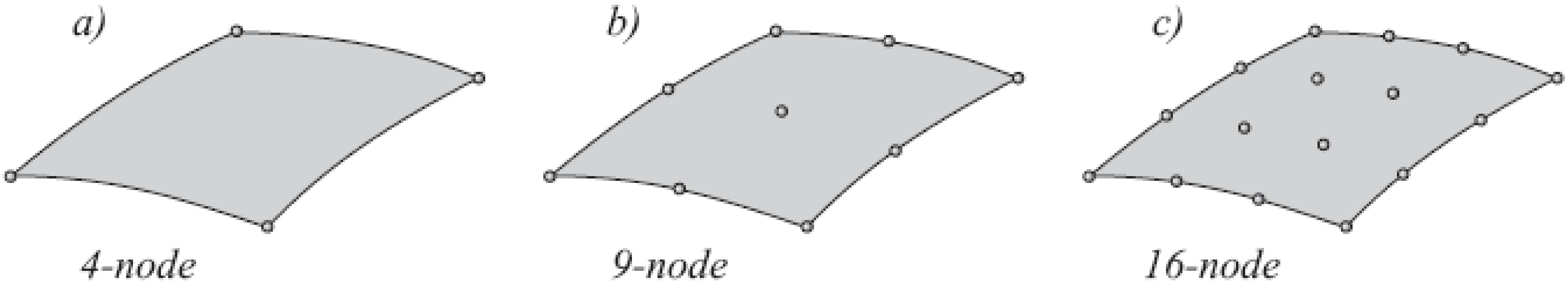


Central deflection for the laminate panel C (h=3.15 mm)

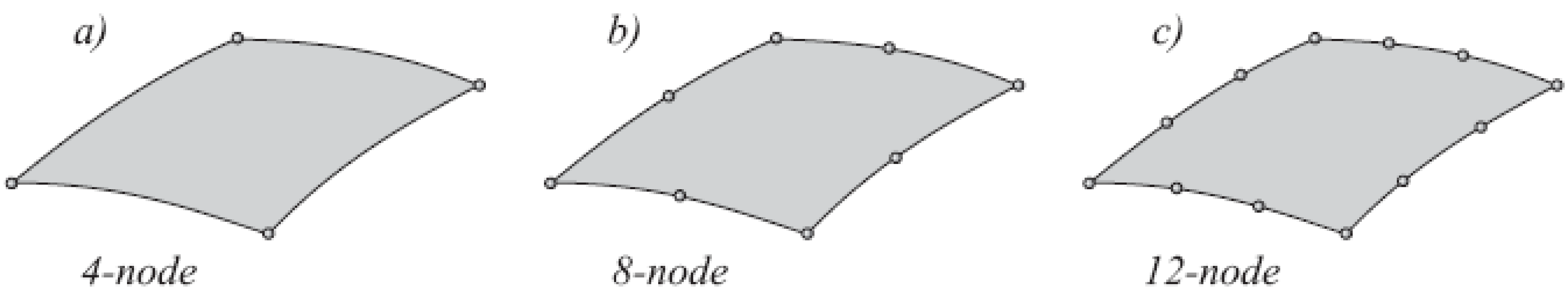


Axial compression of composite cylindrical panel





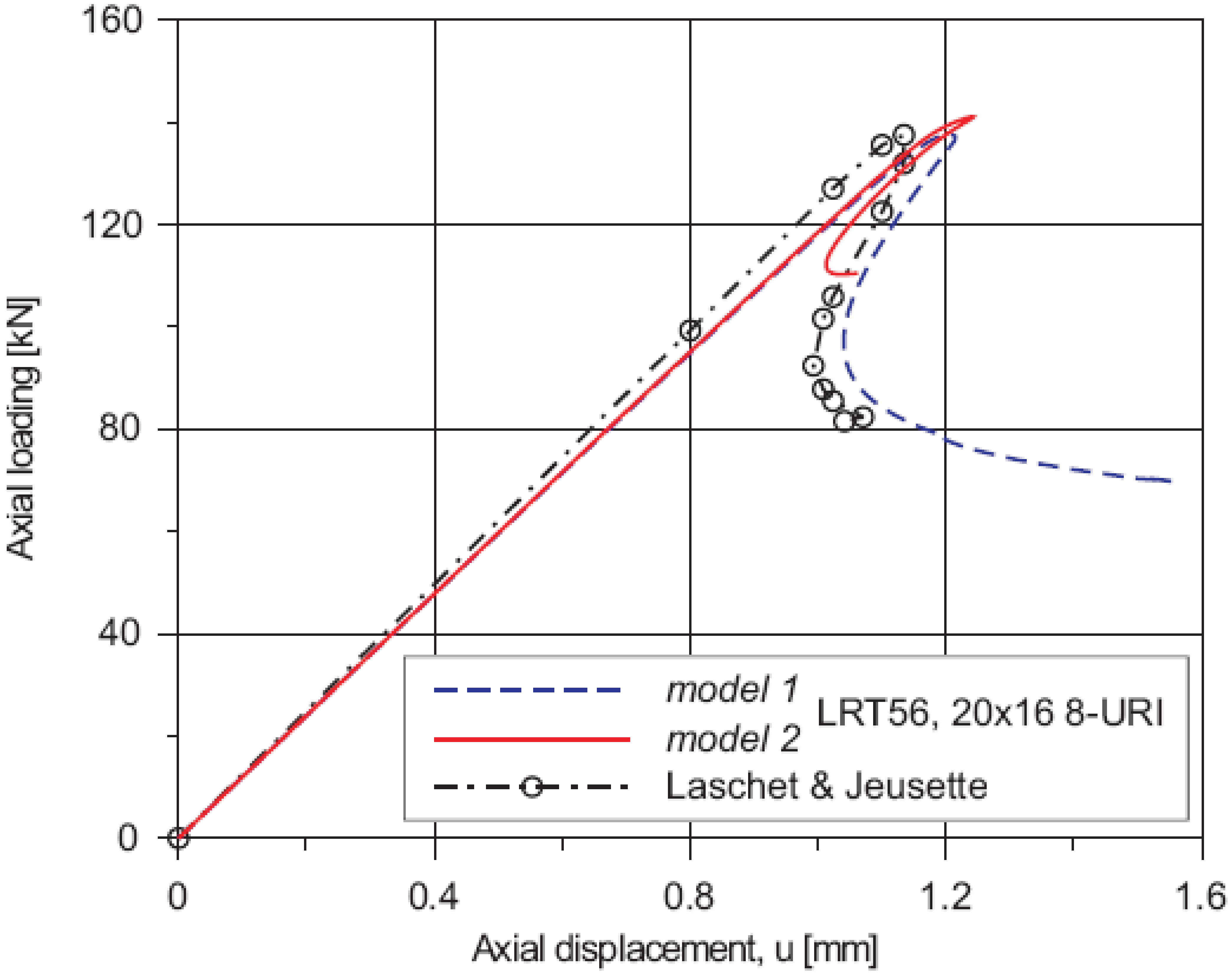
Lagrange family of shell elements: a) linear; b) quadratic; c) cubic interpolation

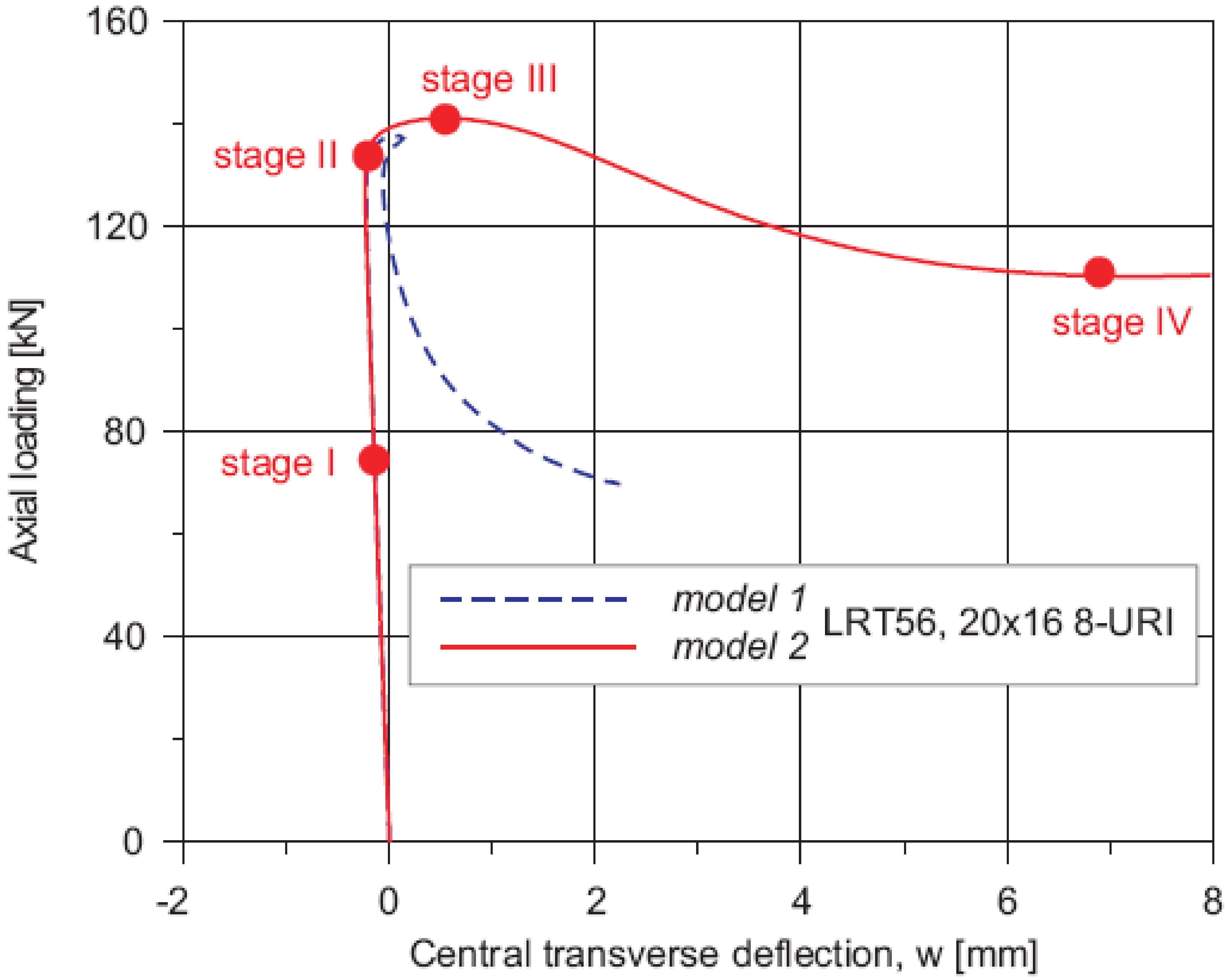


Serendipity family of shell elements: a) linear; b) quadratic; c) cubic interpolation

Model	Mesh	Critical load [kN]	
		Linear buckling	Incremental analysis
model 1	10×8	-	138.9
LRT56	16×8	-	137.8
8-URI elements	20×16	-	137.7
model 2	10×8	-	143.5
LRT56	16×8	-	143.3
8-URI elements	20×16	-	140.9
8-node elements Jun & Hong	8×10	-	143.2
16-node elements Laschet & Jeusette	8×10	143.9	137.8
	12×18	140.3	-

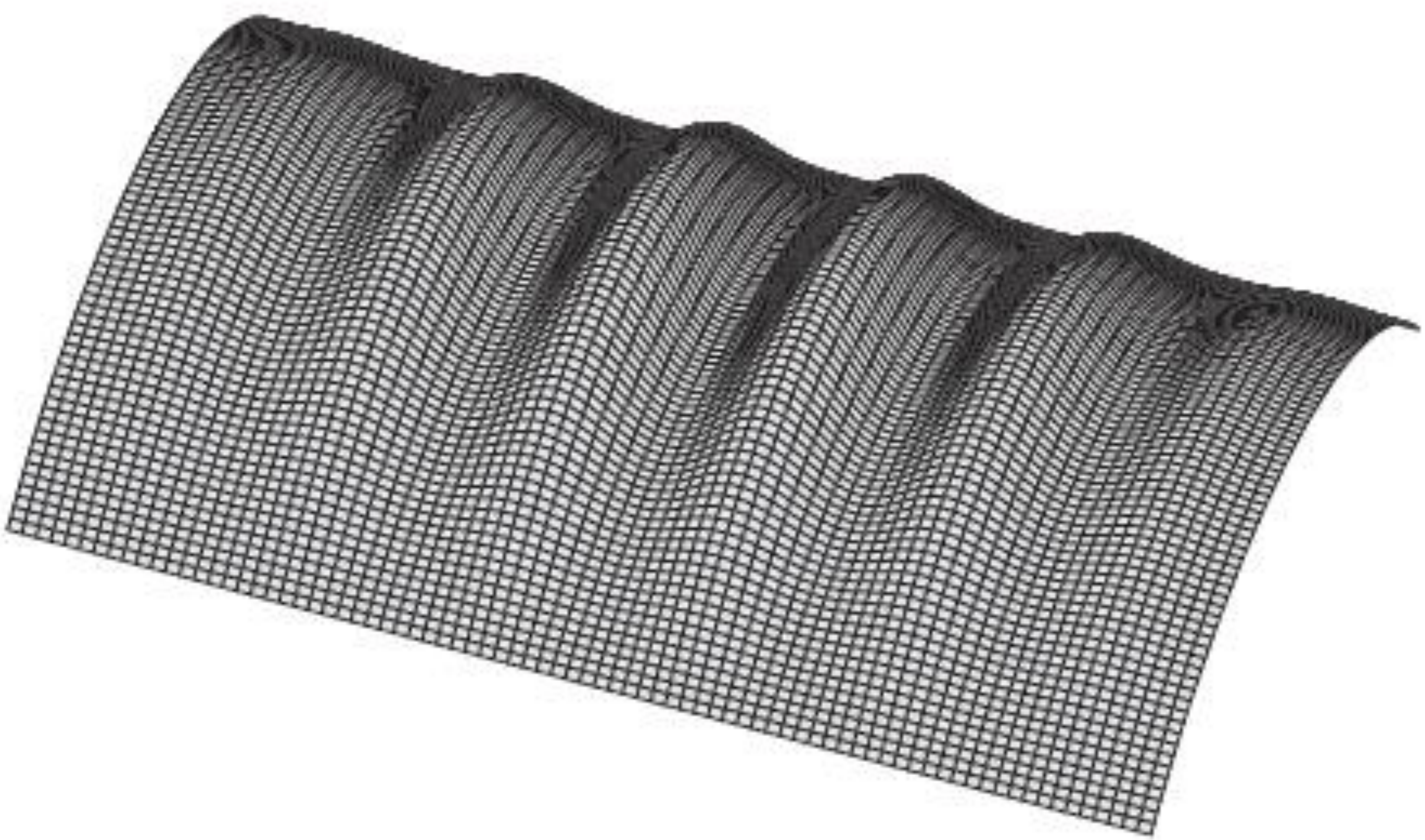
4-node elements Wagner	4×12	145.6	-
	4×16	142.2	-
	4×20	140.8	-
	4×40	140.0	-
	4×80	139.6	-
4-node elements Brank & Carrera	32×32	-	150
MSC Nastran QUAD4 elements	20×20	144.56	144.35
	40×40	141.56	142.34
Krej[illegible]]	80×80	140.34	140.38
Snell & Morley	Experiment	134	


160
120
80
40
0
Axial loading [kN]
0
0.4
0.8
1.2
1.6
Axial displacement, u [mm]
model 1
model 2
Laschet & Jeusette
LRT56, 20x16 8-URI

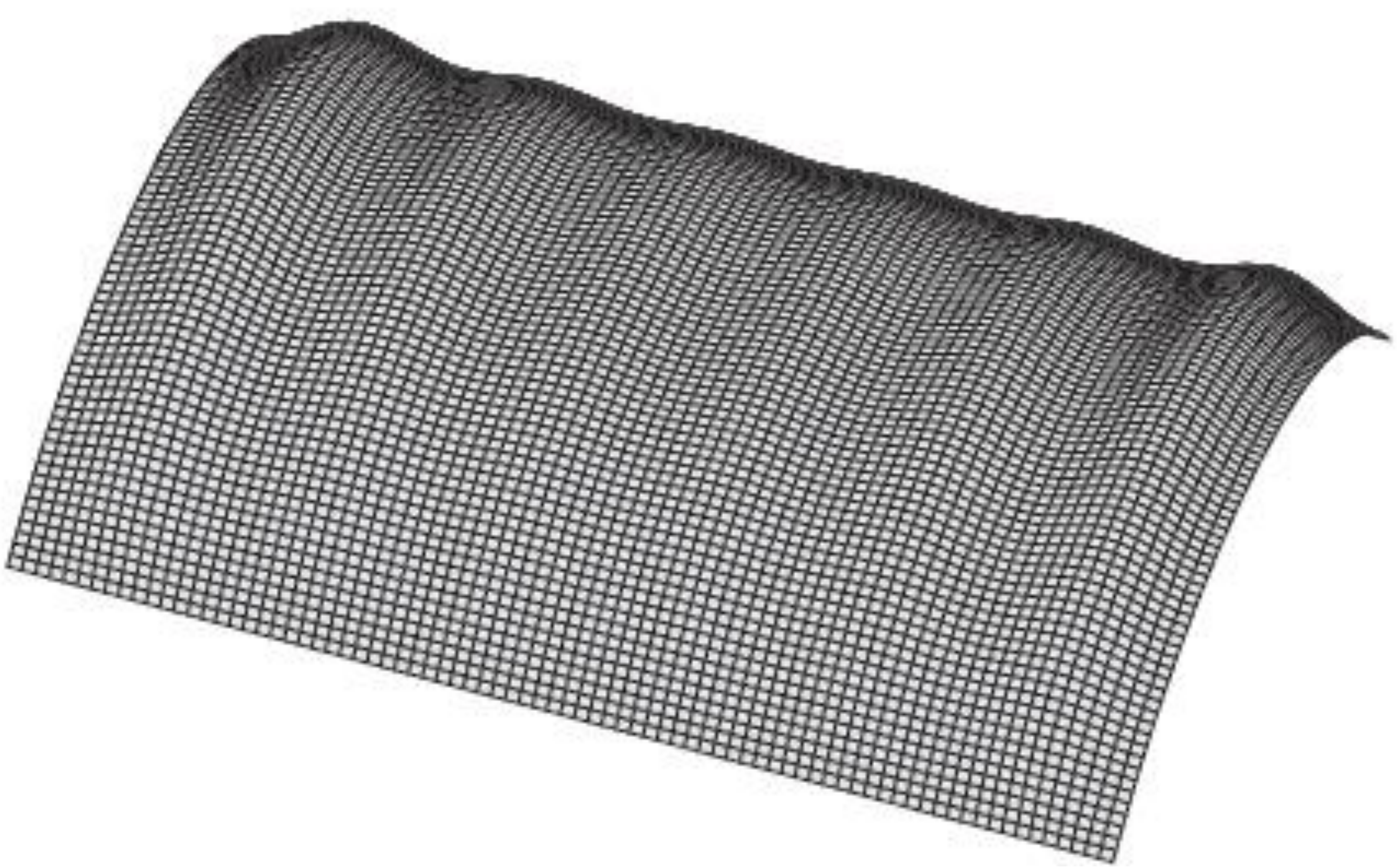


160
120
80
40
0
stage III
stage II
stage I
stage IV
Axial loading [kN]
model 1
model 2
LRT56, 20x16 8-URI
-2
0
2
4
6
8
Central transverse deflection, w [mm]

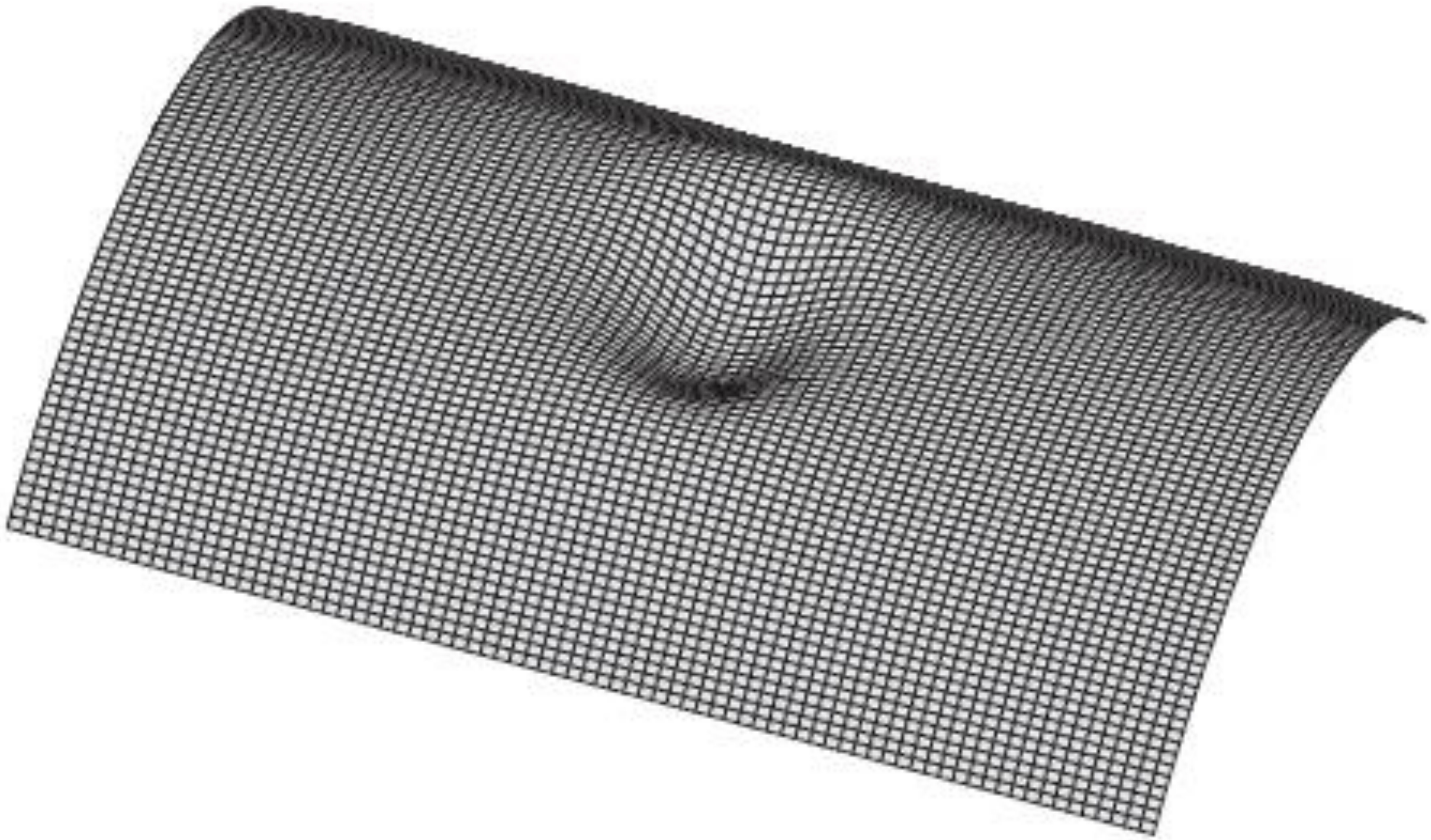
The first buckling mode (MSC Nastran 80X80 QUAD4 elements, Pcrit = 140.34 kN)



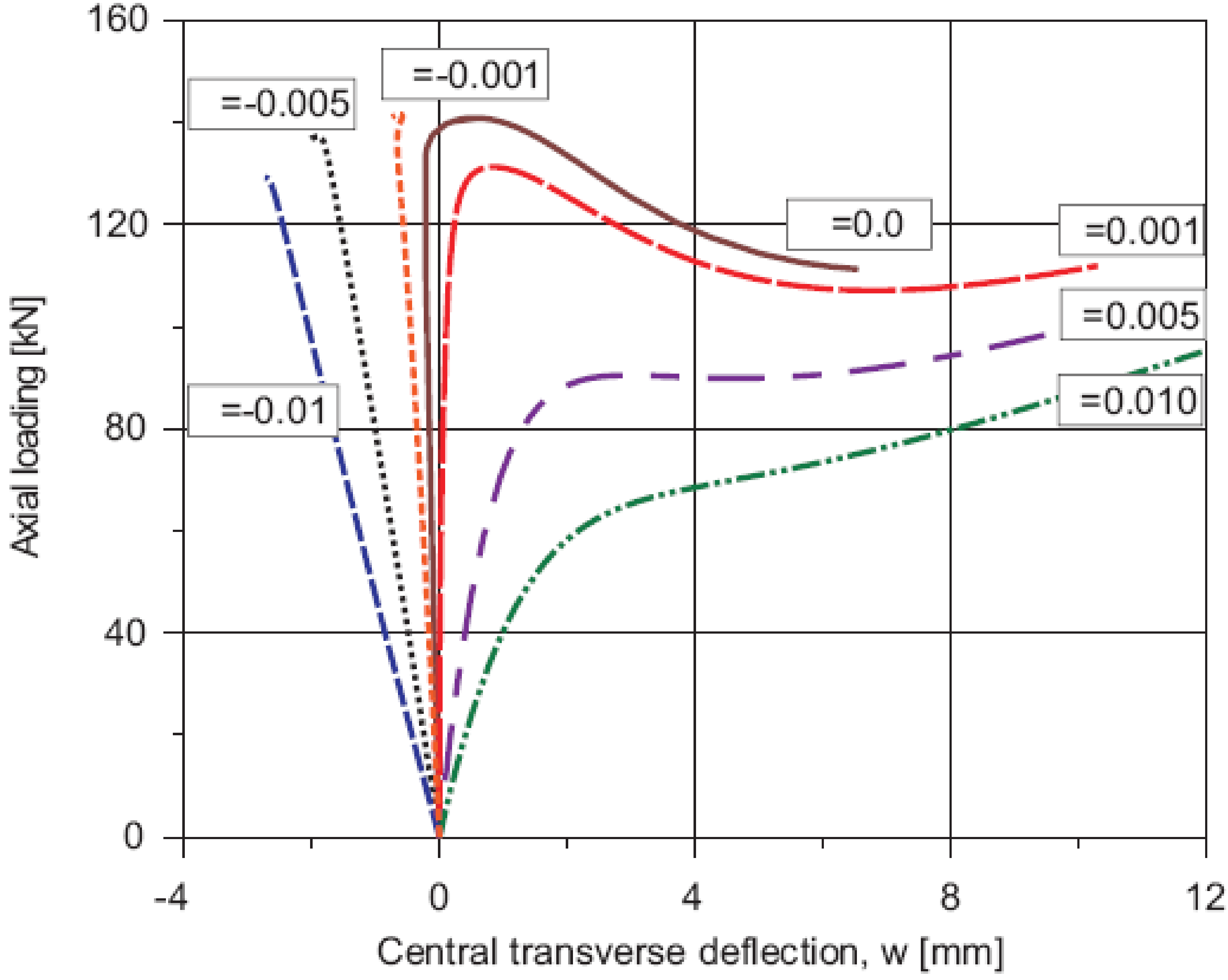
Pre-buckling deformation at P=136.60 kN (MSC Nastran 80x80 QUAD4 elements)



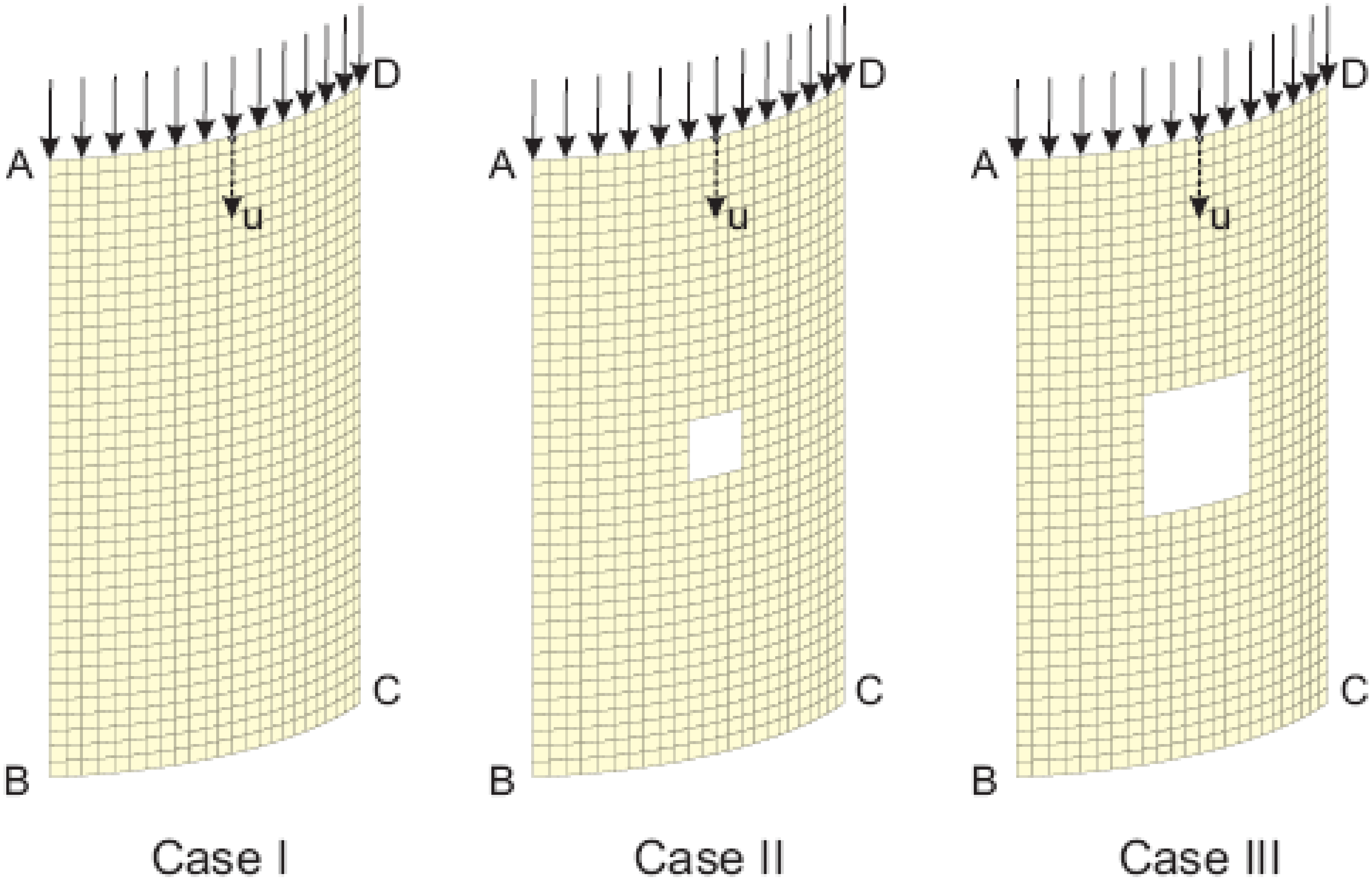
Post-buckling deformation at P=110.26 kN (MSC Nastran 80x80 QUAD4 elements)



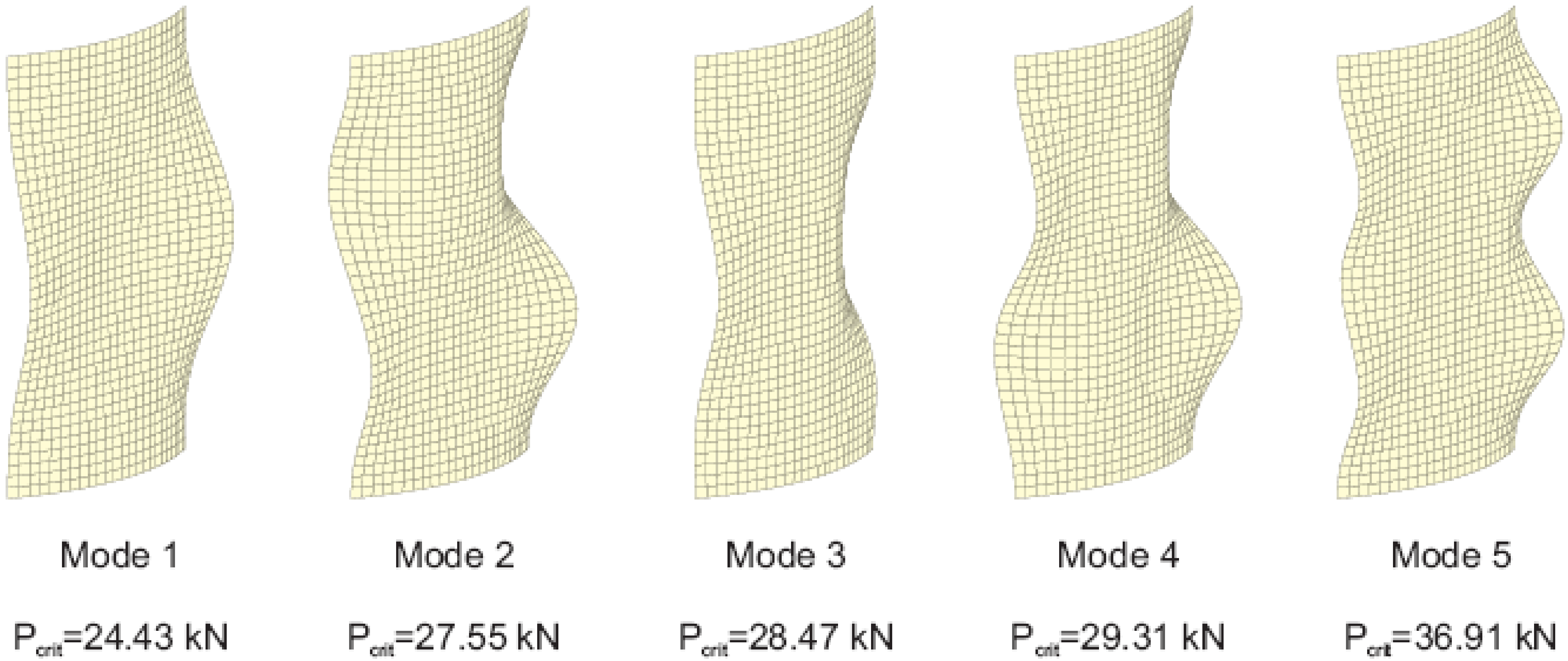
Imperfection sensitivity of the layered panel buckling

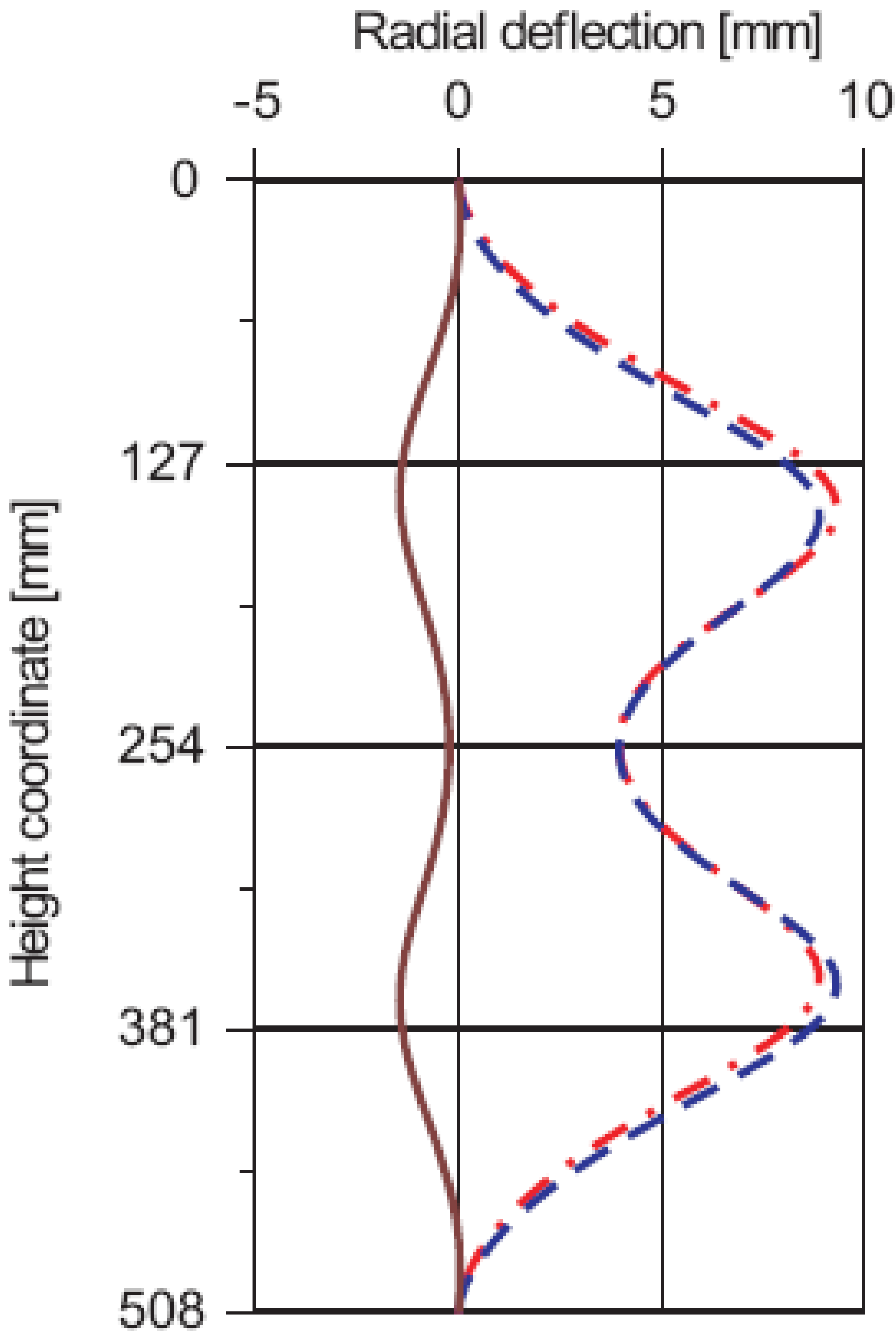


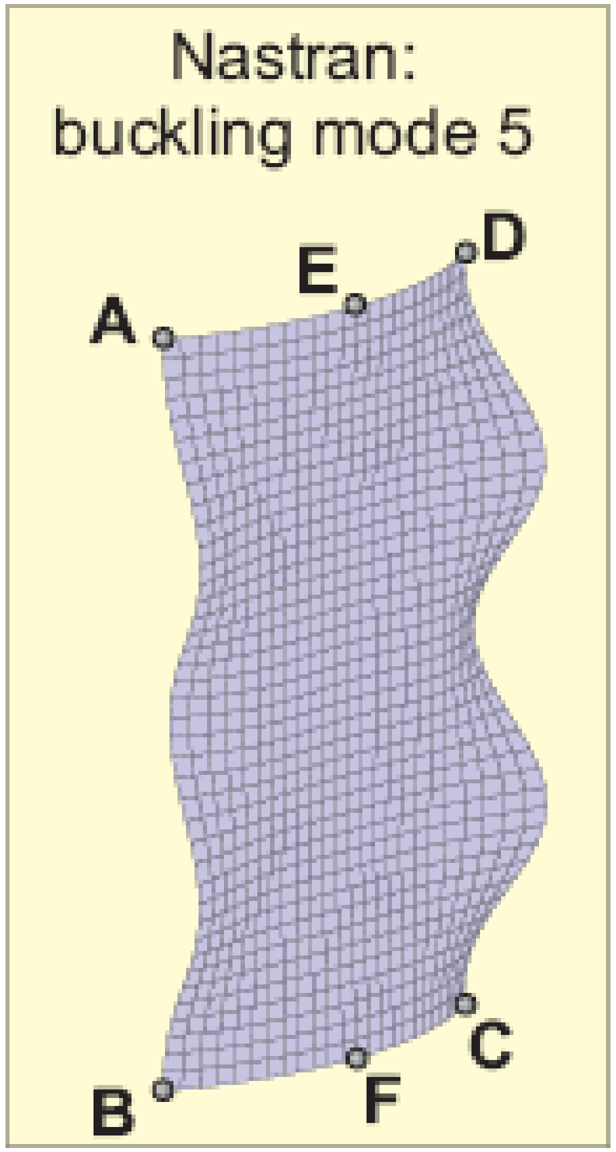
Buckling of composite cylindrical panels with square cut-outs



Linear buckling modes obtained for Case I with model 24X40 QUAD4 elements

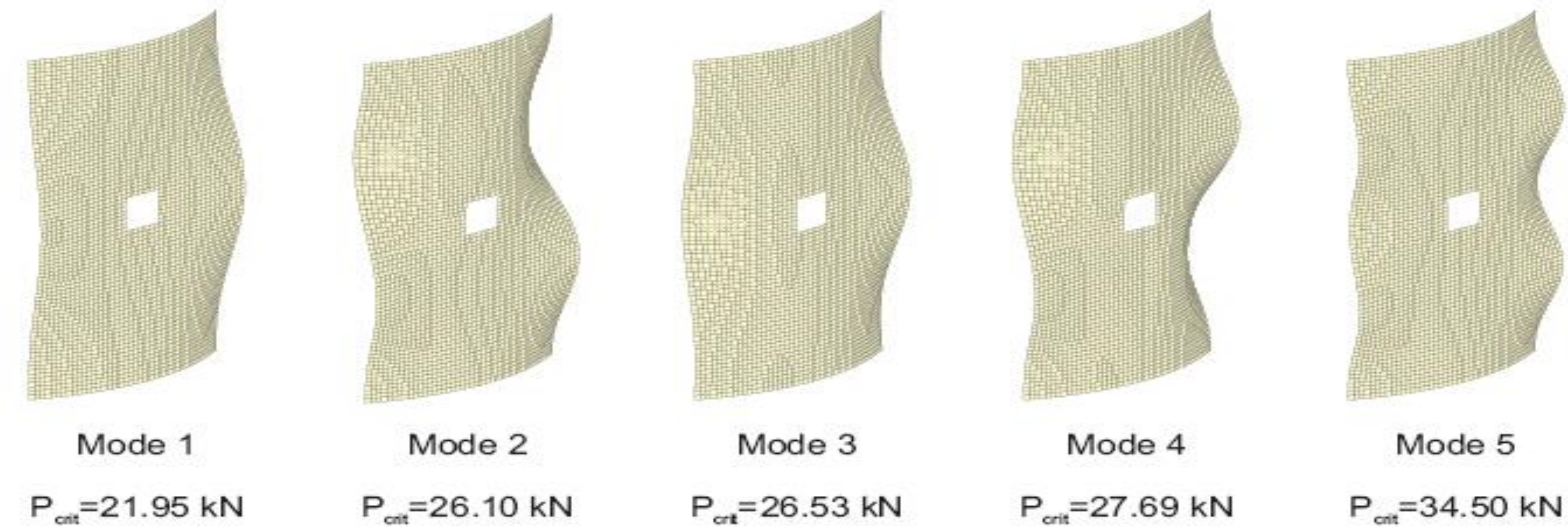


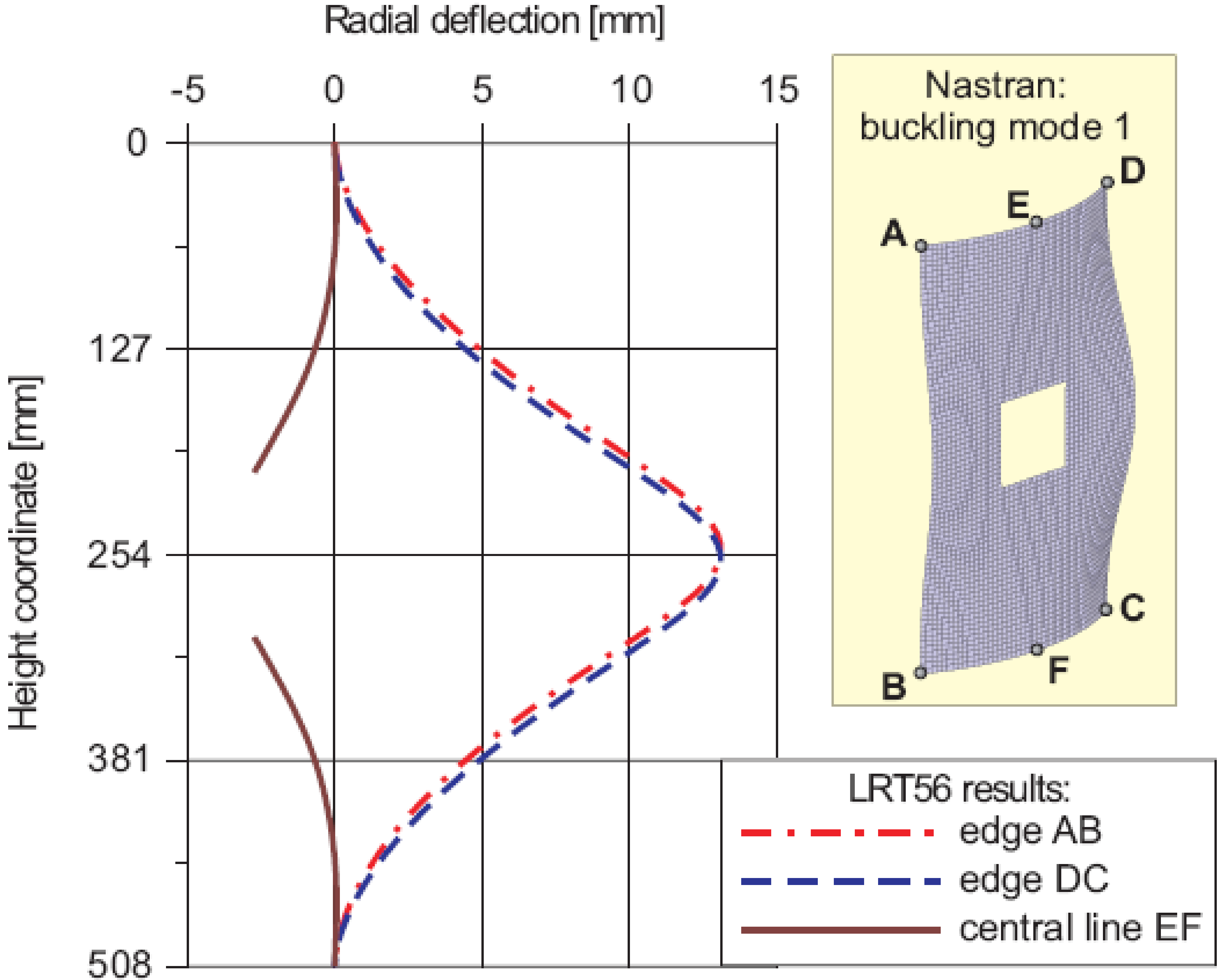

Radial deflection [mm]
-5
0
5
10
0
127
254
381
508
Height coordinate [mm]
LRT56 results:
edge AB
edge DC
central line EF

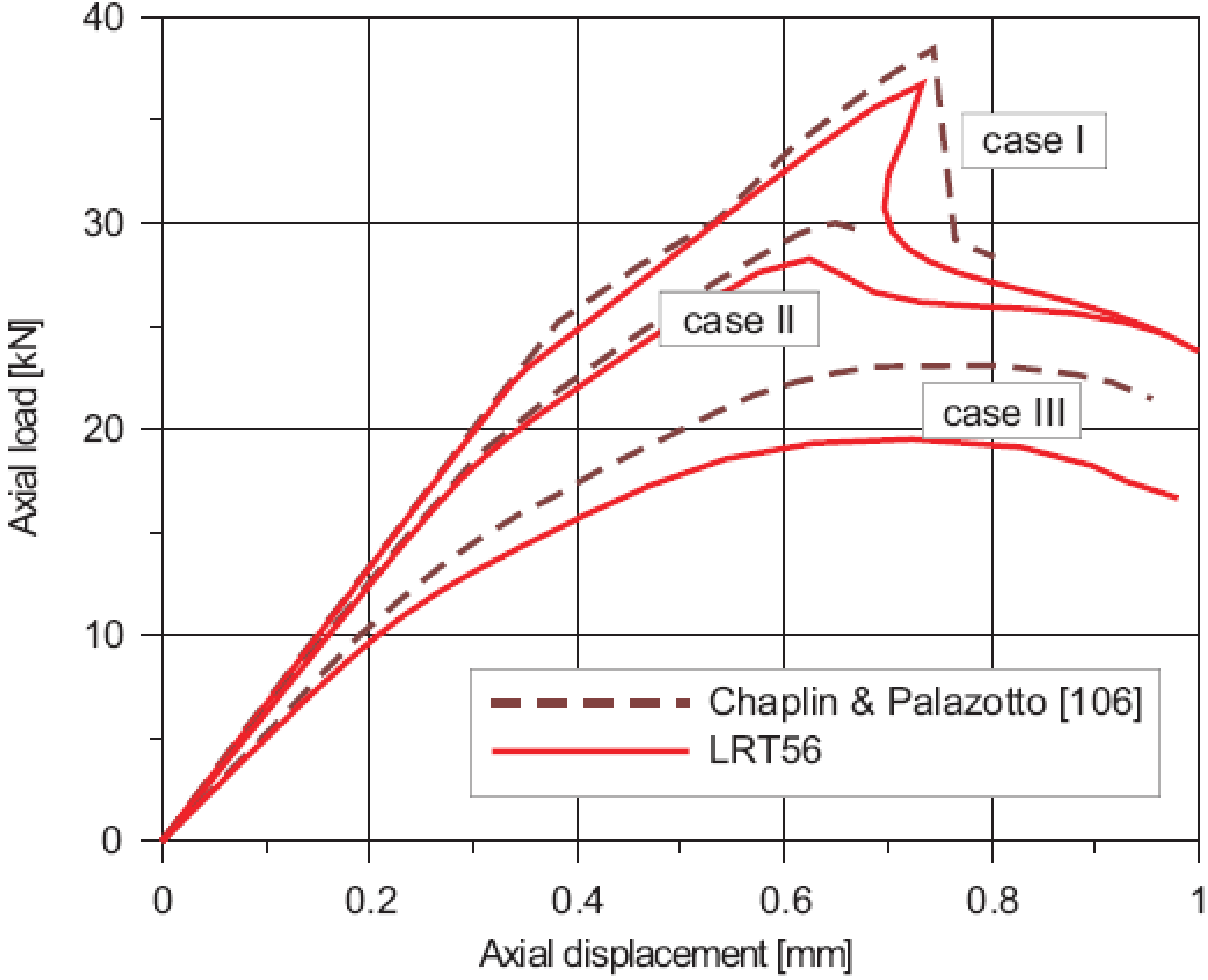

Nastran:
buckling mode 5
A
E
D
B
F
C

Linear buckling modes obtained for Case II with 37X76

QUAD4 elements




Radial deflection [mm]
-5
0
5
10
15
Height coordinate [mm]
0
127
254
381
508
Nastran:
buckling mode 1
A
B
C
D
E
F
LRT56 results:
edge AB
edge DC
central line EF



40
30
20
10
0
Axial load [kN]
0
0.2
0.4
0.6
0.8
1
Axial displacement [mm]
case I
case II
case III
Chaplin & Palazotto [106]
LRT56