



# Structural Stability and Limit Analysis of Structures (Instabilità delle strutture e calcolo a rottura)

> **Lezione 18**

Snap-through buckling of von Mises truss  
(lo snap-through dell'arco a tre cerniere)

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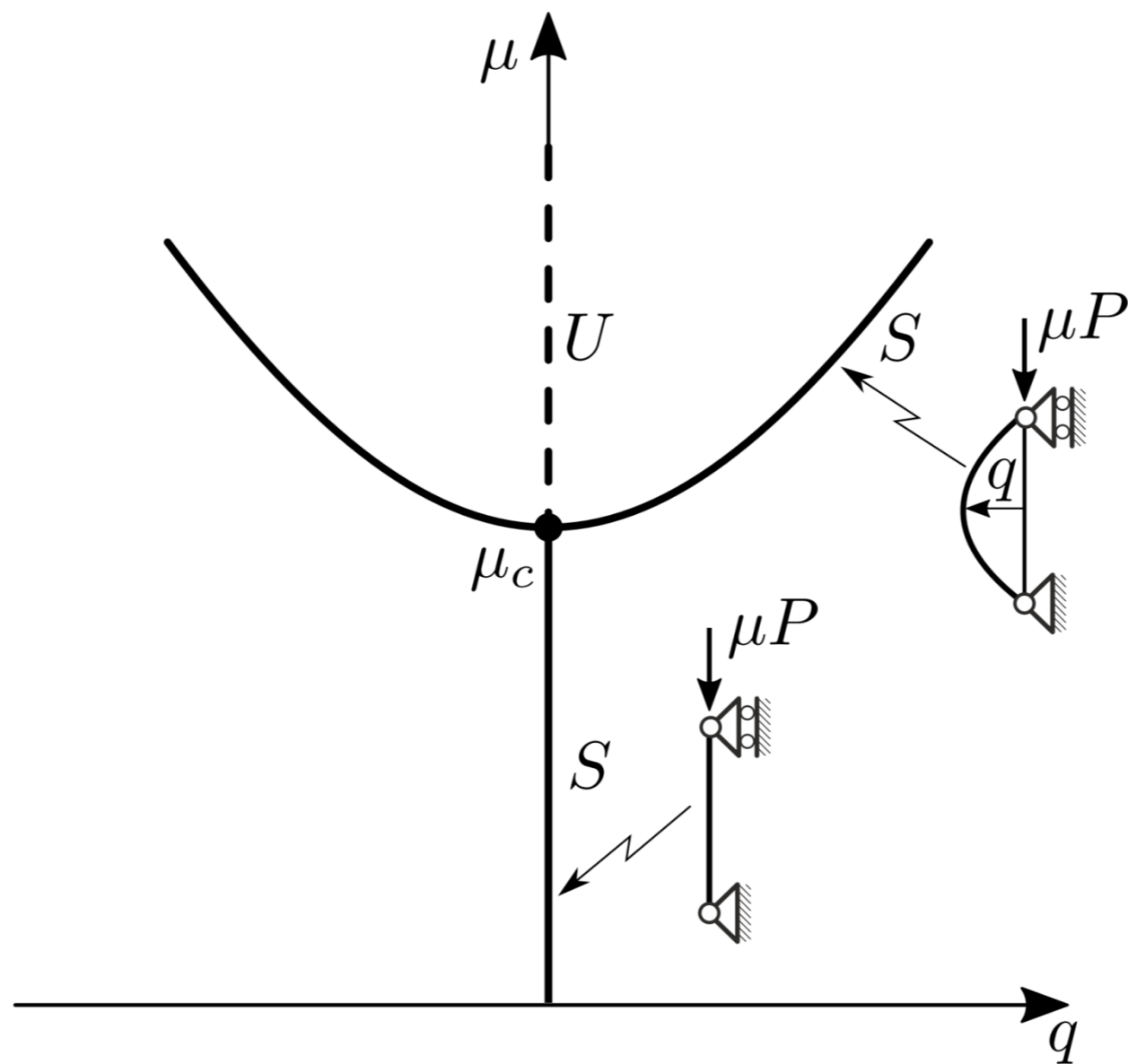


Diagramma di biforcazione della trave di Eulero;  $S$  stabile,  $U$  instabile.

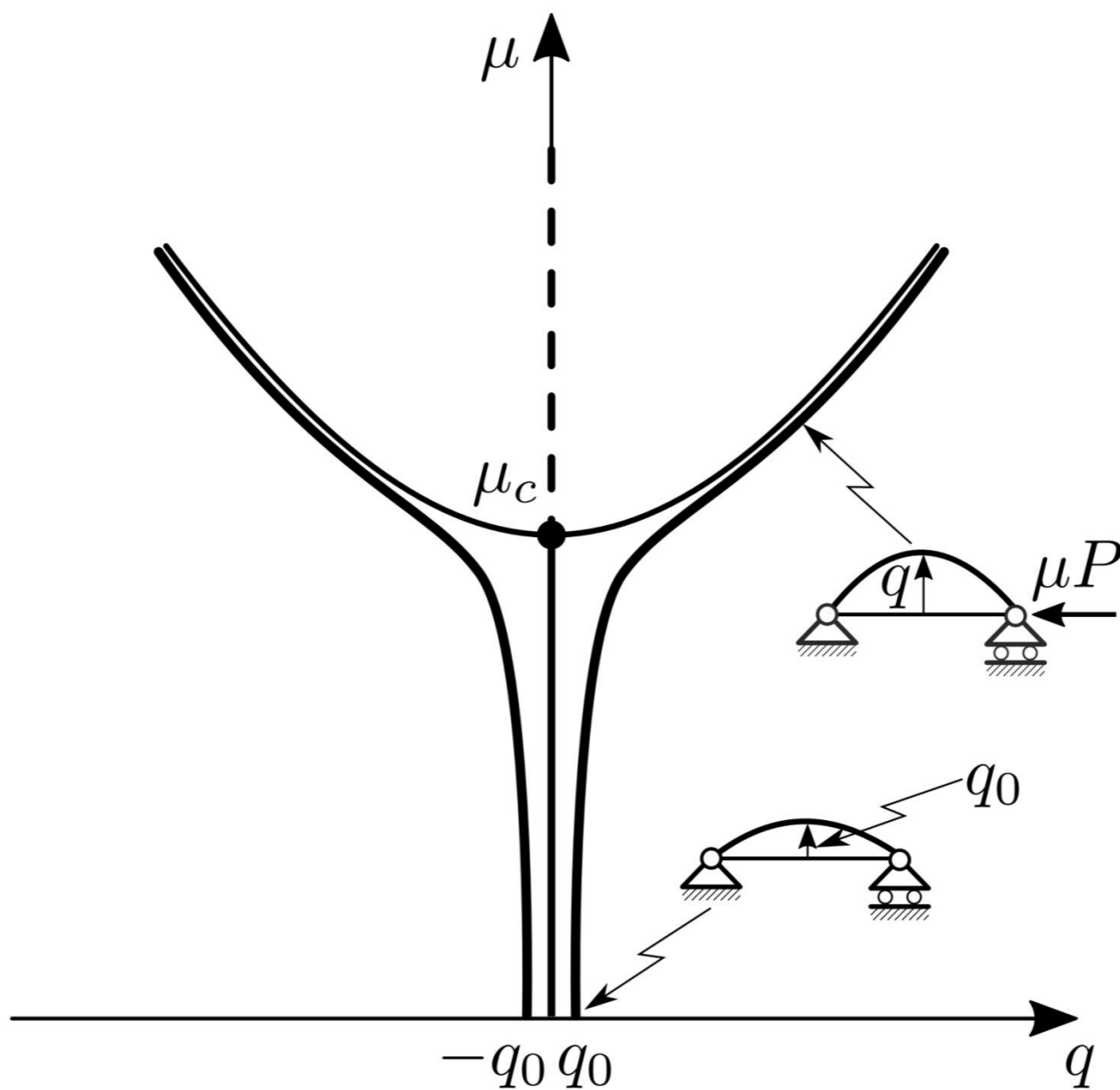
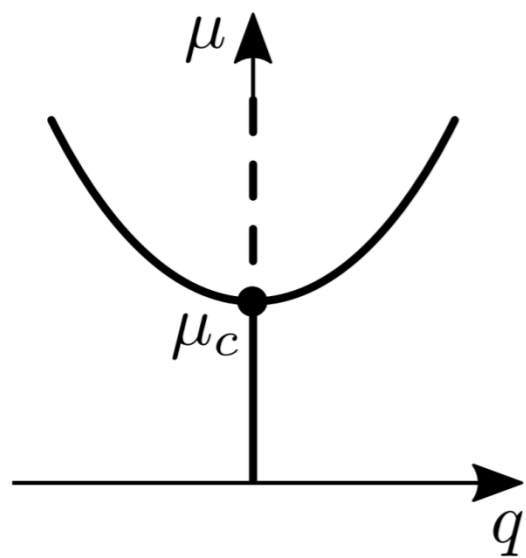
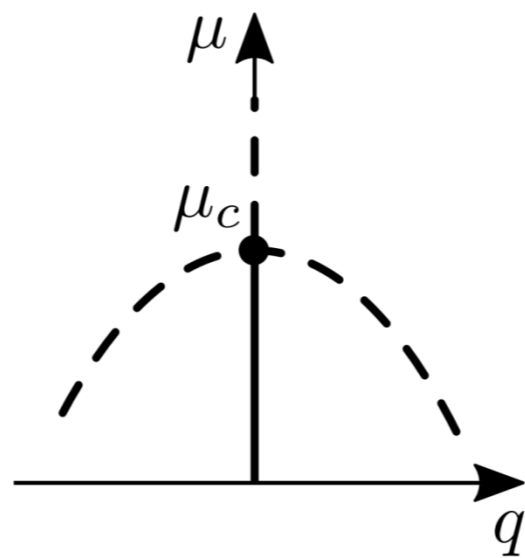


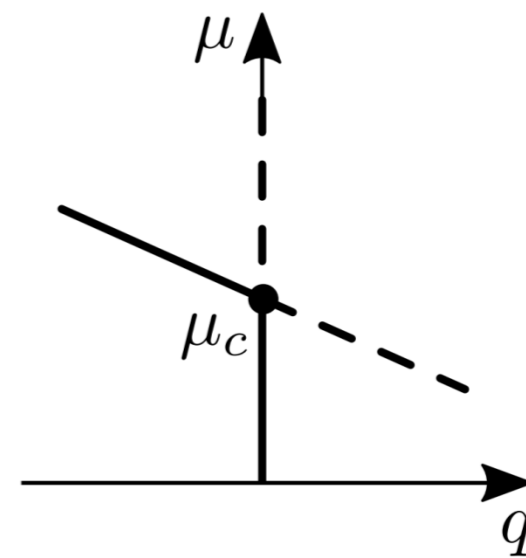
Diagramma di biforcazione della trave di Eulero imperfetta.



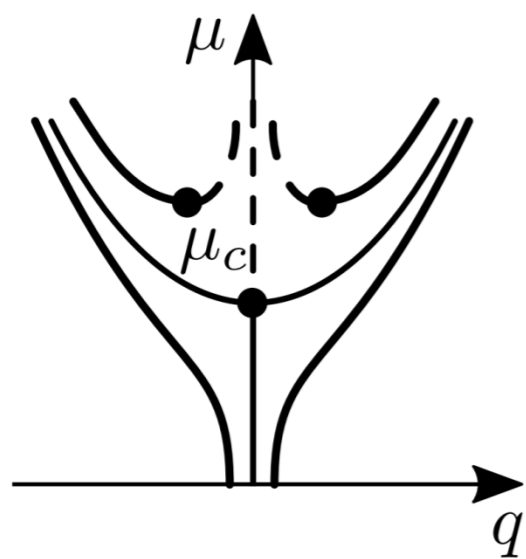
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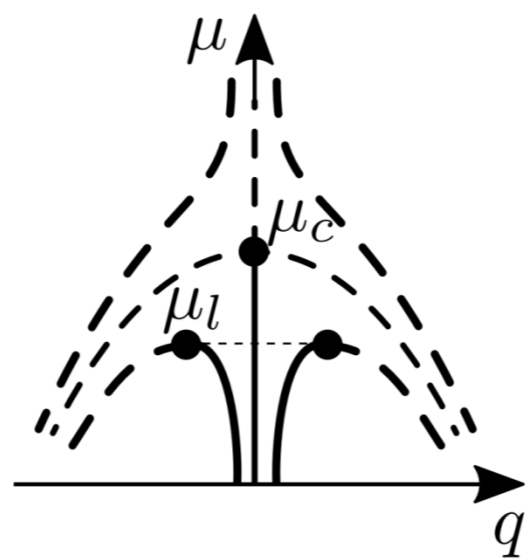
(b)



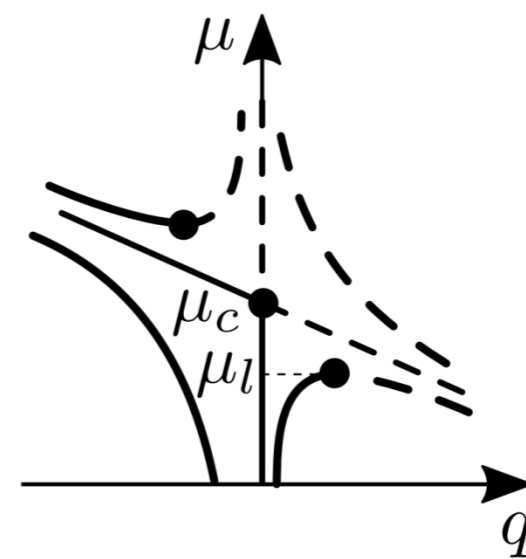
(c)



(d)



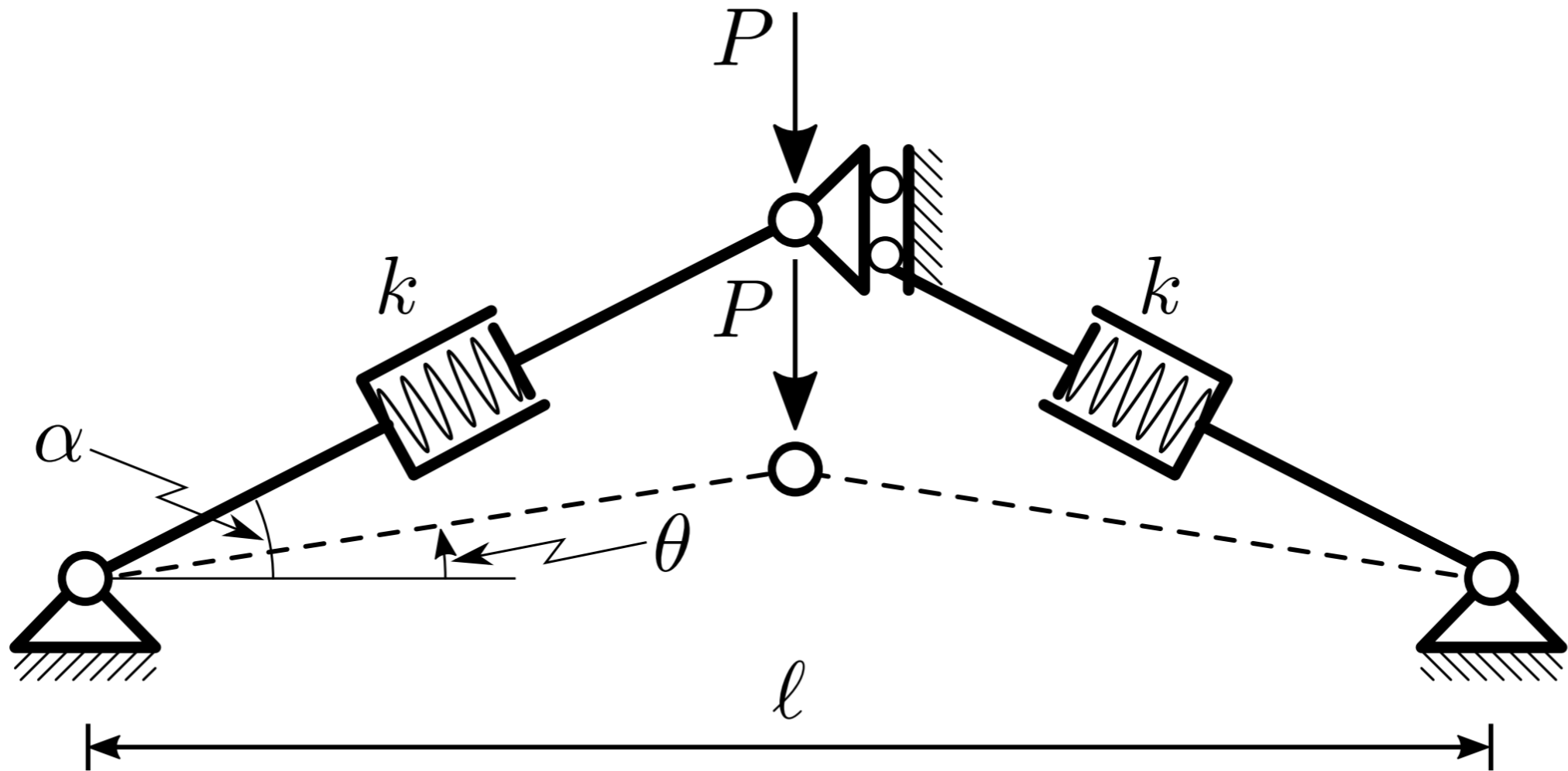
(e)



(f)

**Figura** : Biforcazioni statiche: (a) forza stabile; (b) forza instabile; (c) transcritica; (d) forza stabile imperfetta; (e) forza instabile imperfetta; (e) transcritica imperfetta.

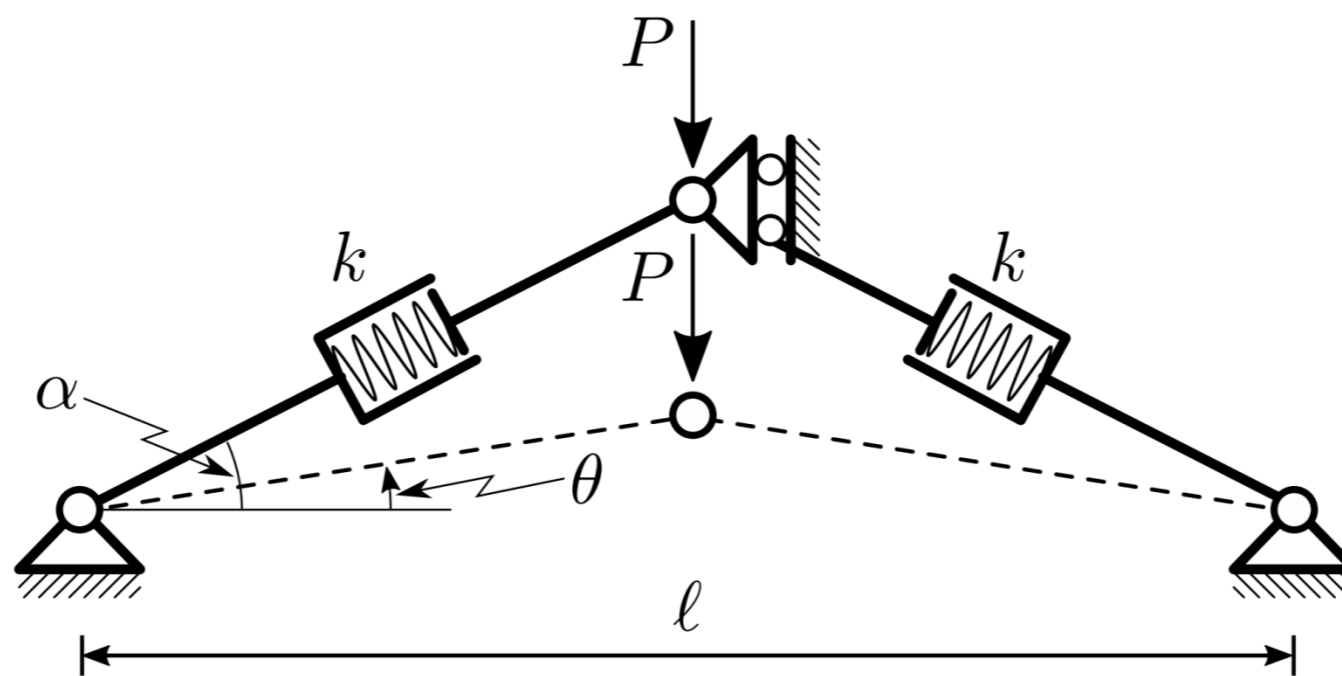
# von Mises truss





UNIVERSITÀ  
DI PADOVA

Si considera un esempio paradigmatico di struttura perfetta, che subisce instabilità per raggiungimento di un carico limite (*biforcazione a piega*). Questo è rappresentato dall'*arco a tre cerniere ribassato*, simmetrico e costretto a deformarsi simmetricamente (Fig. ) L'arco è costituito da due aste identiche, flessionalmente rigide, ma assialmente elastiche, di rigidezza  $k$ , inizialmente inclinate sull'orizzontale di un angolo  $\alpha$ , soggetto ad una forza verticale di intensità  $P$ , che induce compressione. Si assume quale parametro lagrangiano l'angolo  $\theta$  che, nella configurazione variata, le aste formano con l'orizzontale



**Figura** Arco a tre cerniere simmetrico, ribassato, vincolato a deformarsi simmetricamente.

# L'analisi esatta

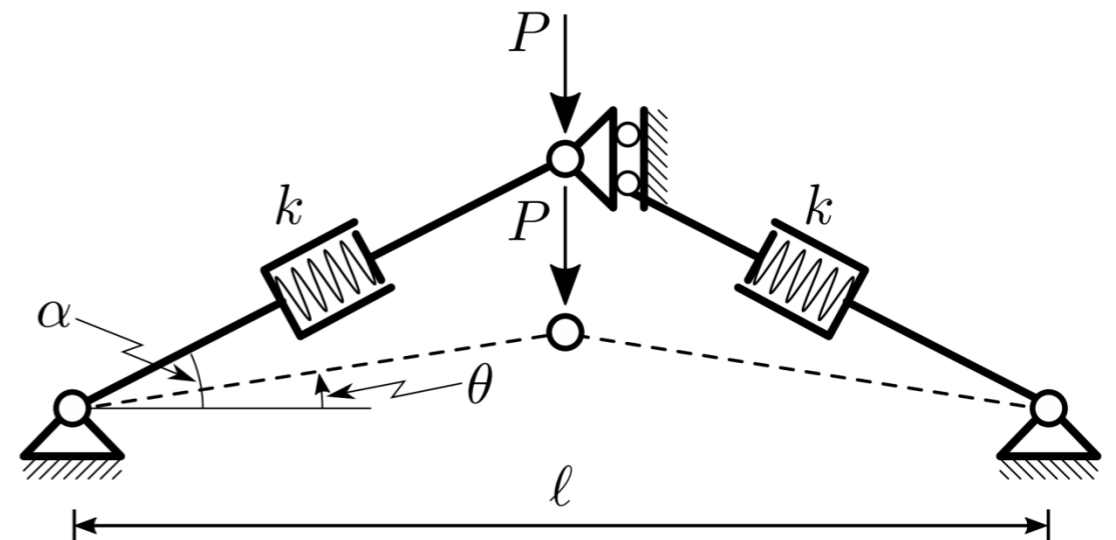
Per questo sistema semplice è possibile condurre un'analisi esatta. L'energia potenziale totale del sistema si scrive:

$$\Pi(\theta) = 2 \frac{1}{2} k \Delta \ell^2 - P \Delta$$

in cui  $\Delta \ell$  è l'allungamento di una singola asta, e  $\Delta$  lo spostamento della chiave dell'arco, positivo verso il basso:

$$\Delta \ell = \frac{\ell}{2} \left( \frac{1}{\cos \theta} - \frac{1}{\cos \alpha} \right)$$

$$\Delta = \frac{\ell}{2} (\tan \alpha - \tan \theta)$$



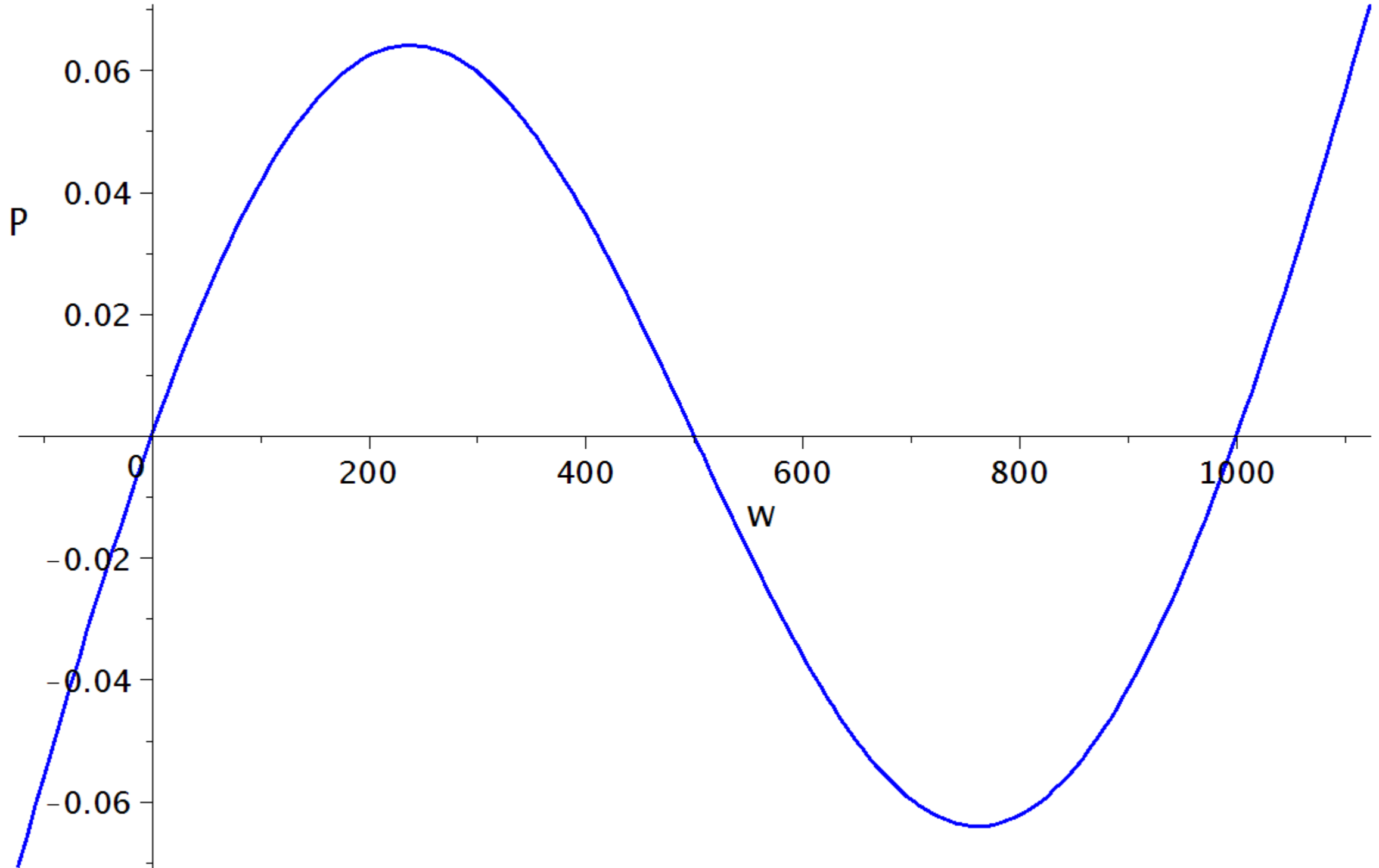


Dalla stazionarietà  $\frac{d\Pi}{d\theta} = 0$  si ricava l'equazione di equilibrio:

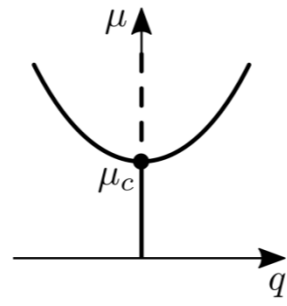
$$\frac{1}{2}\ell \frac{1}{\cos \theta} \left[ k\ell \tan \theta \left( \frac{1}{\cos \theta} - \frac{1}{\cos \alpha} \right) + P \frac{1}{\cos \theta} \right] = 0$$

la cui soluzione fornisce un unico percorso

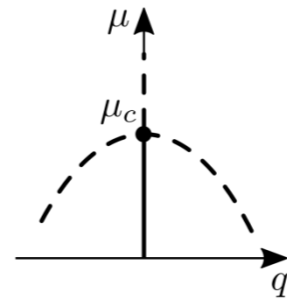
$$P = k\ell \sin \theta \left( \frac{1}{\cos \alpha} - \frac{1}{\cos \theta} \right)$$



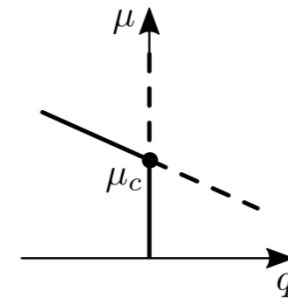
**$\alpha = \pi/4, \quad \mathbf{w}=\Delta l, \quad P = P/k$**



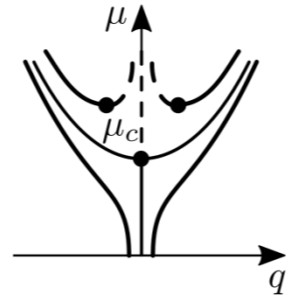
(a)



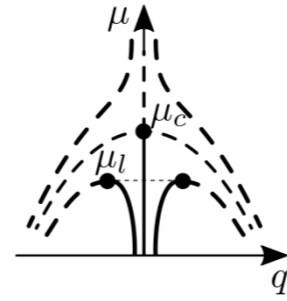
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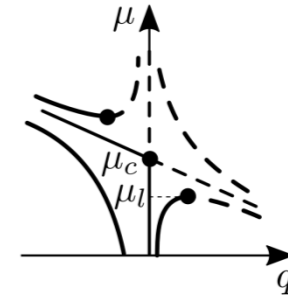
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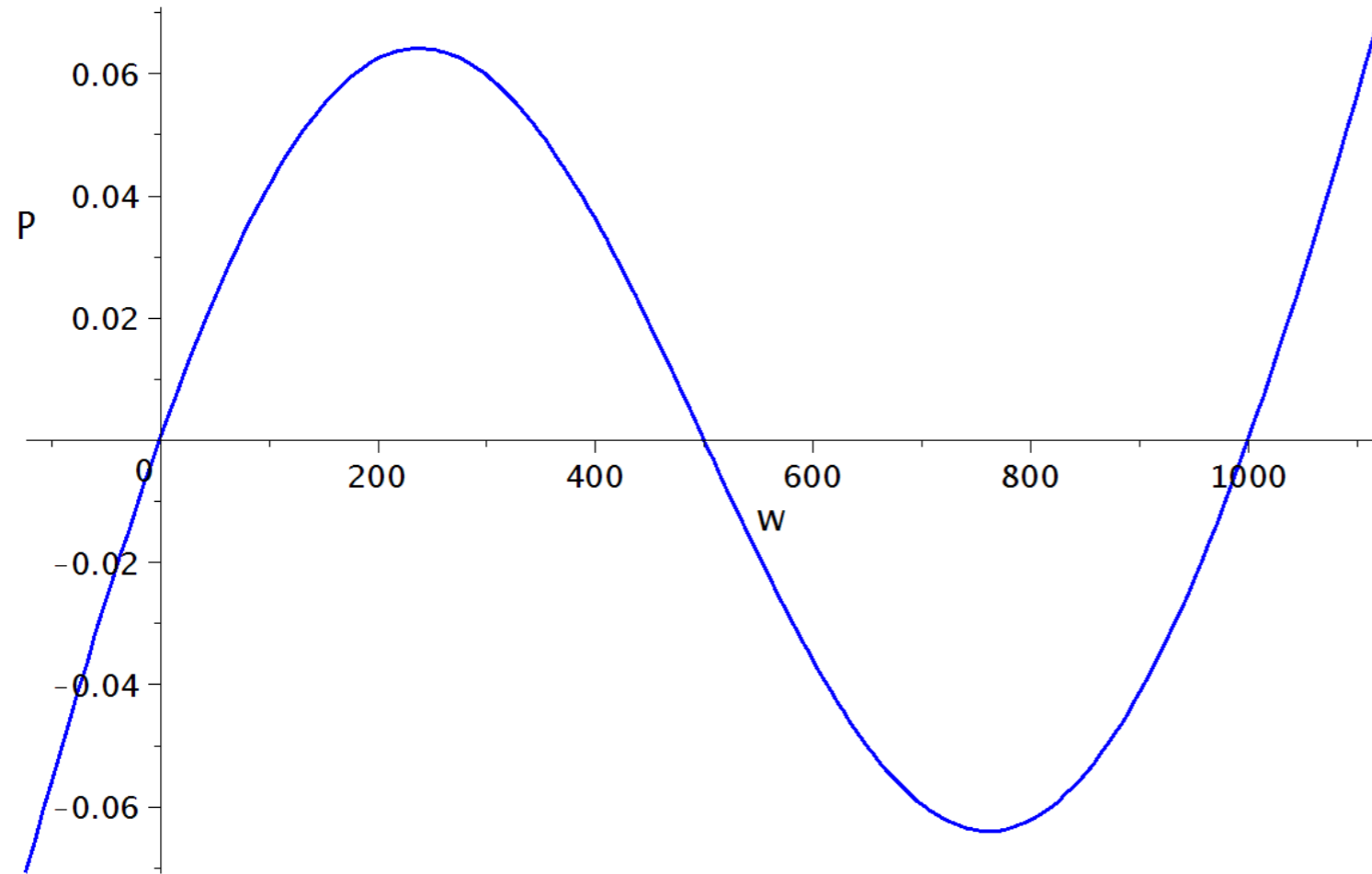
(d)



(e)



(f)



Per analizzare la stabilità dei diversi rami del percorso, si valuta la derivata seconda dell'energia potenziale:

$$\frac{d^2 \Pi}{d\theta^2} = \frac{1}{4} \ell \frac{1}{\cos^2 \alpha} \left[ k\ell \left( \frac{\cos(2\theta) - 3}{\cos \alpha \cos \theta} + 6 \frac{1}{\cos^2 \alpha} - 4 \right) + 4P \tan \theta \right]$$

che, espresso  $\cos \alpha$  dal percorso di equilibrio diventa:

$$\frac{d^2 \Pi}{d\theta^2} = \frac{k\ell^2}{\sin(2\theta)} \left( \tan^3 \theta - \frac{P}{k\ell} \right)$$



Per determinare i punti limite si impone  $\frac{dP}{d\theta} = 0$ , che implica:

$$\cos^3 \theta = \cos \alpha$$

Questa fornisce due radici uguali ed opposte,  $\theta = \theta_l^\pm$ , associate ai carichi limite:

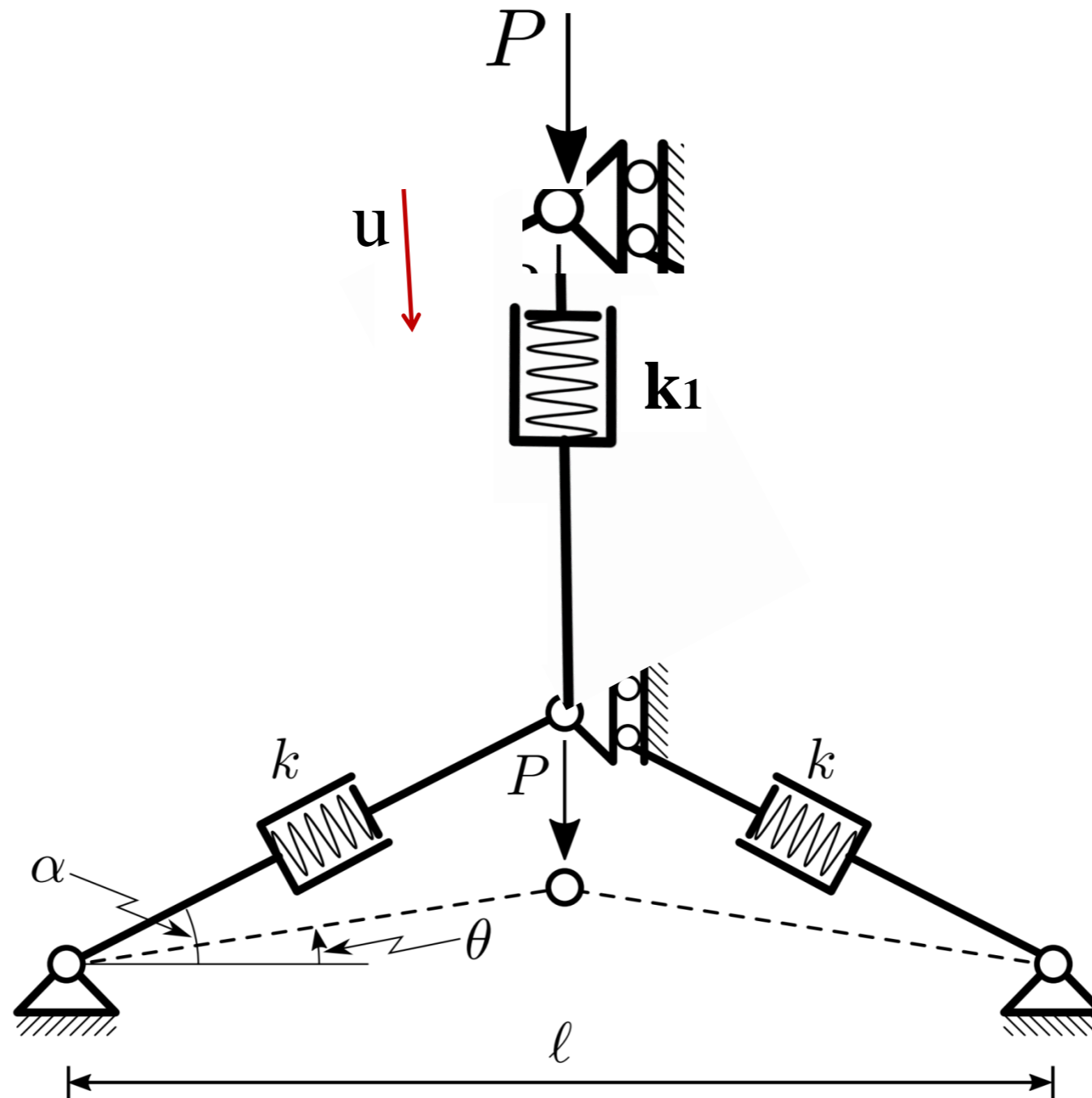
$$P_l^\pm = \pm k\ell \tan^3 \theta_l$$

Da questa si evince che:

$$|\theta| > \theta_l \Rightarrow \frac{d^2 \Pi}{d\theta^2} > 0$$

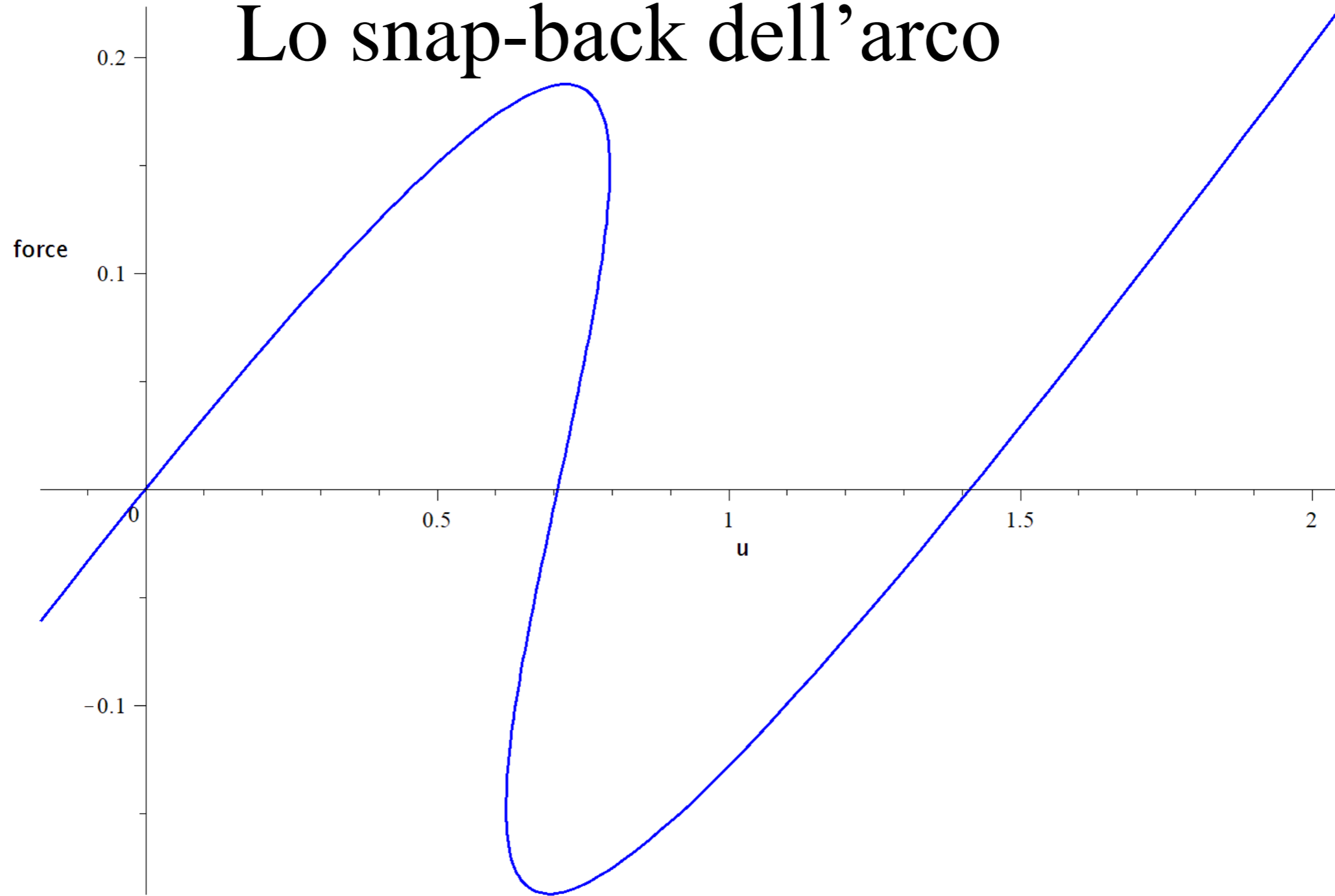
$$|\theta| < \theta_l \Rightarrow \frac{d^2 \Pi}{d\theta^2} < 0$$

# Snap-back buckling





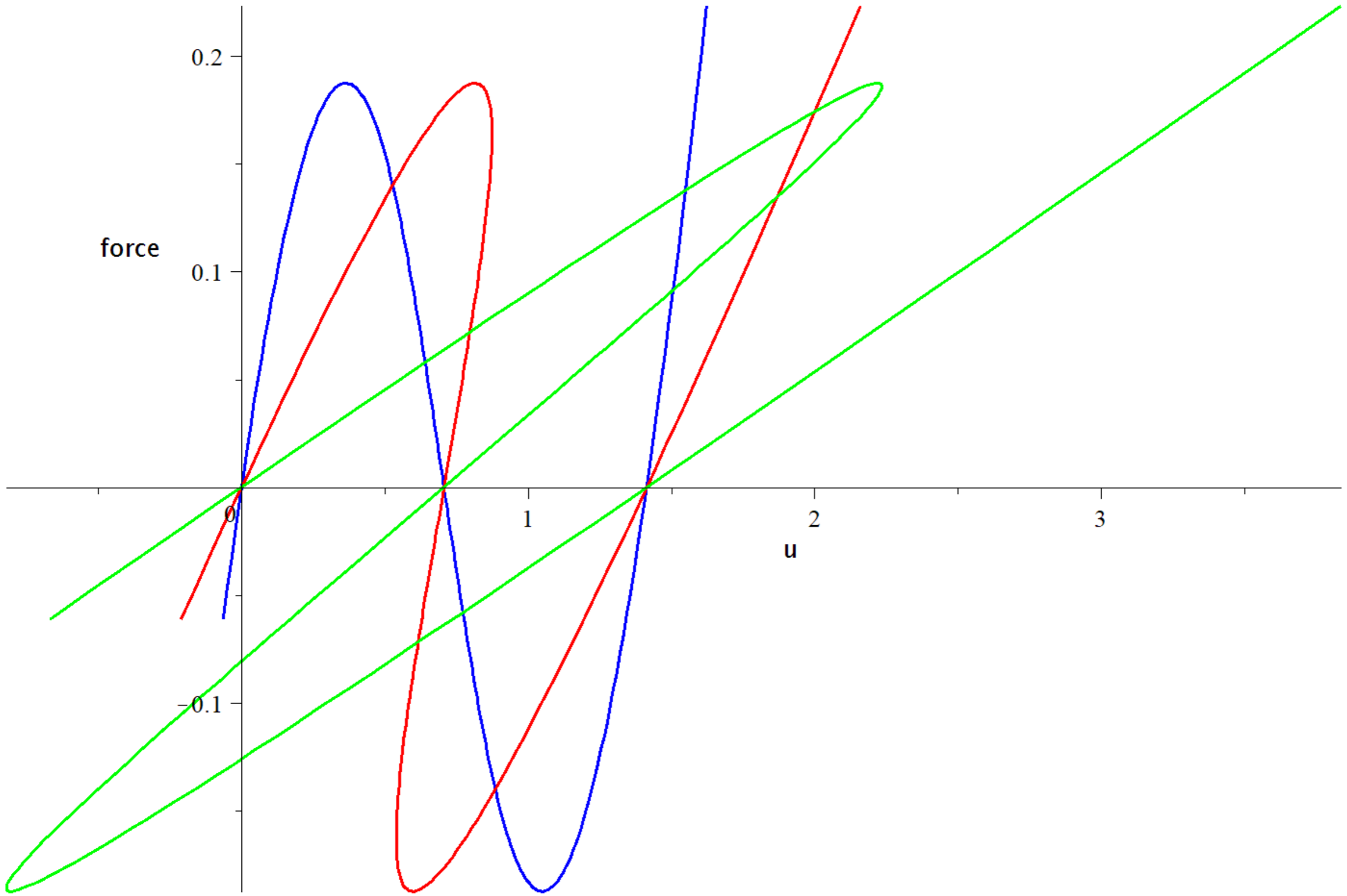
# Lo snap-back dell'arco



$$\alpha = \pi/4, \quad k_1 = 0.5k, \quad P = P/k$$



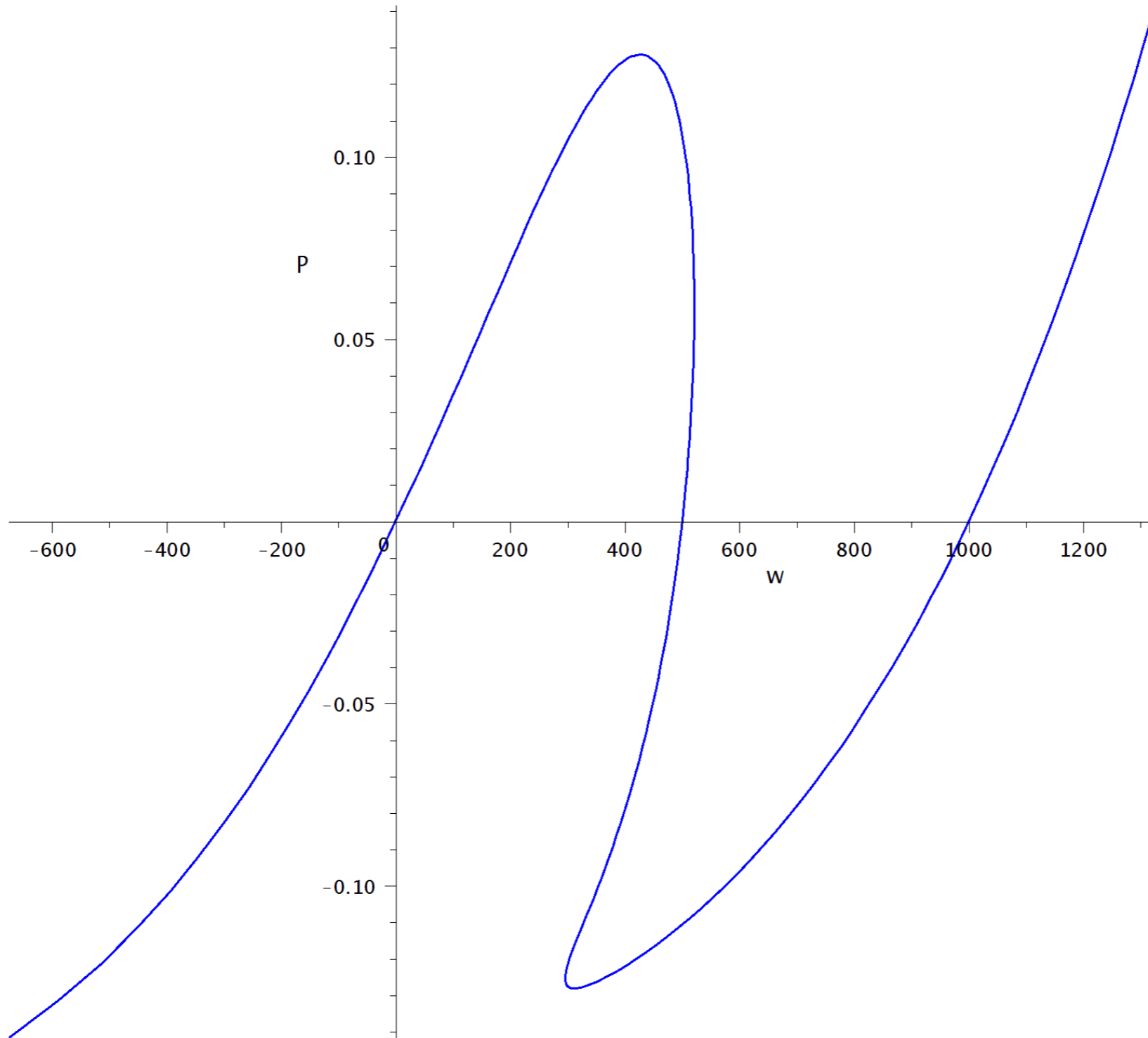
# Lo snap-back dell'arco





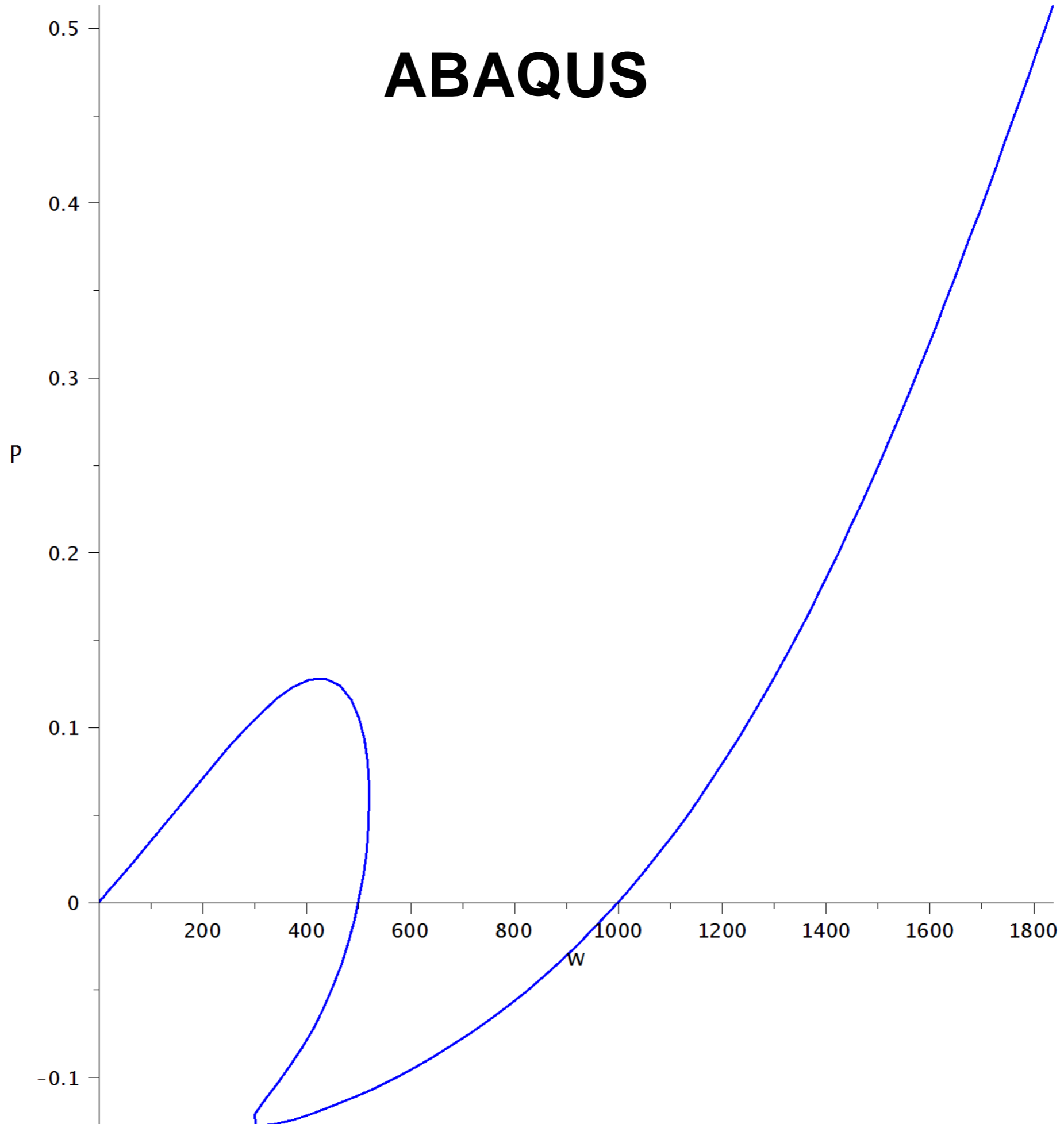
# Snap-back of the truss with Hensky's strain

$\epsilon = \ln(l/l_0)$





# ABAQUS

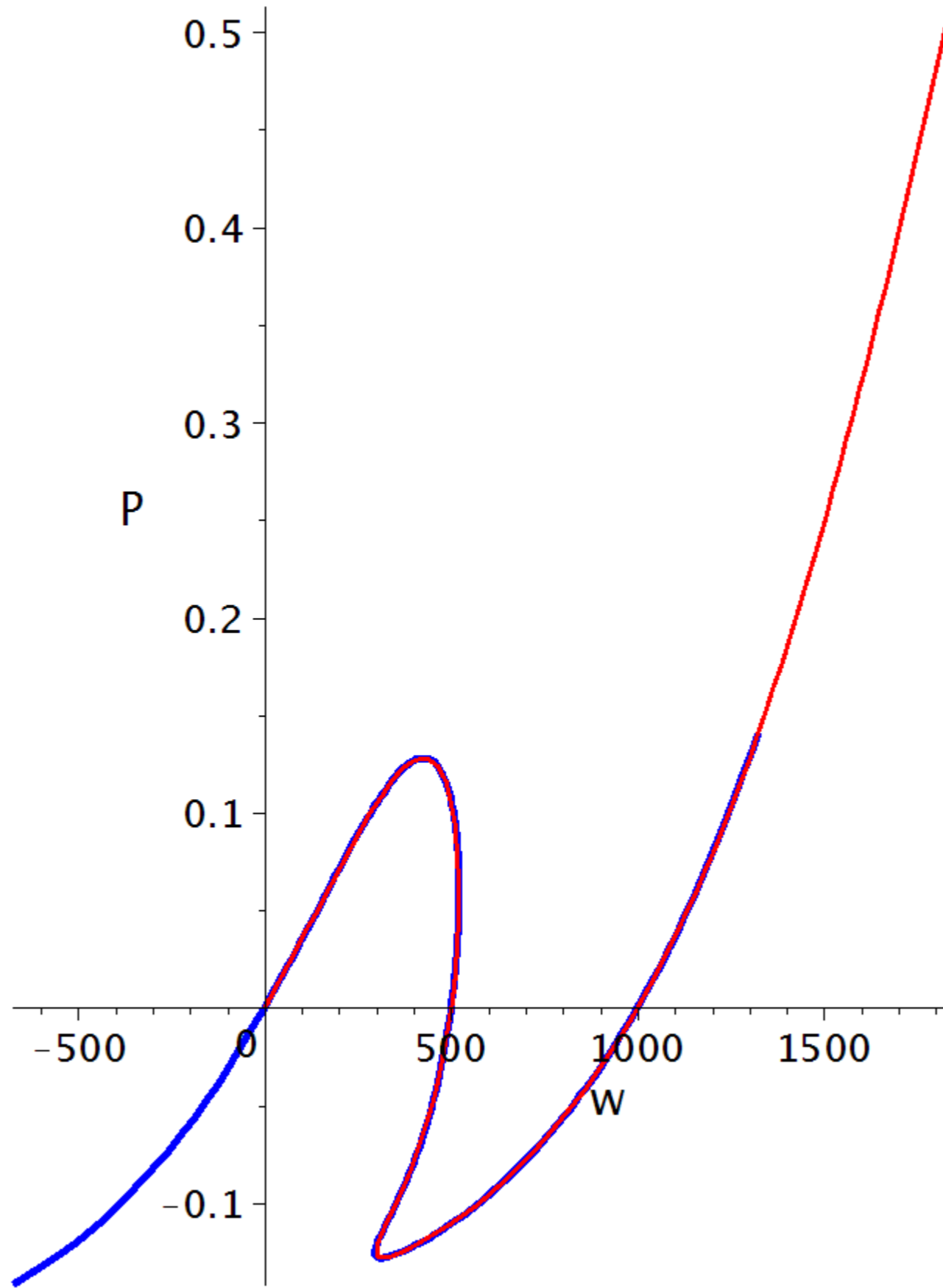


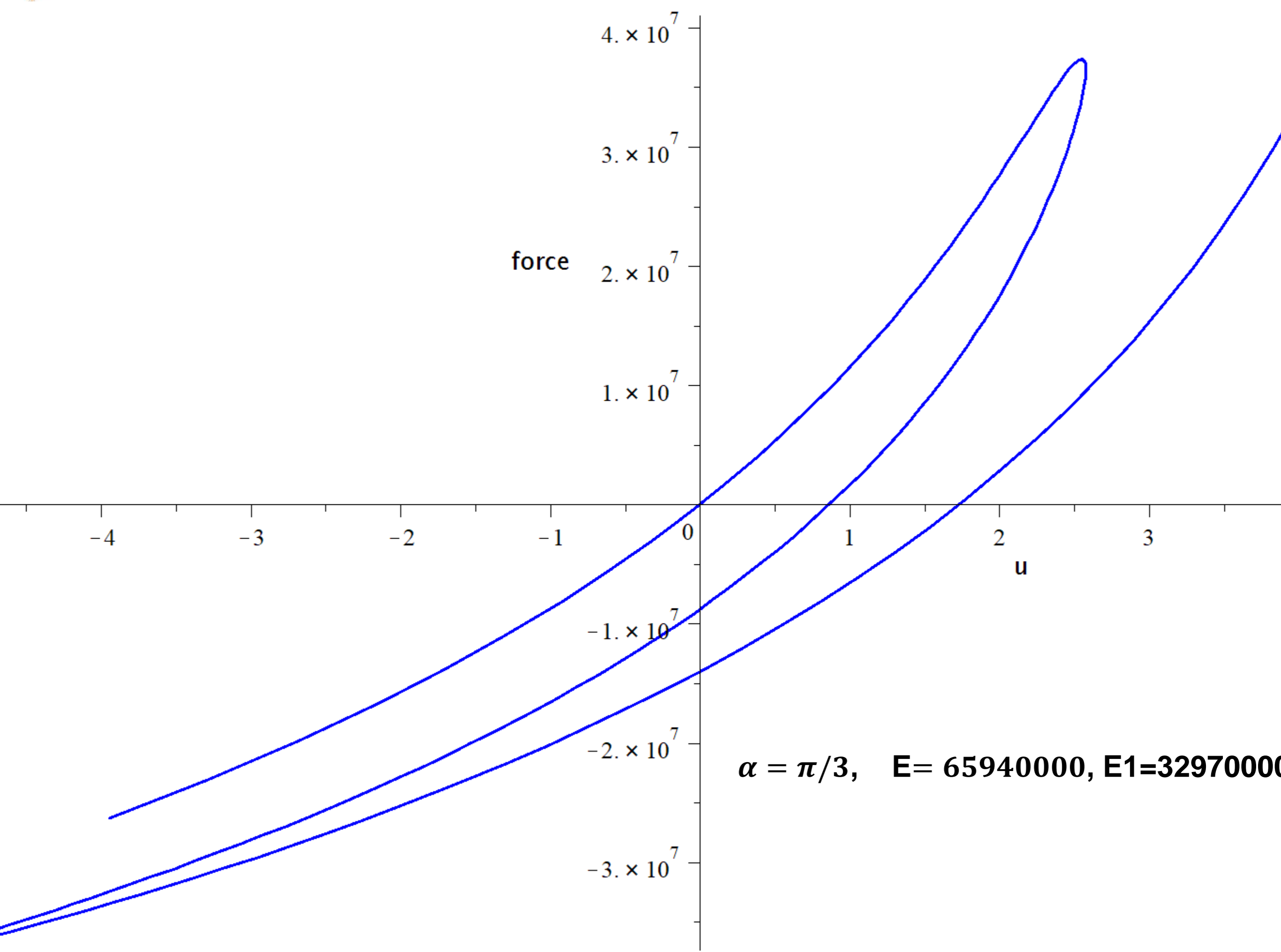


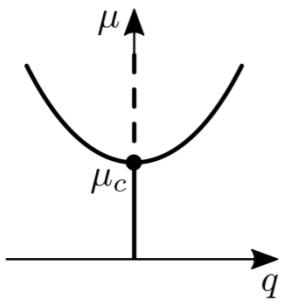
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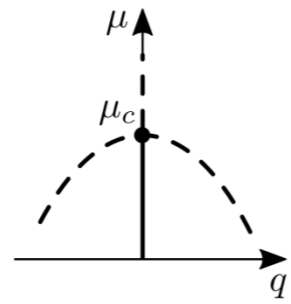
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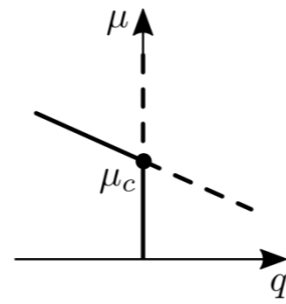




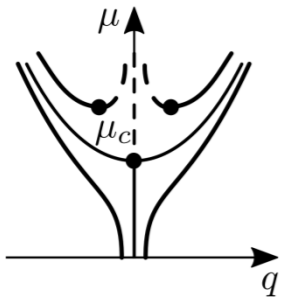
(a)



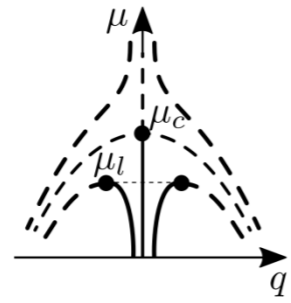
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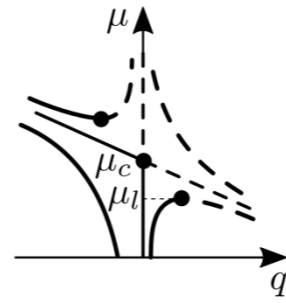
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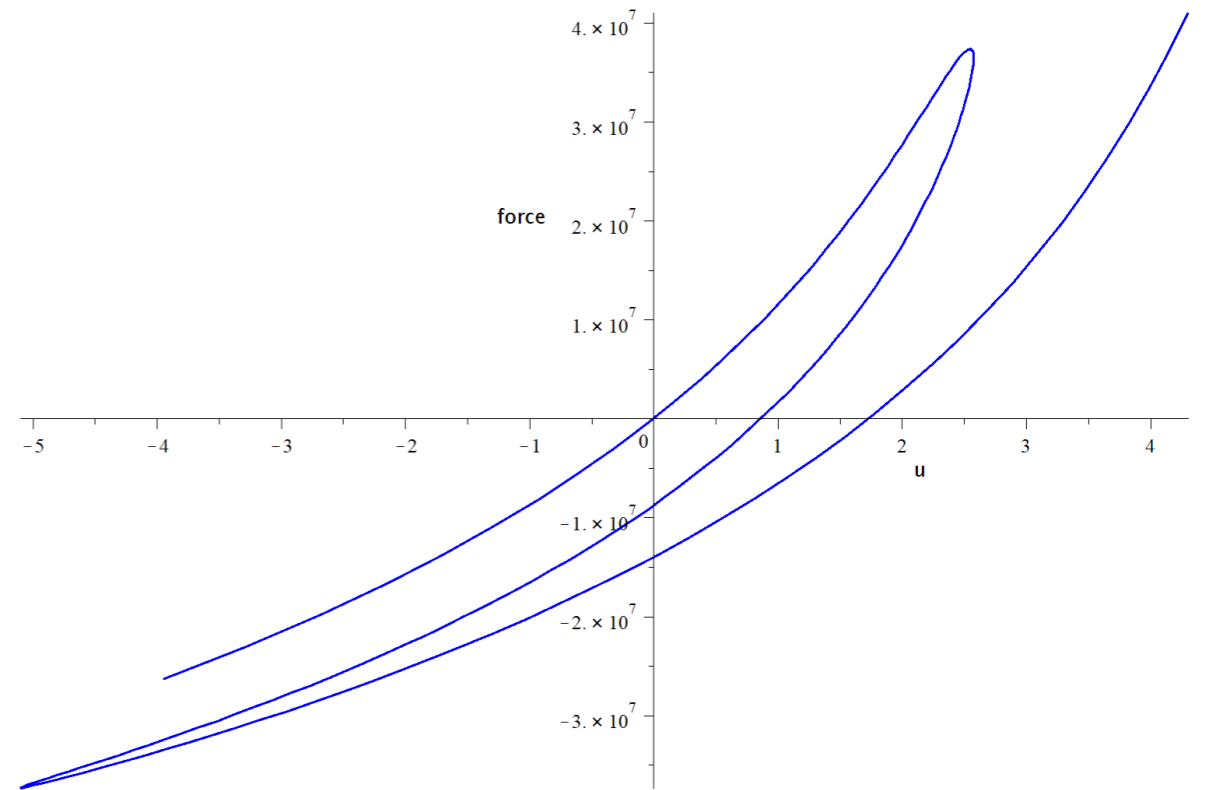
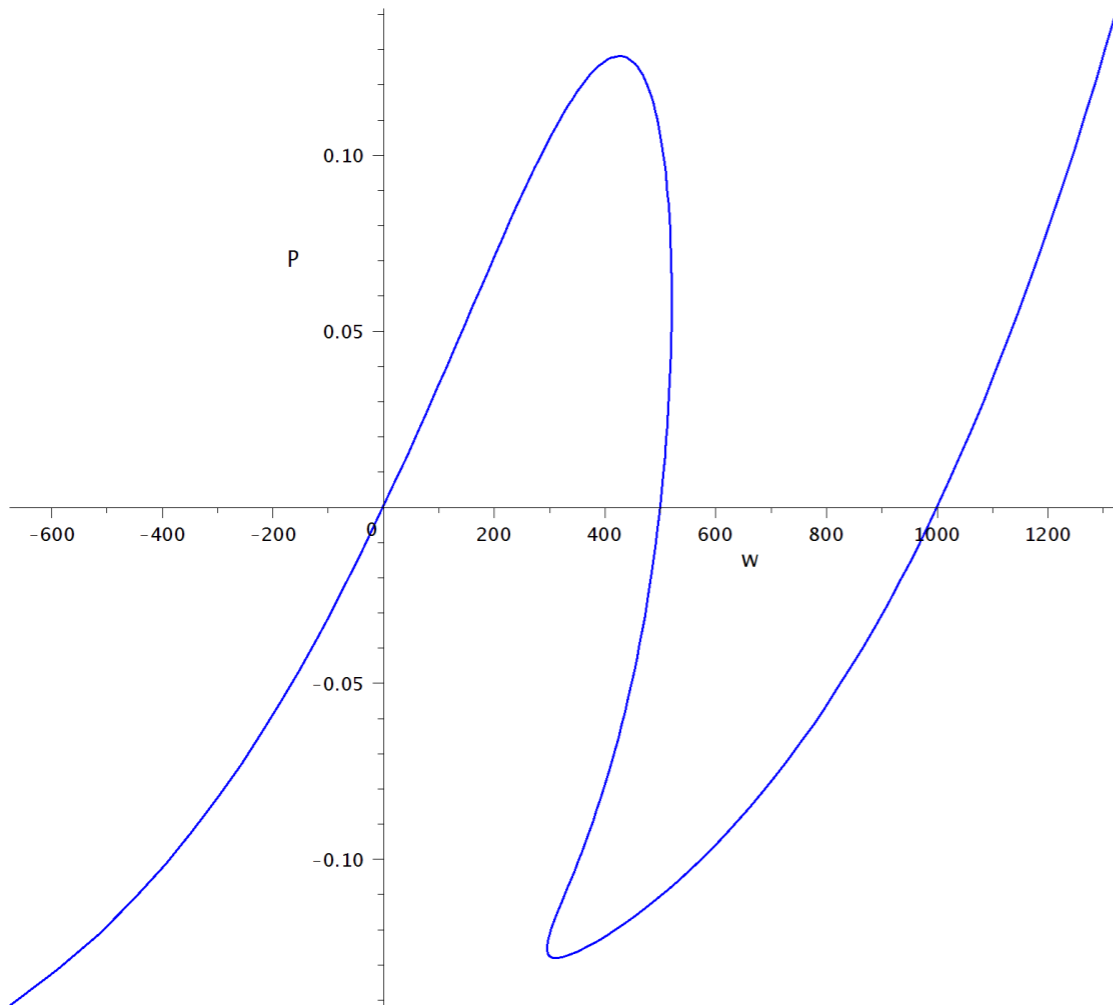
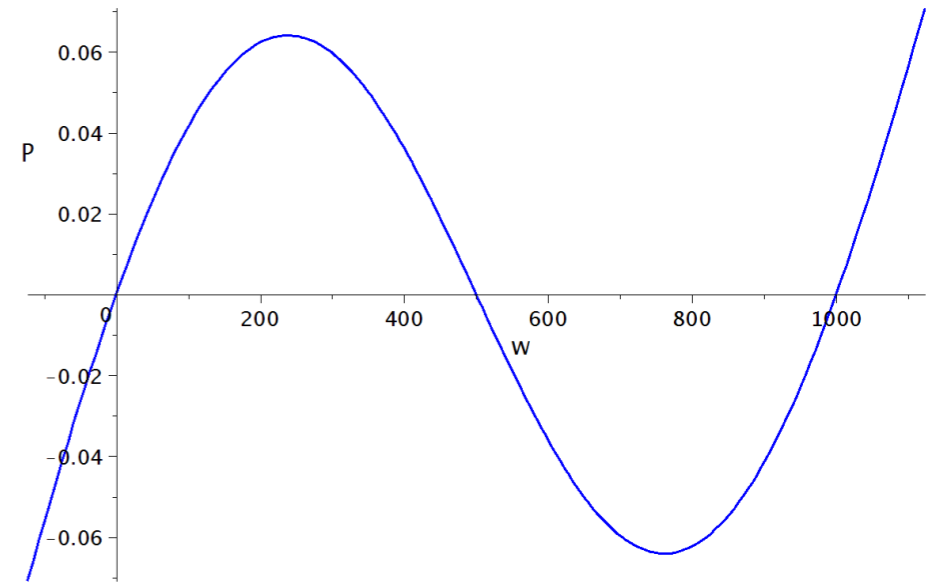
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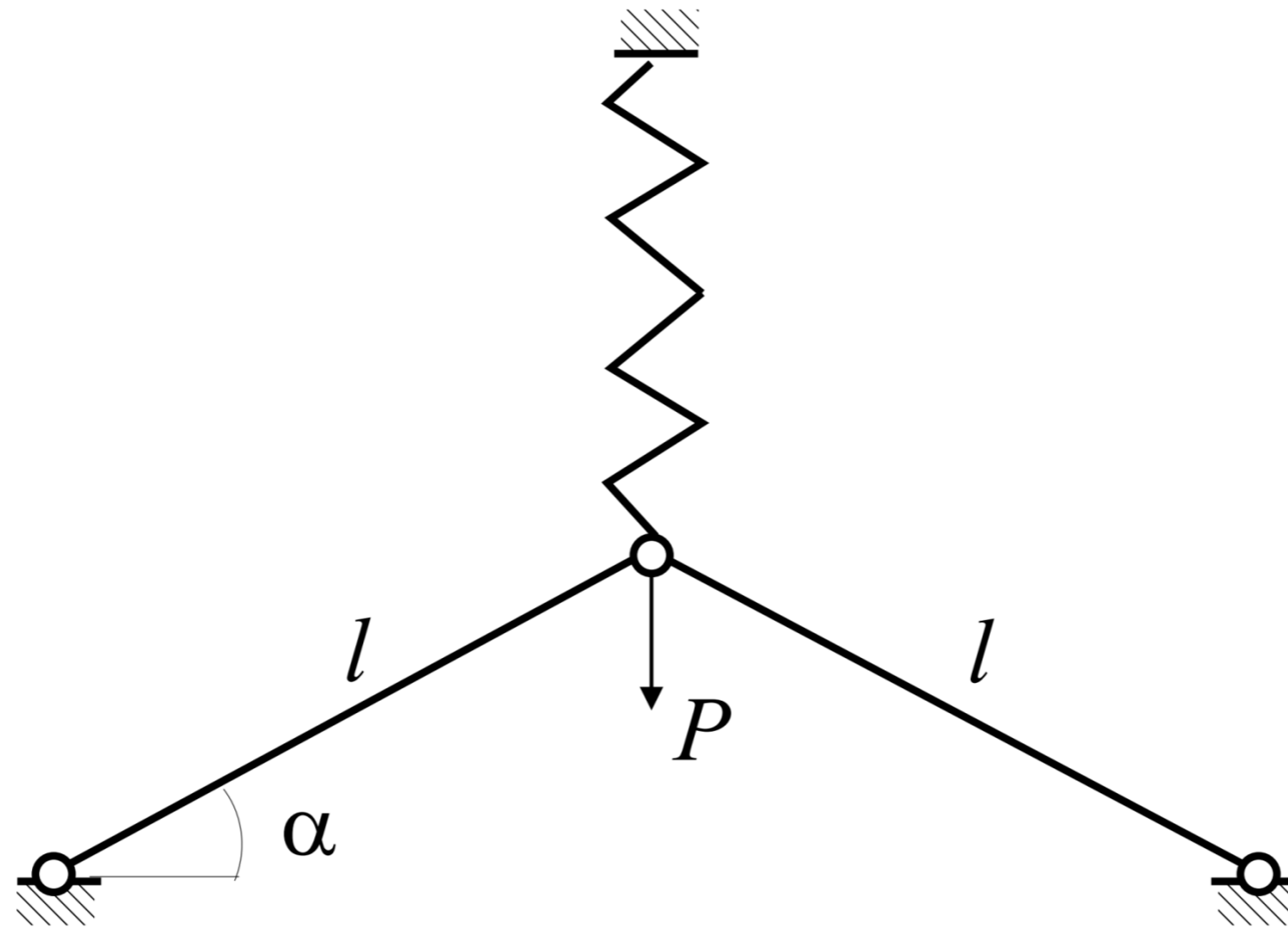


(f)





# Von Mises truss with a spring





# Esempio

Classic von Mises truss with a spring

