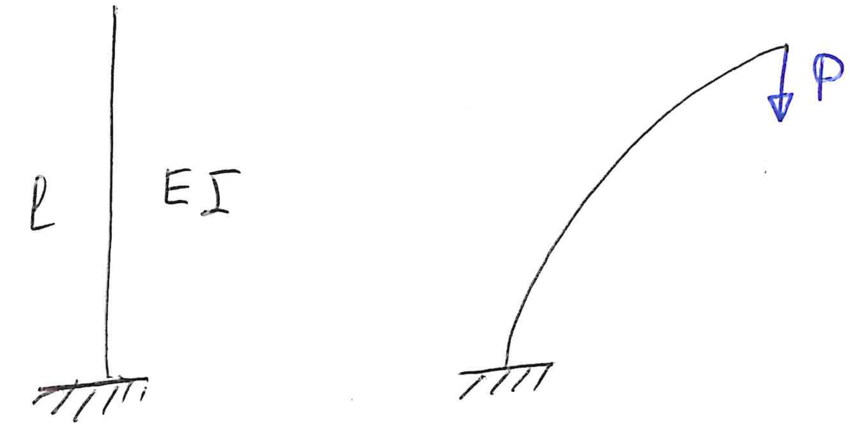
Nonlinear analysis

Nothing is small



$$\boxed{P_c = \frac{\pi^2 EI}{4l^2}}$$

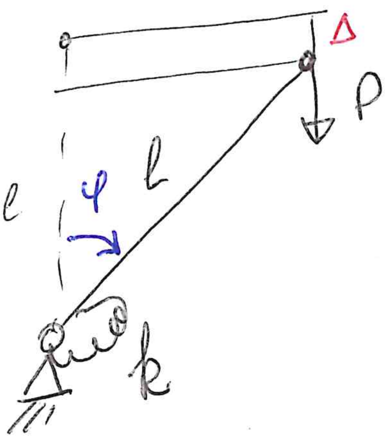
linearized theory gives a value of critical force and a mode of deformation.

Nonlinear theory should give us also the non-trivial solution.

real

"real beams"

Reminder.



Total potential energy

$$\Pi = U - V$$

Elastic energy $U = \frac{1}{2} k \varphi^2$

Work of P is given by $V = P\Delta$

$$\Delta = l - l\cos\varphi$$

So Π became

$$\Pi = \frac{1}{2} k\varphi^2 - Pl(1 - \cos\varphi)$$

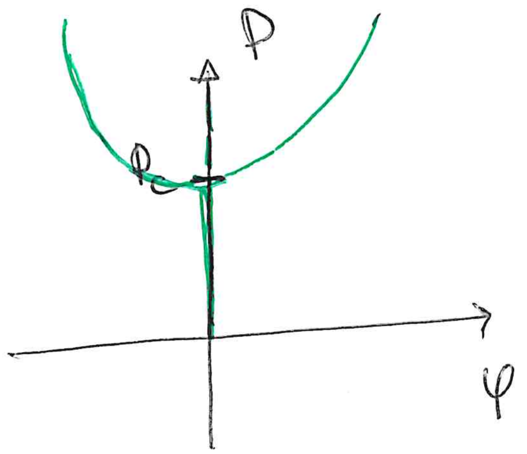
Equilibrium : $\frac{d\Pi}{d\varphi} = 0 \qquad k\varphi - Pl\sin\varphi = 0$

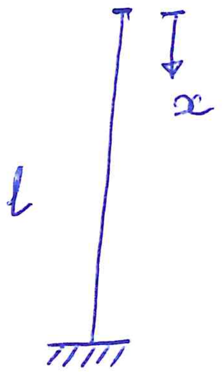
Two solutions : 1. Trivial $\varphi = 0$

2. Non-trivial

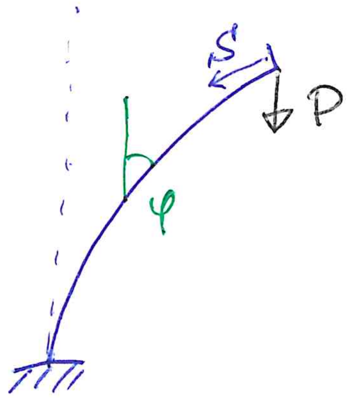
$$P = \frac{k}{l} \frac{\varphi}{\sin\varphi}$$

$$P_c = \frac{k}{l}$$





initial



current state

s – arc-length

$0 \le s \le l$

Angle $\varphi = \varphi(s)$

Curvature $\varkappa = \varphi'(s)$

Constitutive equation $M \sim \varkappa$

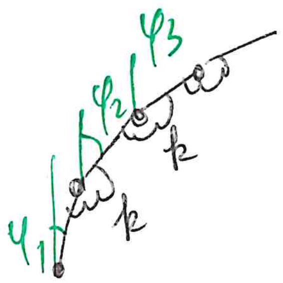
Elastic energy $U = \frac{1}{2} \int_0^l EI \varkappa^2 \, ds$

strain (deformation) energy density

$\frac{1}{2} EI \varkappa^2$

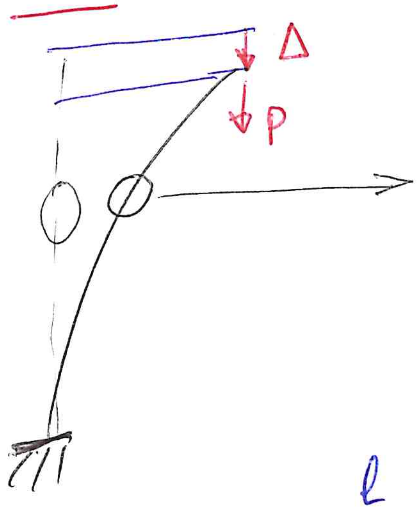
Discrete analog

$$U = \sum_{i=1}^{N} \frac{1}{2} k (\varphi_{i+1} - \varphi_i)^2$$



$V - ?$ Former case $V = P\Delta$

$\Delta - ?$



ds

$$d\Delta(s) = ds - ds\cos\varphi(s)$$

$$\Delta = l - \int_0^l \cos\varphi(s)\,ds$$

$$V = P\int_0^l [1 - \cos\varphi(s)]\,ds$$

As a result, we get Π in the form

$$\Pi = \frac{1}{2}\int_0^l EI\,\varkappa^2\,ds - P\int_0^l [1 - \cos\varphi(s)]\,ds$$

$$\Pi = \int_0^l \left[\frac{1}{2}EI\,\varphi'^2(s) - P(1 - \cos\varphi(s))\right]ds$$

Equilibrium equation

$$\delta \Pi = 0 \qquad \Big/ \text{instead of } \frac{d\Pi}{d\varphi} = 0$$

$$\delta \Pi = \int_0^{\ell} \left[EI\,\varphi'(s)\,\delta\varphi'(s) - P \sin\varphi(s)\,\delta\varphi(s) \right] ds$$

Integrating by parts we get

$$\delta \Pi = \int_0^{\ell} \left[-EI\,\varphi''(s)\,\delta\varphi(s) - P \sin\varphi(s)\,\delta\varphi(s) \right] ds$$

$$+ \; EI\,\varphi'(s)\,\delta\varphi(s) \Big|_0^{\ell}$$

Boundary conditions :

$\varphi = 0$

$\Rightarrow \delta\varphi(0) = 0$

$\Rightarrow$ Equilibrium equation

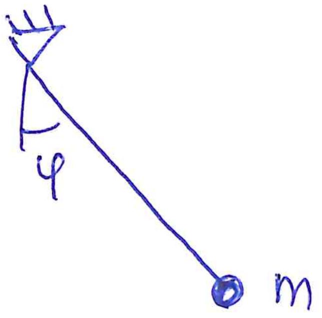
$$-EI\,\varphi''(s) - P \sin\varphi(s) = 0$$

\+ natural boundary condition

$$EI\,\varphi'(\ell) = 0$$

\+ kinematic (essential) BC : $\varphi(0) = 0$

Note:



Pendulum has the same equation!

$\varphi = \varphi(t)$

$$EI\varphi'' + P\sin\varphi = 0$$

$$\varphi'' + \beta^2 \sin\varphi = 0 \qquad \beta^2 = \frac{P}{EI}$$

First observations: two solutions

1. trivial $\varphi = 0$
2. non-trivial $\varphi \neq 0$?

$$\varphi'' + \beta^2 \sin\varphi = 0 \quad | \times \varphi'$$

$$\varphi''\varphi' + \beta^2 \sin\varphi\,\varphi' = 0$$

$$\left(\frac{1}{2}\varphi'^2\right)' - \beta^2 (\cos\varphi)' = 0$$

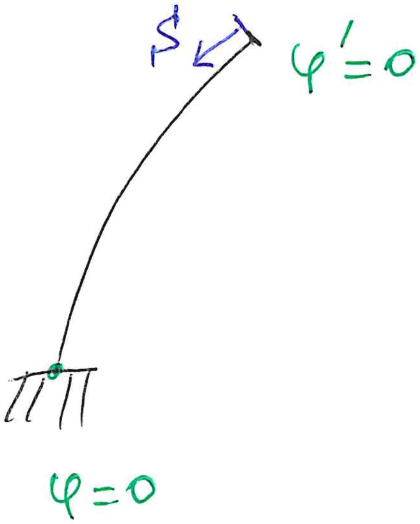
$$\left[\frac{1}{2}\varphi'^2 - \beta^2\cos\varphi\right]' = 0$$

$\Rightarrow$

$$\frac{1}{2}\varphi'^2 - \beta^2\cos\varphi = C, \qquad C = \text{const}$$

unknown.

first integral of eq. equations

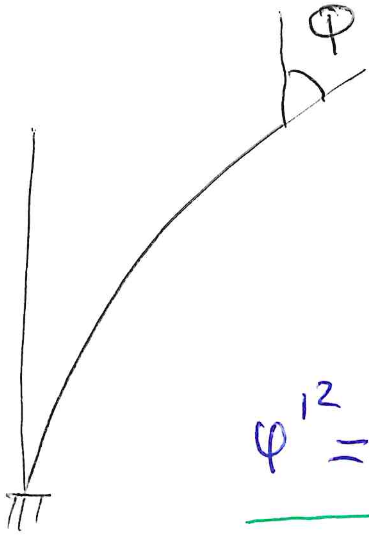


$s = 0$ at the end

$$\frac{1}{2}\underbrace{\varphi'(0)^2}_{=0} - \beta^2\cos\varphi(0) = C$$

$$C = -\beta^2\cos\varphi(0)$$

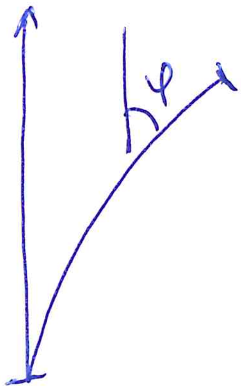
$$\Phi = \varphi(0)$$

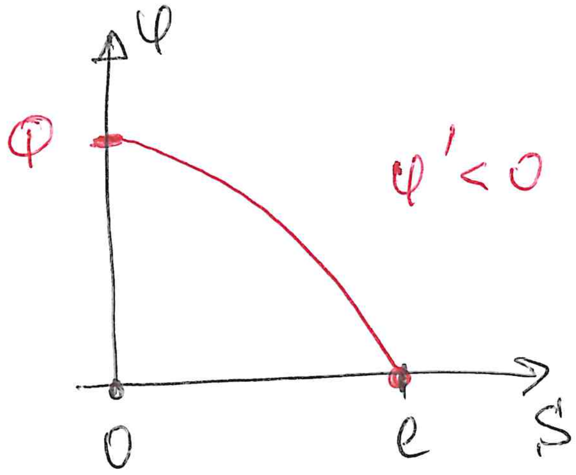


$$\frac{1}{2}\varphi'^2 - \beta^2\cos\varphi = -\beta^2\cos\Phi$$

$$\varphi'^2 = \beta^2\left[\cos\varphi - \cos\Phi\right]$$

$$\varphi' = \pm\beta\sqrt{\cos\varphi - \cos\Phi}$$





$$\Rightarrow \quad \varphi' = -\beta\sqrt{\cos\varphi - \cos\Phi}$$

$$\frac{d\varphi}{ds} = -\beta\sqrt{\cos\varphi - \cos\Phi}$$

$$-\frac{d\varphi}{\beta\sqrt{\cos\varphi - \cos\Phi}} = ds$$

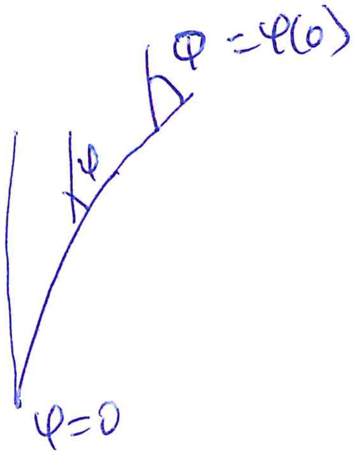
$$\boxed{-\int_{\Phi}^{\varphi(s)} \frac{d\varphi}{\beta\sqrt{\cos\varphi - \cos\Phi}} = s}$$

$$\cos\varphi = \cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2} = 1 - 2\sin^2\frac{\varphi}{2}$$

$$\cos\Phi = 1 - 2\sin^2\frac{\Phi}{2}$$

$$\cos\varphi - \cos\Phi = 2\sin^2\frac{\Phi}{2} - 2\sin^2\frac{\varphi}{2}$$

$$-\int_{\Phi}^{\varphi(s)} \frac{d\varphi}{\sqrt{2}\,\beta\sqrt{\sin^2\frac{\Phi}{2} - \sin^2\frac{\varphi}{2}}} = s$$



$\varphi(s) \leq \Phi$

$$\int_{\varphi(s)}^{\Phi} \frac{d\varphi}{\beta\sqrt{2}\sqrt{\sin^2\frac{\Phi}{2} - \sin^2\frac{\varphi}{2}}} = s$$

BC: $\varphi(\ell) = 0$: at $s = \ell$ $\varphi = 0$

$$\boxed{\int_{0}^{\Phi} \frac{d\varphi}{\beta\sqrt{2}\sqrt{\sin^2\frac{\Phi}{2} - \sin^2\frac{\varphi}{2}}} = \ell}$$

$$p = \sin\frac{\varphi}{2} \quad , \quad p^2 = \sin^2\frac{\varphi}{2}$$

$$\eta : \quad \sin\frac{\varphi}{2} = p \sin\eta$$

φ varies from 0 until φ

$\sin\eta$ varies from 0 until 1

η varies from 0 until $\pi/2$

$$d\varphi \to d\eta$$

$$d\sin\frac{\varphi}{2} = \frac{1}{2}\cos\frac{\varphi}{2}\, d\varphi$$

$$d\sin\frac{\varphi}{2} = d(p\sin\eta) = p\cos\eta\, d\eta$$

$$\frac{1}{2}\cos\frac{\varphi}{2}\, d\varphi = p\cos\eta\, d\eta$$

$$d\varphi = \frac{p\cos\eta\, d\eta}{\cos\frac{\varphi}{2}} = \frac{2p\cos\eta\, d\eta}{\sqrt{1-\sin^2\frac{\varphi}{2}}}$$

$$= \frac{2p\cos\eta\, d\eta}{\sqrt{1-p^2\sin^2\eta}}$$

$$l = \frac{1}{2\beta}\int_0^{\pi/2} \frac{2\cancel{p}\,\cancel{\cos\eta}}{\sqrt{1-p^2\sin^2\eta}}\;\frac{1}{\cancel{p}}\;\frac{1}{\underset{\cancel{\cos^2\eta}}{\sqrt{1-\sin^2\eta}}}\,d\eta$$

$$l = \frac{1}{\beta}\int_0^{\pi/2} \frac{d\eta}{\sqrt{1-p^2\sin^2\eta}}$$

$$\beta l = \int_0^{\pi/2} \frac{d\eta}{\sqrt{1-p^2\sin^2\eta}} \equiv K(p)$$

$K(p)$ is the elliptic integral of first kind

$$\boxed{\beta l = K(p)}$$

$\sim$ dependence of $P(\beta)$ on $\Phi = \max_s \varphi$

$$\underline{\beta \ell = K(p)} \qquad \beta^2 = \frac{P}{EI}$$

$$\varphi \to p \equiv \sin\frac{\varphi}{2}$$

$$P = K^2(p)\,\frac{EI}{\ell^2}$$

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if $\psi \ll 1$, $\varphi \ll 1$, $\eta \leq 1$

Linear theory

$$\sin\frac{\psi}{2} = \sin\frac{\varphi}{2}\sin\eta$$

$$\int_0^{\pi/2} \frac{d\eta}{\sqrt{1-\underbrace{p^2}_{}\underbrace{\sin^2\eta}_{\leq 1}}} \approx \int_0^{\pi/2} d\eta = \frac{\pi}{2}$$

$$\beta\ell = \frac{\pi}{2} \implies \boxed{P_c = \frac{\pi^2}{4}\frac{EI}{\ell^2}}$$

$$\boxed{P = \frac{4}{\pi^2} P_c K^2(p)}$$

$$P \sim \varphi \quad \text{or} \quad P \sim \bar{v} = \max_{s} v(s)$$