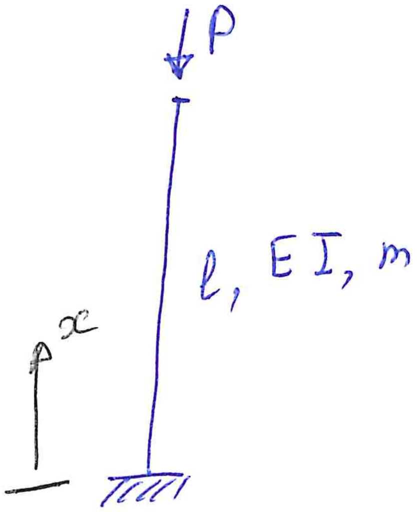
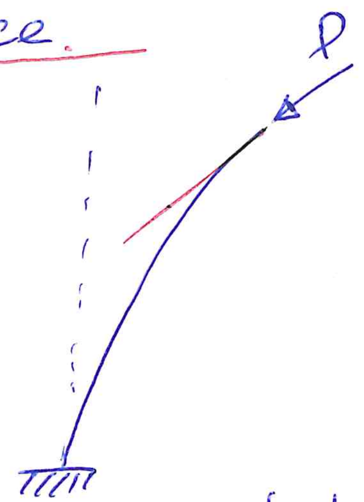
Instability of a beam under follower (non-conservative) force.



initial state



current state

Equation of motion

$$EI\,v^{(4)} + P v'' + m \ddot{v} = 0$$

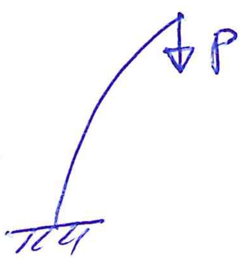
$()' = \frac{\partial}{\partial x}$, $\dot{()} = \frac{\partial}{\partial t}$, m is a linear mass density

Boundary conditions are

at $x = 0$: $v(0,t) = 0$, $v'(0,t) = 0$

at $x = \ell$: $EI\,v''(\ell,t) = 0$; $\underline{EI\,v'''(\ell,t) = 0}$

new

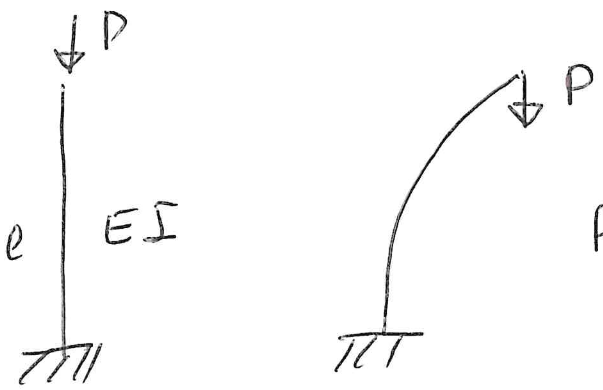


$$EI\,v''' \boxed{+ P v'} = 0$$ ← it was

Statics

Let us assume static behavior.

Reminder



„dead" force

$$P_c = \frac{\pi^2 EI}{4\ell^2}$$

$$EIv^{(4)} + Pv'' = 0$$

$$v(0) = 0, \quad v'(0) = 0, \quad v''(\ell) = 0, \quad v'''(\ell) = 0$$

$$\beta = \sqrt{\frac{P}{EI}}$$

$$v(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 x + C_4$$

$C_1, \dots C_4$ – consts

$$v' = -C_1\beta \sin\beta x + C_2\beta \cos\beta x + C_3$$

$$v'' = -C_1\beta^2 \cos\beta x - C_2\beta^2 \sin\beta x$$

$$v''' = C_1\beta^3 \sin\beta x - C_2\beta^3 \cos\beta x$$

BCs: $v(0) = C_1 + C_4 = 0$

$$v'(0) = C_2\beta + C_3 = 0$$

$$v''(l) = -C_1\beta^2\cos\beta l - C_2\beta^2\sin\beta l = 0$$

$$v'''(l) = C_1\beta^3\sin\beta l - C_2\beta^3\cos\beta l = 0$$

$$\begin{cases} C_1\beta^2\cos\beta l + C_2\beta^2\sin\beta l = 0 \\ C_1\beta^3\sin\beta l - C_2\beta^3\cos\beta l = 0 \end{cases}$$

$$\begin{cases} C_1\cos\beta l + C_2\sin\beta l = 0 \\ C_1\sin\beta l - C_2\cos\beta l = 0 \end{cases}$$

$$A\cdot\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$A = \begin{pmatrix} \cos\beta l & \sin\beta l \\ \sin\beta l & -\cos\beta l \end{pmatrix}$$

$$\underline{\det A = -\cos^2\beta l - \sin^2\beta l = -1}$$

$\Rightarrow C_1, C_2 = 0, \quad C_3, C_4 = 0$

$\Rightarrow \underline{v(x) \equiv 0}$!!! $\boxed{P_c^{(static)} = \infty}$

So conclusion: we should use dynamics

$v = v(x,t)$

$$EI\, v^{(4)} + P v'' + m \ddot{v} = 0$$

$$v^{(4)} + \beta^2 v'' + \rho \ddot{v} = 0 \quad , \quad \rho = \frac{m}{EI}$$

Form of solution is

$$v(x,t) = V(x)\, T(t)$$

Substituting it into equation of motion we get

$$\frac{V^{(4)} + \beta^2 V''}{V(x)} = \underset{\text{const}}{\lambda} = -\rho \frac{\ddot{T}}{T}$$

For $T = T(t)$: $\rho \ddot{T} + \lambda T = 0$

$\boxed{\lambda > 0}$ $\qquad \omega^2 = \frac{\lambda}{\rho}$ $\qquad T = C \sin(\omega t + \varphi)$

$\boxed{T \sim e^{i\omega t}}$

For $V = V(x)$

$$V^{(4)} + \beta^2 V'' - \lambda V = 0$$

Characteristic equation is $\qquad \Big| \; V \sim e^{\alpha x}$

$$\alpha^4 + \beta^2 \alpha^2 - \lambda = 0$$

$$\gamma^2 + \beta^2 \gamma - \lambda = 0, \quad \gamma = \alpha^2$$

$\Rightarrow \gamma_\pm$ – two solutions

$\Rightarrow$ four solutions for α:

$$\gamma_+ = \sqrt{\frac{\beta^4}{4} + \lambda} - \frac{\beta^2}{2} > 0$$

$$\gamma_- = -\sqrt{\frac{\beta^4}{4} + \lambda} - \frac{\beta^2}{2} < 0$$

So $\alpha_{1,2} = \pm s_1, \qquad s_1 = \sqrt{\gamma_+}$

$\alpha_{3,4} = \pm i s_2, \qquad s_2 = \sqrt{-\gamma_-}$

$$V(x) = C_1 \cosh s_1 x + C_2 \sinh s_1 x$$
$$+ C_3 \cos s_2 x \quad + C_4 \sin s_2 x$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \; ; \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$
(ch) (sh)

Final solution for v is given by

$$\boxed{v(x,t) = \sin(\omega t + \varphi)\, V(x) \quad (e^{i\omega t})}$$

$C_1, \dots C_4$ are constants

$\sin' = \cos \qquad \sinh' = \cosh$

$\cos' = -\sin \qquad \cosh' = \sinh$

BCs : $v(0,t) = 0, \quad v'(0,t) = 0$
$v''(l,t) = 0, \quad v'''(l,t) = 0$

So we can use them for V :
$V(0) = 0, \quad V'(0) = 0, \quad V''(l) = 0, \quad V'''(l) = 0$

$$V'(x) = C_1 s_1 \cancel{c}\sinh s_1 x + C_2 s_1 \cosh s_1 x$$
$$- C_3 s_2 \sin s_2 x + C_4 s_2 \overset{\cos}{\cancel{\sin}}\, s_2 x$$

$$V''(x) = C_1 s_1^2 \cosh s_1 x + C_2 s_1^2 \sinh s_1 x$$
$$- C_3 s_2^2 \cos s_2 x - C_4 s_2^2 \sin s_2 x$$

$$V'''(x) = C_1 s_1^3 \sinh s_1 x + C_2 s_1^3 \cosh s_1 x$$
$$+ C_3 s_2^3 \sin s_2 x - C_4 s_2^3 \sin s_2 x$$

$\boxed{\text{BCs :}}$ $\quad V(0) = \underline{C_1 + C_3 = 0}$

$$V'(0) = \underline{C_2 s_1 + C_4 s_2 = 0}$$

$$V''(\ell) = \underline{C_1 s_1^2 \cosh s_1 \ell + C_2 s_1^2 \sinh s_1 \ell}$$
$$\underline{- C_3 s_2^2 \cos s_2 \ell - C_4 s_2^2 \sin s_2 \ell = 0}$$

$$V'''(\ell) = \underline{C_1 s_1^3 \sinh s_1 \ell + C_2 s_1^3 \cosh s_1 \ell}$$
$$\underline{+ C_3 s_2^3 \sin s_2 \ell - C_4 s_2^3 \cos s_2 \ell = 0}$$

$$\underline{C_3 = -C_1} \quad : \quad \underline{C_4 S_2 = -C_2 S_1}$$

$$\begin{cases} C_1 S_1^2 \cosh S_1 l + C_2 S_1^2 \sinh S_1 l \\ + C_1 S_2^2 \cos S_2 l + C_2 S_1 S_2 \sin S_2 l = 0 \\ C_1 S_1^3 \sinh S_1 l + C_2 S_1^3 \cosh S_1 l \\ - C_1 S_2^3 \sin S_2 l + C_2 S_1 S_2^2 \cos S_2 l = 0 \end{cases}$$

$$A \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$A = \begin{pmatrix} S_1^2 \cosh S_1 l + S_2^2 \cos S_2 l & \begin{array}{l} S_1^2 \sinh S_1 l \\ + S_1 S_2 \sin S_2 l \end{array} \\ \\ S_1^3 \sinh S_1 l - S_2^3 \sin S_2 l & \begin{array}{l} S_1^3 \cosh S_1 l + \\ + S_1 S_2^2 \cos S_2 l \end{array} \end{pmatrix}$$

$$\det A = \left[S_1^2 \cosh S_1 l + S_2^2 \cos S_2 l \right] \left[S_1^3 \cosh S_1 l + S_1 S_2^2 \overset{\cos}{\cancel{\sin}} S_2 l \right]$$

$$- \left[S_1^2 \sinh S_1 l + S_1 S_2 \sin S_2 l \right] \left[S_1^3 \sinh S_1 l - S_2^3 \sin S_2 l \right]$$

$$\det A = s_1^5 \cosh^2 s_1 l + s_1 s_2^4 \cos^2 s_2 l$$
$$+ s_1^3 s_2^2 \cos s_2 l \cosh s_1 l + s_1^3 s_2^2 \cosh s_1 l \cos s_2 l$$
$$- s_1^5 \sinh^2 s_1 l + s_1 s_2^4 \sin^2 s_2 l$$
$$- s_1^4 s_2 \sin s_2 l \sinh s_1 l + s_1^2 s_2^3 \sinh s_1 l \sin s_2 l$$

$$\boxed{\begin{aligned}\det A &= s_1^5 + s_1 s_2^4 \\ &+ s_1^3 s_2^2 \cos s_2 l \cosh s_1 l \\ &+ (s_1^2 s_2^3 - s_1^4 s_2) \sin s_2 l \sin s_1 l\end{aligned}}$$

$$\cos^2 x + \sin^2 x = 1$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 =$$
$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} -$$
$$- \frac{\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}}{4} =$$
$$= 1$$

Now we have to substitute here s_1 and s_2

As a result we get

$$\det A = F(\underset{\sqrt{\frac{P}{EI}}}{\underset{\shortparallel}{\beta}}, \underset{\omega^2 \rho}{\underset{\shortparallel}{\lambda}}) = G(P, \omega)$$

$G(P, \omega) = 0$

for any P we are looking for ω.

Let us take $P = 0$

$P = 0 \Rightarrow \beta = 0$

$\gamma_+ = \sqrt{\lambda}$, $\gamma_- = -\sqrt{\lambda}$

$s_1 = \sqrt{\gamma_+} = \lambda^{1/4}$, $s_2 = \sqrt{-\gamma_-} = \lambda^{1/4}$

$\lambda = \rho\,\omega^2 \Rightarrow$

$$\frac{1}{s_1}\det A = s_1^4 + s_2^4 + s_1^2 s_2^2 \cos s_2 l \cosh s_1 l + \frac{(s_1 s_2^3 - s_1^3 s_2)\sin s_2 l \sin s_1 l}{s_1 s_2 (s_2^2 - s_1^2)}$$

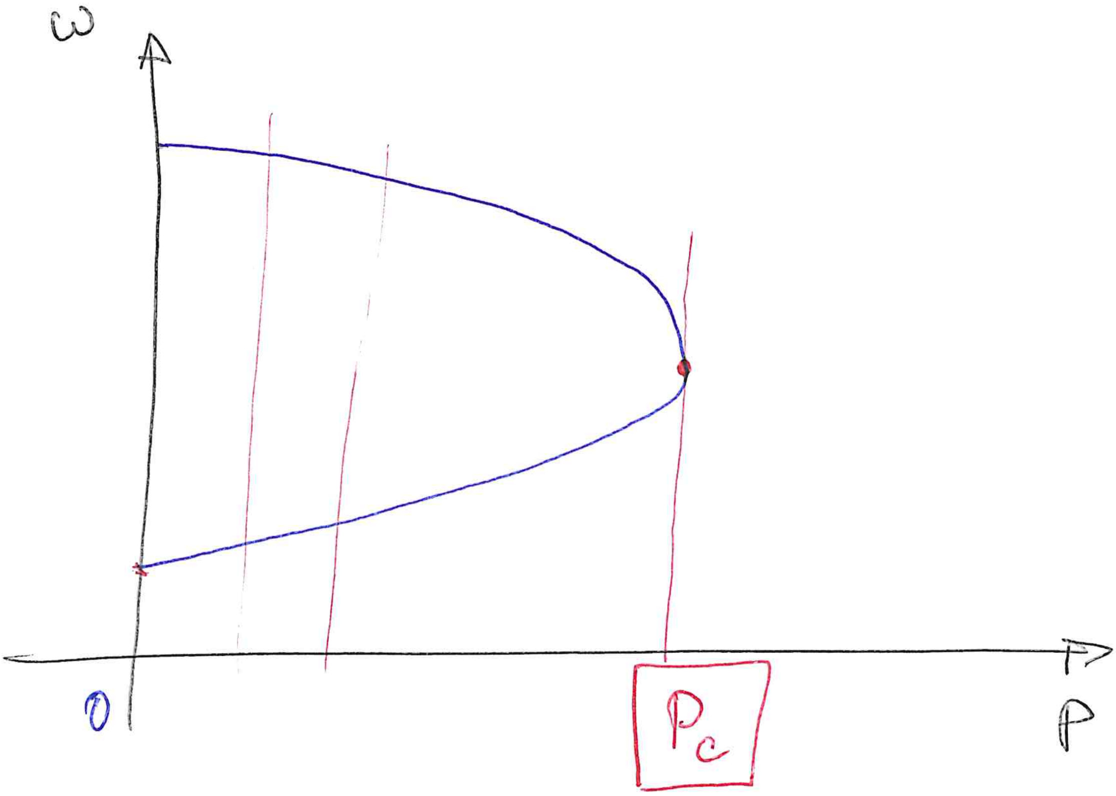
In this case $s_1 = s_2$, $s_1^4 = \lambda$

$$2\lambda + \lambda \cos s_2 l \cosh s_1 l = 0$$

$$2 + \cos s_2 l \cos s_1 l = 0$$ $\rightarrow$ it has real roots

Typical form of $G(P, \omega) = 0$

implicit curve $\omega = \omega(P)$



For $P > P_c$ we have not real ω!

ω becomes complex => asymptotic instability

$$\boxed{P_c \approx 20.05 \frac{EI}{\ell^2}}$$

Dynamic critical force is 8 times larger than static one.

Dead force : $P_c = \dfrac{\pi^2 EI}{4\ell^2}$