Dynamics and stability

$$EI\,v^{(4)} + P v'' = \underline{p(x)}$$

external load

Transition to dynamics

$$v = v(x,t) \qquad p \to \underline{-m\ddot{v}}$$

inertia force

Equation of motion

$$\boxed{EI\,v^{(4)} + P v'' + m\ddot{v} = 0}$$

m – linear mass density, if M is a mass
ℓ is a length
$m = M/\ell$

$$(\,)' = \frac{\partial}{\partial x}, \qquad \dot{(\,)} = \frac{\partial}{\partial t}$$

$$v^{(4)} + \beta^2 v'' + \rho\ddot{v} = 0,$$

$$\beta = \sqrt{\frac{P}{EI}}, \qquad \rho = \frac{m}{EI}$$

Separation of variables

$$v(x,t) = V(x)\,T(t)$$

$$V^{(4)}(x)\,T(t) + \beta^2 V''(x)\,T(t) + \rho V(x)\ddot{T}(t) = 0$$

Let us divide by VT:

$$\underbrace{\frac{V^{(4)}(x)}{V(x)} + \beta^2 \frac{V''(x)}{V(x)}}_{x} + \underbrace{\rho\frac{\ddot{T}(t)}{T(t)}}_{t} = 0$$

$$\underbrace{\frac{V^{(4)}(x) + \beta^2 V''(x)}{V(x)}}_{x} = \underbrace{-\rho\frac{\ddot{T}(t)}{T(t)}}_{t}$$

$$\underline{f(x) = g(t)} \quad ?$$

$$\Rightarrow \quad f(x) = \underset{\lambda}{\underset{\shortparallel}{const}} = g(t)$$

$$-\rho\frac{\ddot{T}(t)}{T(t)} = \lambda \quad \text{or} \quad \underline{\rho\ddot{T} + \lambda T = 0}$$

$$\ddot{T} + \frac{\lambda}{\rho} T = 0$$

$\rho > 0$ is positive, sign of λ - ?

$\lambda > 0$ because $T(t)$ - oscillations

$$\boxed{\omega^2 = \frac{\lambda}{\rho}}$$

$$T(t) = A \cos\omega t + B \sin\omega t$$

For negative λ a solution has exponential form

$$T(t) = A e^{\omega t} + B e^{-\omega t}$$

Another form of $T(t) = C \sin(\omega t + \varphi)$

Fourier transform $T(t) \sim e^{i\omega t}$

$V(x)$: $\dfrac{V^{(4)}(x) + \beta^2 V''(x)}{V(x)} = \lambda$

$$V^{(4)} + \beta^2 V'' - \lambda V = 0$$

$V \sim e^{\alpha x}$

Characteristic equation for α

$$\boxed{\alpha^4 + \beta^2 \alpha^2 - \lambda = 0}$$

α - ?

$\gamma = \alpha^2$ $\qquad \gamma^2 + \beta^2 \gamma - \lambda = 0$

$$\gamma = \frac{-\beta^2 \pm \sqrt{\beta^4 + 4\lambda}}{2}$$

$(\lambda > 0)$

$\beta^4 + 4\lambda > 0$

$$\gamma_+ = \sqrt{\frac{\beta^4}{4} + \lambda} - \frac{\beta^2}{2} > 0$$

$$\gamma_- = -\sqrt{\frac{\beta^4}{4} + \lambda} - \frac{\beta^2}{2} < 0$$

two roots

$$\boxed{\alpha^2 = \gamma}$$

$$\boxed{\begin{aligned} \alpha_{1,2} &= \pm\sqrt{\gamma_+} \\ \alpha_{3,4} &= \pm i\sqrt{-\gamma_-} \end{aligned}}$$

$$\alpha_{1,2} = \pm \underbrace{\sqrt{\sqrt{\frac{\beta^4}{4} + \lambda} - \frac{\beta^2}{2}}}_{s_1} \equiv \pm s_1$$

$$\alpha_{3,4} = \pm i \underbrace{\sqrt{\sqrt{\frac{\beta^4}{4} + \lambda} + \frac{\beta^2}{2}}}_{s_2} \equiv \pm i s_2$$

$$V(x) \sim e^{\alpha x}$$

$\alpha_{1,2}$ $\quad V \sim e^{\pm s_1 x}$

$\alpha_{3,4}$ $\quad V \sim e^{\pm i s_2 x}$

$$V(x) = C_1 \cosh s_1 x + C_2 \sinh s_1 x + C_3 \cos s_2 x + C_4 \sin s_2 x$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

Finally, we have

$$v(x,t) = V(x)\, T(t) = \sin(\omega t + \varphi)\left[C_1 \cosh s_1 x + C_2 \sinh s_1 x + C_3 \cos s_2 x + C_4 \sin s_2 x\right]$$

Simply supported beam

BCs: $v(0,t)=0$

$v''(0,t)=0$

$v(\ell,t)=0$

$v''(\ell,t)=0$

$\Rightarrow$ $\underline{V(0)=0\,,\; V''(0)=0\,,\; V(\ell)=0\,,\; V''(\ell)=0}$

$$\boxed{\begin{aligned} \sin'' &= -\sin \\ \cos'' &= -\cos \\ \sinh'' &= \sinh \\ \cosh'' &= \cosh \end{aligned}}$$

$V(0) = C_1 + C_3 = 0$

$V''(0) = S_1^2 C_1 - S_2^2 C_3 = 0$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ S_1^2 & -S_2^2 \end{pmatrix}}_{A} \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = 0$$

$\det A = 0 \qquad -S_2^2 - S_1^2 = 0$?

$\underline{S_1^2 + S_2^2 \neq 0}$

$\underline{\boxed{\det A \neq 0}}$ So $\underline{C_1 \text{ and } C_3 = 0}$

Now

$$V(x) = C_2 \sinh s_1 x + C_4 \sin s_2 x$$

BCs at $x = l$:

$$\left.\begin{aligned} V(l) &= C_2 \sinh s_1 l + C_4 \sin s_2 l = 0 \\ V''(l) &= C_2 s_1^2 \sinh s_1 l - C_4 s_2^2 \sin s_2 l = 0 \end{aligned}\right\}$$

for C_2 and C_4

$$\underbrace{\begin{pmatrix} \sinh s_1 l & \sin s_2 l \\ s_1^2 \sinh s_1 l & -s_2^2 \sin s_2 l \end{pmatrix}}_{B} \begin{pmatrix} C_2 \\ C_4 \end{pmatrix} = 0$$

$\underline{\det B - ?}$ $\qquad$ $\underline{\det B = 0}$

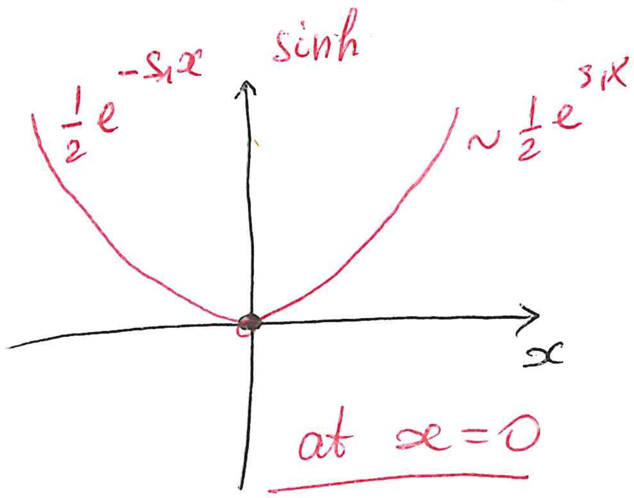
$$-\sinh s_1 l \, s_2^2 \sin s_2 l - \sin s_2 l \, s_1^2 \sinh s_1 l = 0$$

$$\sin s_2 l \sinh s_1 l \underbrace{\left[s_2^2 + s_1^2 \right]}_{\neq 0} = 0$$

$$\sin s_2 l \, \sinh s_1 l = 0$$

$$\Rightarrow \sin s_2 l = 0$$

$$\sinh s_1 l = 0$$



$$\Rightarrow \underline{\sin s_2 l = 0}$$

$$s_2 l = \pi n, \quad n = 1, 2, \ldots$$

$$s_2 = \frac{\pi n}{l}$$

$$s_2 = \sqrt{\sqrt{\frac{\beta^4}{4} + \lambda} + \frac{\beta^2}{2}} \qquad (*)$$

We have to analyze : λ, $\beta \sim P$

Let us find λ from $(*)$.

$$s_2^2 = \sqrt{\frac{\beta^4}{4} + \lambda} + \frac{\beta^2}{2}$$

$$s_2^2 - \frac{\beta^2}{2} = \sqrt{\frac{\beta^4}{4} + \lambda}$$

$$\left(s_2^2 - \frac{\beta^2}{2}\right)^2 = \frac{\beta^4}{4} + \lambda$$

$$\lambda = \left(s_2^2 - \frac{\beta^2}{2} \right)^2 - \frac{\beta^4}{4}$$

$$\lambda = s_2^4 - s_2^2 \beta^2 + \cancel{\frac{\beta^4}{4}} - \cancel{\frac{\beta^4}{4}}$$

$$= s_2^2 \left(s_2^2 - \beta^2 \right)$$

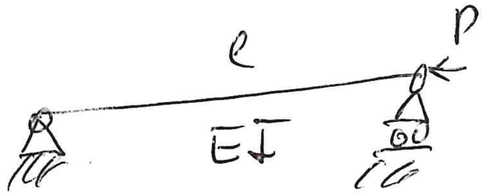
$$\boxed{s_2 = \frac{\pi n}{\ell}}$$

$$\lambda = \frac{\pi^2 n^2}{\ell^2} \left[\frac{\pi^2 n^2}{\ell^2} - \frac{P}{EJ} \right]$$

$$\boxed{\beta^2 = \frac{P}{EJ}}$$

$$\lambda = \frac{\pi^2 n^2}{\ell^2} \cdot \frac{\pi^2 n^2}{\ell^2} \left[1 - \frac{P}{EJ} \frac{\ell^2}{\pi^2 n^2} \right]$$

$$\boxed{\lambda = \frac{\pi^4 n^4}{\ell^4} \left(1 - \frac{P}{P_n} \right)}$$



$$P_n = \frac{\pi^2 n^2 EJ}{\ell^2}$$

$$v(x,t) = V(x)\,T(t)$$

$$= \left[c_2 \sinh s_1 x + c_4 \sin s_2 x\right] \sin(\omega t + \varphi)$$

$$\boxed{\omega = \sqrt{\frac{\lambda}{\rho}}}$$

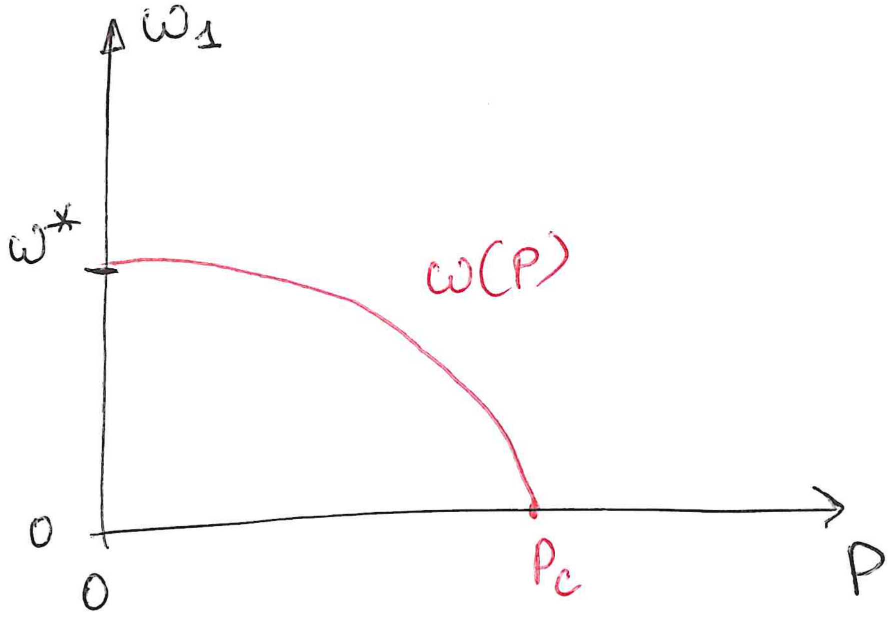
$$\omega = \omega_n = \frac{\pi^2 n^2}{\ell^2 \sqrt{\rho}} \sqrt{1 - \frac{P}{P_n}}$$

Ordered set of critical forces : $P_1, P_2, P_3 \dots$

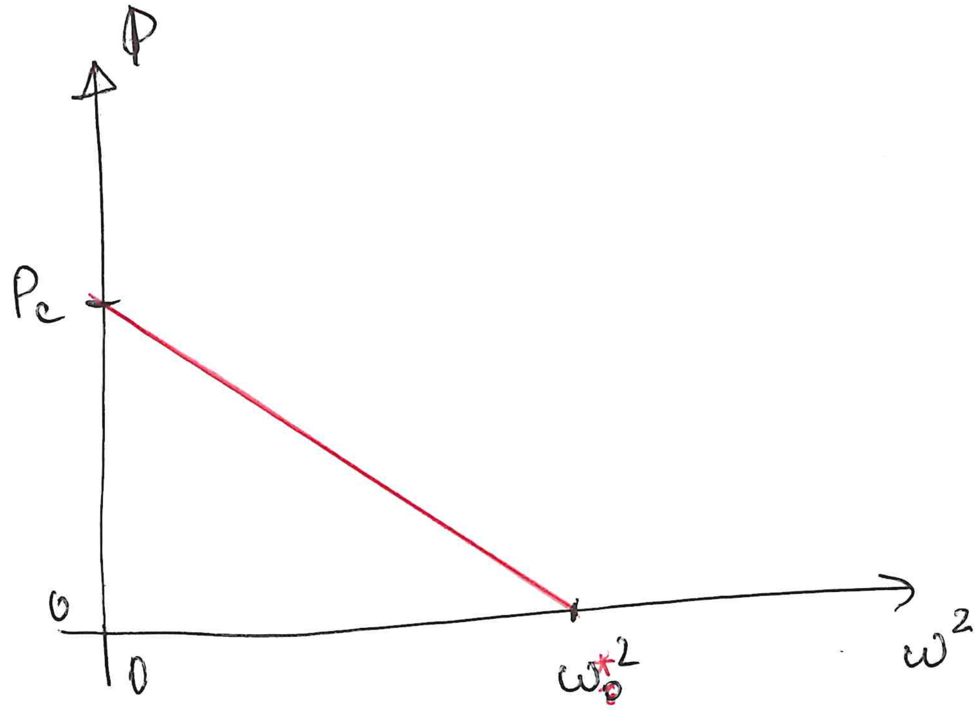
$$\underline{\min_n P_n = P_c} \quad , \quad P_c = P_1 = \frac{\pi^2 EJ}{\ell^2}$$

$$\boxed{\omega_1 = \frac{\pi^2}{\ell^2 \sqrt{\rho}} \sqrt{1 - \frac{P}{P_c}}}$$

<u>How ω depends on P ?</u>



$$\omega^* = \frac{\pi^2}{\ell^2\sqrt{\rho}}$$



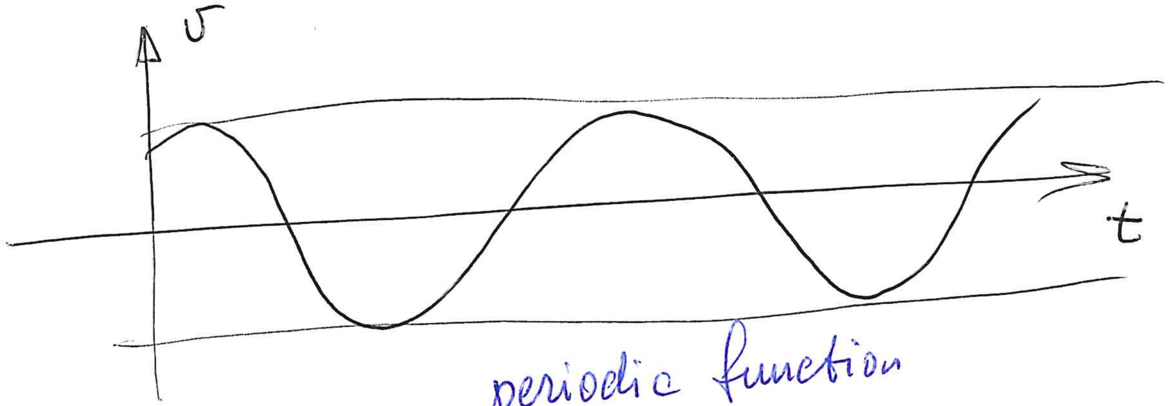
$$\omega = \frac{\pi^2}{l^2\sqrt{\rho}}\sqrt{1-\frac{P}{P_c}}$$

ω is a real number if $P < P_c$

ω is zero $P = P_c$

ω is an imaginery number if $P > P_c$

Let us consider $v(\frac{l}{2},t) = V(\frac{l}{2})\sin(\omega t + \varphi)$



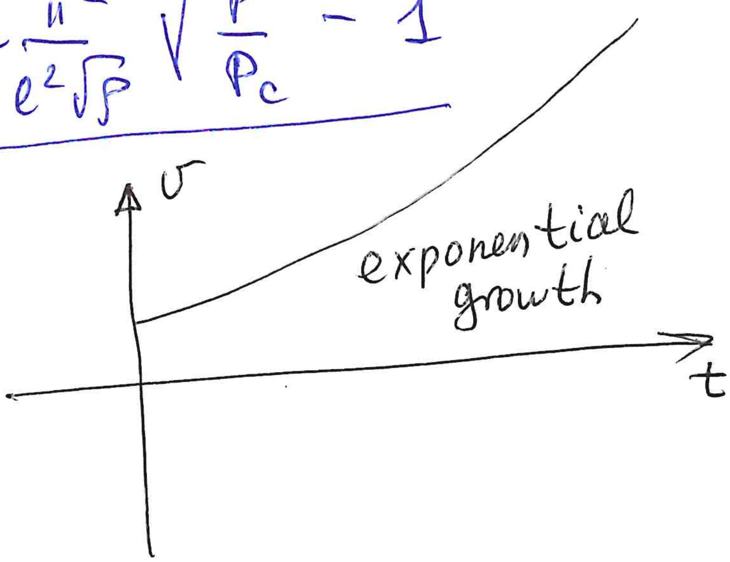
periodic function

$$v(\tfrac{l}{2},t) \sim e^{i\omega t}$$

$P > P_c$ $\qquad \omega = i\frac{\pi^2}{l^2\sqrt{\rho}}\sqrt{\frac{P}{P_c} - 1}$

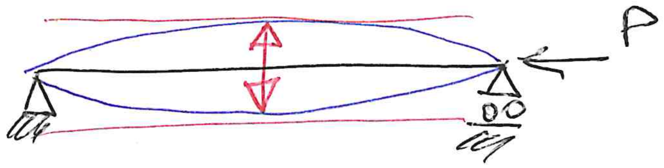
$$v(\tfrac{l}{2},t) \sim e^{|\omega| t}$$

$\Rightarrow$ instability!

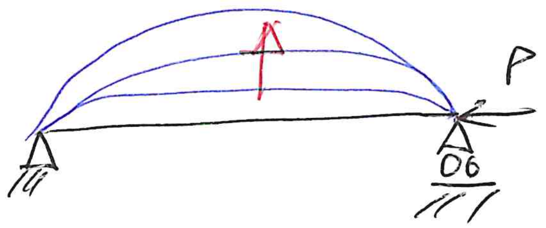


1. $P < P_c$ $\quad v \sim e^{i\omega t}$

2. $P > P_c$ $\quad v \sim e^{|\omega| t}$

1. 

bounded oscillations

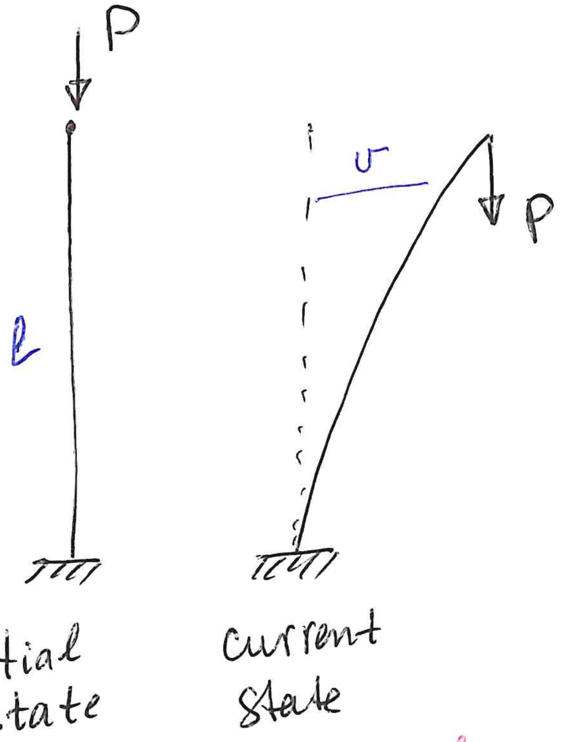
2. 

exponential growth

Conclusions: Static and dynamic criteria are equivalent.

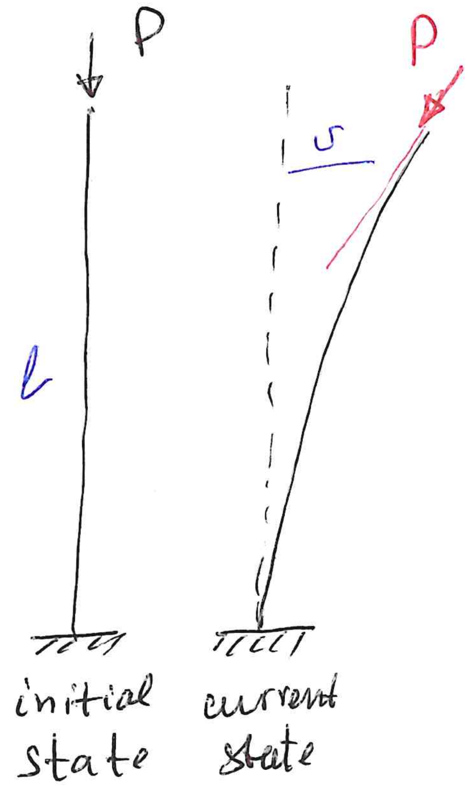
This is true if and only if our system is conservative.

Follower forces



force is vertical

dead force

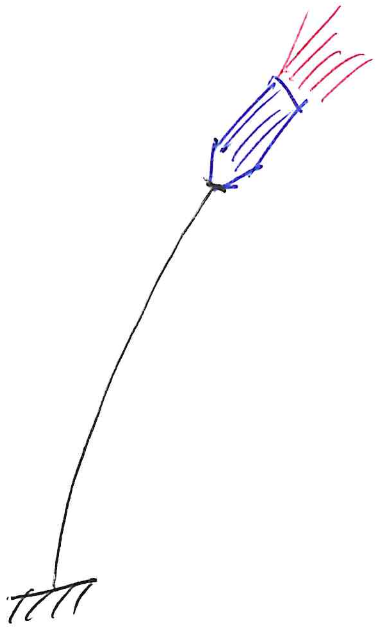


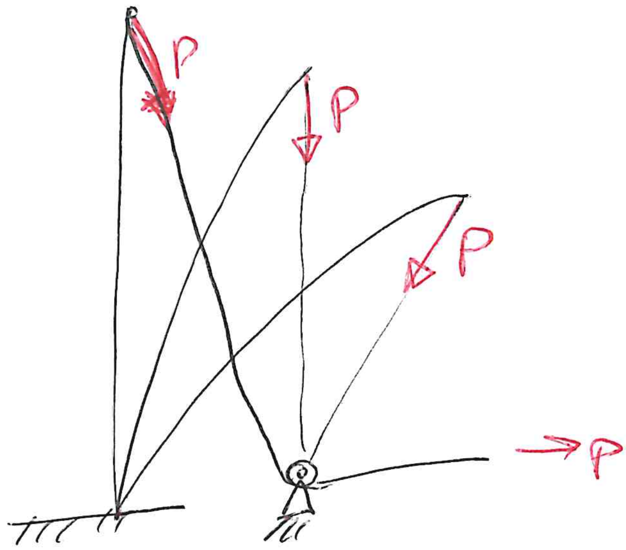
force is tangent

follower force

$EIv^{(4)} + Pv'' = 0 / -m\ddot{v}$	$EIv^{(4)} + Pv'' = 0 / -m\ddot{v}$
BCs: $x=0$: $v(0)=0,\ v(0)=0$	$x=0$; $v(0)=0,\ v'(0)=0$
$x=l$: $EIv''(l)=0$	$x=l$: $EIv''(l)=0$
$EIv'''(l) + \underline{Pv'(l)} = 0$	$EIv'''(l) = 0$

Examples of follower forces





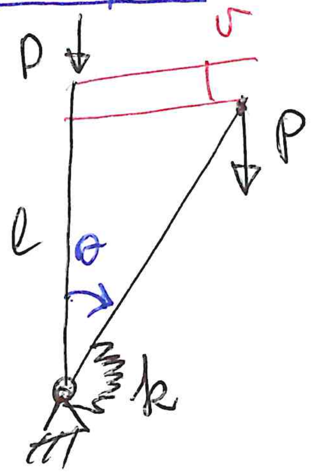
follower forces

dead forces

non-conservative

conservative

Example



1. Energy

$$\Pi = U + V$$

$$U = \frac{k}{2}\theta^2$$

$$V = -Pv$$

$$v = l - l\cos\theta \approx l(1-\cos\theta)$$

$$\approx l\frac{\theta^2}{2}$$

$\delta\Pi = 0$ $\quad \delta\Pi = \Pi'\delta\theta \Rightarrow \Pi' = 0$

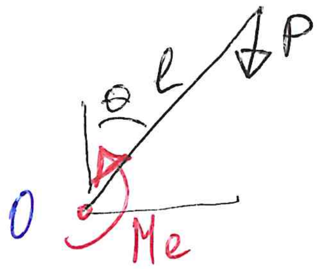
$$k\theta - Pl\sin\theta = 0,$$

linearized equation

$$(k - Pl)\theta = 0$$

$$\boxed{P_c = \frac{k}{l}}$$

2. Static criterium



$$M_e = M_P$$

$$M_e = k\theta$$

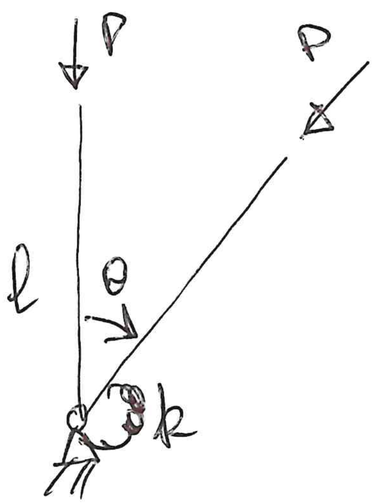
$$M_P = Pl\sin\theta$$

Equilibrium equation

$$k\theta = Pl\sin\theta$$

Linearization $\rightarrow$ $(k - Pl)\theta = 0$

$$\Rightarrow \boxed{P_c = \frac{k}{l}}$$



1. Energy ?

$$U = \frac{1}{2} k \theta^2$$

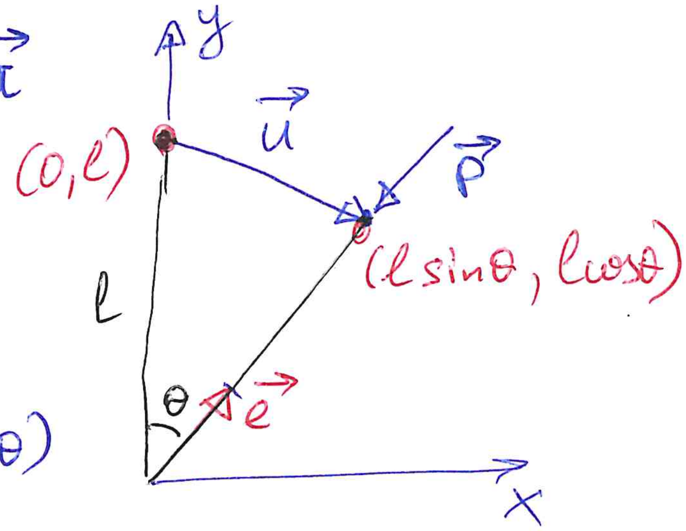
V - ?

Former time $V = -P v$

v was a displacement in direction of P

Now as the bar ~~is~~ rigid there is no displacement in direction of P. $\Delta = 0 \Rightarrow V = 0$

Proof. $\delta V = -\vec{P} \cdot \delta \vec{u}$



$\vec{e}$: $|\vec{e}| = 1$

$\vec{e} = (\sin\theta, \cos\theta)$

$\vec{P} = -P\vec{e} = -P(\sin\theta, \cos\theta)$

$\vec{u} = l(\sin\theta, \cos\theta - 1)$

$\delta\vec{u} = l(\cos\theta\,\delta\theta, -\sin\theta\,\delta\theta) = l\,\delta\theta(\cos\theta, -\sin\theta)$

$\delta V = -\vec{P} \cdot \delta\vec{u} = P l\,\delta\theta(\sin\theta, \cos\theta) \cdot (\cos\theta, -\sin\theta)$

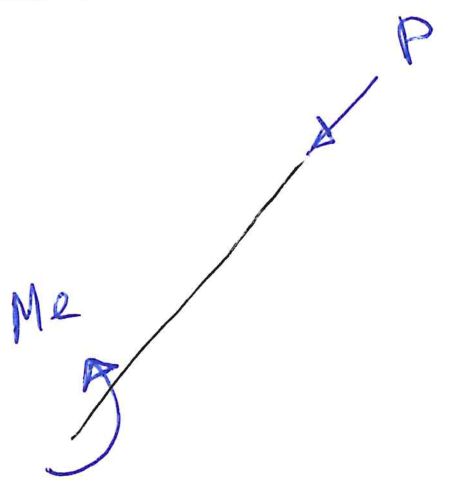
$= P l\,\delta\theta(\sin\theta\cos\theta - \cos\theta\sin\theta) = 0$

$V = 0$

$$\Pi = U = \frac{1}{2} k \theta^2$$

$$\Pi' = k\theta \quad , \quad \Pi' = 0 \quad \Rightarrow \quad \boxed{\theta = 0}$$

2. Static criterium



$$M_P \equiv 0$$

$$M_e = 0$$

$$M_e = k\theta = 0$$

$$\boxed{\theta = 0}$$