Variational approach.

$$EI v^{(4)} + P v'' = 0 \qquad x \in [0, l]$$

BCs: kinematic : $v = 0$, $v' = 0$ at some points

static : $M = 0$, $V = 0$

if we have nonhomogeneous beam the equilibrium equation becomes

$$(EI v'')'' + P v'' = 0$$

If $N = N(x)$ we have even more ..

$$(EI v'')'' - (N v')' = 0 \qquad \boxed{N = -P}$$

Let us assume that we have P_c and v_c

P_c - critical force

v_c - buckling mode

Integral identity :

$$EI\,v^{(4)} + Pv'' = 0$$

$v = v_c$, $P = P_c$ it is identity

Let us multiply by v_c and integrate ~~by~~ ~~per~~ over $(0,\ell)$ and then integrate by parts.

$$\int_0^\ell \left(EI\,v_c'''' + P_c v_c''\right) v_c\,dx = 0$$

$$f'g \rightarrow -fg'$$

$$\int_0^\ell \left[EI(v_c'')^2 - P_c(v_c')^2\right] dx = 0$$

$$\int_0^\ell EI(v_c'')^2 dx = P_c \int_0^\ell v_c'^2 dx$$

$$\Rightarrow \quad \boxed{P_c = \frac{\int_0^\ell EI(v_c'')^2 dx}{\int_0^\ell (v_c')^2 dx}}$$

Rayleigh quotient

First conclusion:

if we know buckling mode we can calculate P_c.

Second conclusion: useful for computations

$$P_{appox} = \frac{\int_0^\ell EI(v'')^2\,dx}{\int_0^\ell (v')^2\,dx}$$

for v: satisfying kinematic BCs.

$$P_c = \min_{v} P_{approx}$$

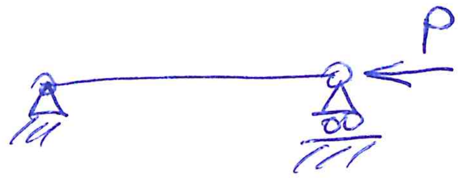
$$\boxed{P_c = \min_{v(x)} \frac{\int_0^\ell EI(v'')^2\,dx}{\int_0^\ell (v')^2\,dx}}$$

$v(x)$: kinematic BCs

$$P_c \leq P_{approx}$$

Example

EI, ℓ



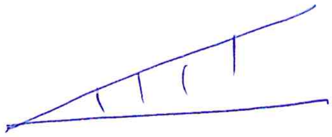
kinematic BCs:

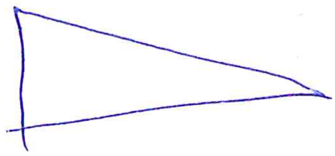
$$v(0) = v(\ell) = 0$$

$$P_c = \frac{\pi^2 EI}{\ell^2}$$

$$\ell = 1$$

1. $v(x) = x(1-x)\cancel{V_0}$





$$v' = [x - x^2]' = 1 - 2x$$

$$v'' = -2$$

$$\int_0^1 (v'')^2 dx = \int_0^1 4\,dx = 4$$

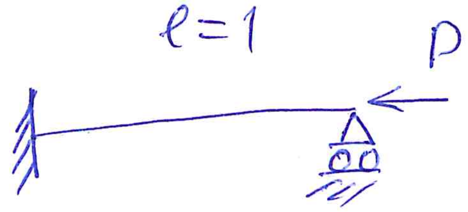
$$\int_0^{\ell=1} (1-2x)^2 dx = \int_0^{\ell=1} (1 - 4x + 4x^2) dx =$$

$$= \left(x - 2x^2 + \frac{4}{3}x^3\right)\Big|_0^1 =$$

$$= 1 - 2 + \frac{4}{3} = -1 + \frac{4}{3} = \frac{1}{3}$$

$$P_{approx} = \frac{EI \cdot 4}{\frac{1}{3}} = 12\,EI$$

$$P_c = P_{exact} = \pi^2 EI \simeq 9.9\,EI$$



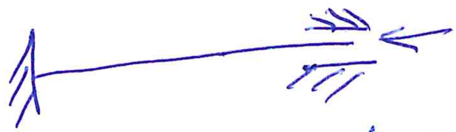
<u>Examples:</u>

trial function:

$v = x^2(1-x)$

$v(0)=0, \; v'(0)=0$

$v(1)=0$



$v = x^2(1-x)^2$

$v(0)=0, \; v'(0)=0$

$v(l)=0, \; v'(l)=0$

$$R(v) = \frac{\int_0^l EI(v'')^2 dx}{\int_0^l (v')^2 dx}$$

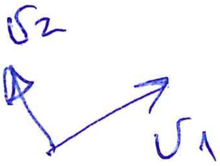
$$P_c = P_1 = \min_v R(v) \qquad v \in C_2(0,l) + \text{kin. BCs}$$

How to find next critical force?

$$P_1,\ P_2,\ P_3, \dots$$

let v_1 – the first eigen-mode.

$$P_2 = \min_v R(v) \qquad v \text{ must be onthogonal to } v_1$$



$$\int_0^l v\, v_1(x)\, dx = 0$$

L_2 orthogonality