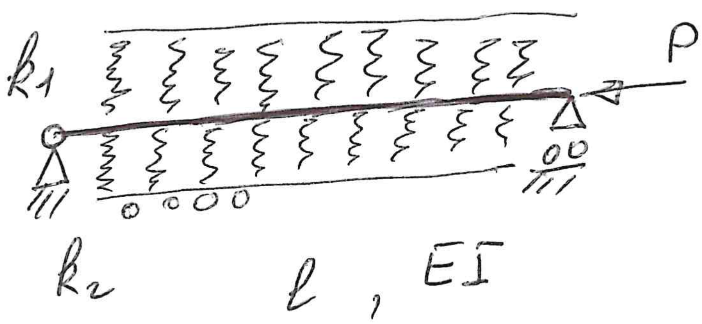
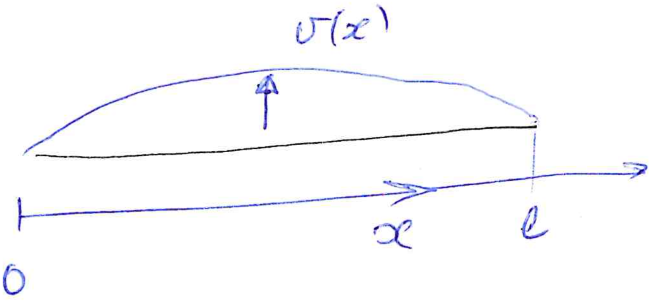
Example 1. Beams with elastic support



$\boxed{P_c - ?}$

Winkler-type model of env-t.



Springs:

$F_{sp} \sim k v$

$$EI v^{(4)} + P v'' + (k_1 + k_2) v = 0$$

BCs: $v(0) = 0$, $v(l) = 0$

$M = EI v''$: $v''(0) = 0$, $v''(l) = 0$

1) Standard way (pure mathematical):

$v \sim e^{\lambda x}$: $(\cdot)' \to \lambda (\cdot)$

$$\underbrace{EI \lambda^4 + P \lambda^2 + (k_1 + k_2) = 0}_{\text{characteristic equation}}$$

four roots: $\lambda_1, \lambda_2, \lambda_3, \lambda_4$

$$\lambda^4 + \frac{P}{EI}\lambda^2 + \frac{\tilde{k}}{EI} = 0, \qquad \tilde{k} = k_1 + k_2$$

$$\lambda^2 = \frac{-\frac{P}{EI} \pm \sqrt{\left(\frac{P}{EI}\right)^2 - 4\frac{\tilde{k}}{EI}}}{2}$$

$$\Rightarrow \lambda_{1,2,3,4} = \pm\sqrt{-\frac{P}{2EI} \pm \sqrt{\left(\frac{P}{EI}\right)^2 / 4 - \frac{\tilde{k}}{EI}}}$$

variants: 1) λ^2 – a) 2 real roots:

$$\lambda^2 = \underbrace{-\frac{P}{2EI}}_{\wedge\, 0} \pm \underbrace{\sqrt{\left(\frac{P}{2EI}\right)^2 - \frac{\tilde{k}}{EI}}}_{V_0}$$

both roots are negative

b) 2 complex roots:

$$\lambda^2 = -\frac{P}{2EI} \pm \underbrace{\sqrt{\left(\frac{P}{2EI}\right)^2 - \frac{\tilde{k}}{EI}}}_{\wedge\, 0}$$

for example, if P is too small

$\Rightarrow$ a) $\lambda_{1,2,3,4}$ – imaginary

b) $\lambda_{1,2,3,4}$ – complex number

The analysis becomes difficult

a) $v \sim \underline{\sin + \cos}$

$\lambda = i\omega \qquad \underline{A \sin\omega x + B \cos\omega x}$

b) $\lambda = \underset{\mathrm{Re}\,\lambda}{\gamma} + i\underset{\mathrm{Im}\,\lambda}{\omega}$

$v \sim e^{\gamma x}(A \sin\omega x + B\cos\omega x)\ldots$

Instead, we try to find a solution

BCs: $v(0)=0, \quad v''(0)=0, \quad v(\ell)=0, \quad v''(\ell)=0$

$$\boxed{v = V_0 \sin\left(\frac{\pi n}{\ell}x\right)}$$

$i\frac{\pi n}{\ell} \sim \lambda$

$$\left[\left(\frac{\pi n}{\ell}\right)^4 - \frac{P}{EJ}\left(\frac{\pi n}{\ell}\right)^2 + \frac{\tilde{k}}{EJ}\right] V_0 \sin\frac{\pi n}{\ell}x = 0$$

$$\boxed{\left(\frac{\pi n}{\ell}\right)^4 - \frac{P}{EJ}\left(\frac{\pi n}{\ell}\right)^2 + \frac{\tilde{k}}{EJ} = 0}$$

$$\frac{P}{EJ} = \left[\left(\frac{\pi n}{l}\right)^4 + \frac{\hat{k}^{\cancel{2}}}{EJ}\right] \Big/ \left(\frac{\pi n}{l}\right)^2 =$$

$$= \left(\frac{\pi n}{l}\right)^2 + \frac{\tilde{k}}{EJ}\frac{l^2}{(\pi n)^2}$$

n is an integer number, $n = 1, 2, 3 \ldots$

$$P = P_n = \frac{\pi^2 EJ}{l^2} n^2 + \frac{\tilde{k}\, l^2}{\pi^2 n^2}, \quad n = 1, 2 \ldots$$

$P_c = \min_n P_n$:

n^* : $P_{n^*} = P_c$, n^* depends on $\tilde{k}$

if $\tilde{k} = 0$ $\quad n^* = 1$.

if $\tilde{k} \gg 1$ $\quad n^* \neq 1$

Critical mode is not the first, in general.

difference

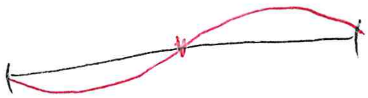
$n = 1$

$\sin \pi x / \ell$

one wave

$n = 2$

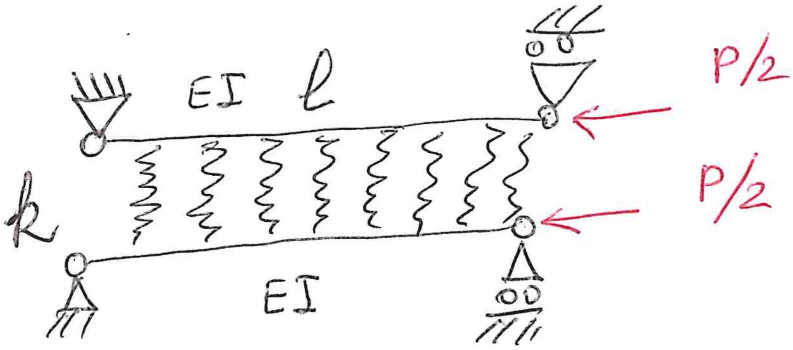
$\sin 2\pi x / \ell$



two waves

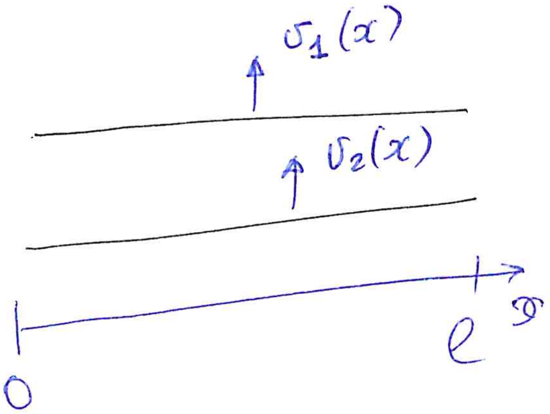
(1)

Example 2. Two beams with elastic core (composite beam)



P_c - ?

Sandwich beam



$\Rightarrow$ $v_1(x)$, $v_2(x)$!

Equilibrium equations

Π – total potential energy. $\delta\Pi = 0$

$$\Pi = \int_0^\ell \Big[\underbrace{\frac{1}{2} EI (v_1'')^2 + \frac{1}{2} EI (v_2'')^2}_{\text{bending energies of the first and } 2^{nd} \text{ beams}} -$$

$$- \underbrace{\frac{P}{2}\frac{1}{2}(v_1')^2 - \frac{P}{2}\frac{1}{2}(v_2')^2}_{\text{from external forces}} +$$

$$+ \underbrace{\frac{k}{2}(v_1 - v_2)^2}_{\text{energy of the core}} \Big] dx$$

(2)

$\delta \Pi - ?$

$$\delta \Pi = \int_0^l \Big[EI\, v_1'' \delta v_1'' + EI\, v_2'' \delta v_2'' - \frac{P}{2} v_1' \delta v_1' - \frac{P}{2} v_2' \delta v_2' + k (v_1 - v_2)(\delta v_1 - \delta v_2) \Big] dx$$

Differentiating by parts :

$$v_1' \delta v_1' \rightarrow - v_1'' \delta v_1$$

$$v_1'' \delta v_1'' \rightarrow - v_1''' \delta v_1' \rightarrow v_1^{(4)} \delta v_1$$

$$\delta \Pi = \int_0^l \Big[EI\, v_1^{(4)} \underline{\delta v_1} + EI\, v_2^{(4)} \delta v_2 + \frac{P}{2} v_1'' \underline{\delta v_1} + \frac{P}{2} v_2'' \delta v_2 + k (v_1 - v_2) \underline{\delta v_1} - k (v_1 - v_2) \delta v_2 \Big] + \ldots = 0$$

for BCs

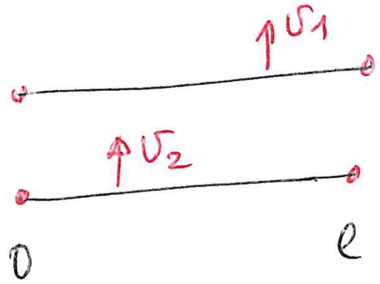
$$\left.\begin{aligned} \delta v_1 :&\quad EI\, v_1^{(4)} + \frac{P}{2} v_1'' + k (v_1 - v_2) = 0 \quad (1) \\ \delta v_2 :&\quad EI\, v_2^{(4)} + \frac{P}{2} v_2'' - k (v_1 - v_2) = 0 \quad (2) \end{aligned}\right\}$$

③ Σ : ①+② − ?

$$EI\left(v_1^{(4)} + v_2^{(4)}\right) + \frac{P}{2}\left(v_1'' + v_2''\right) = 0$$

$$w(x) = v_1 + v_2 \quad :$$

$$EI w^{(4)} + \frac{P}{2} w'' = 0$$



BCs:

$v_1(0) = 0, \quad v_1(l) = 0$

$v_1''(0) = 0, \quad v_1''(l) = 0$

$v_2(0) = 0 = v_2(l)$

$v_2''(0) = 0 = v_2''(l)$

$w = v_1 + v_2$:

$w(0) = 0 = w(l)$

$w''(0) = 0 = w''(l)$

$w = W_0 \sin \frac{\pi x}{l}$ | $w = W_n \sin \frac{\pi n x}{l}$

$n = 1, 2, 3 \ldots$

$$\boxed{P_c = 2\frac{\pi^2 EI}{l^2}}$$

④ ① − ② = ?

$$\boxed{EI\left(v_1^{(4)} - v_2^{(4)}\right) + \frac{P}{2}\left(v_1'' - v_2''\right) + 2k\left(v_1 - v_2\right) = 0}$$

$u(x) = v_1 - v_2$:

$$EI\, u^{(4)} + \frac{P}{2} u'' + 2ku = 0$$

$\sim$ beam on a Winkler foundation

BCs: $u(x)$

$$\begin{cases} u(0) = 0 = u(\ell) \\ u''(0) = 0 = u''(\ell) \end{cases}$$

$$u(x) = \bar{U}_n \sin\left(\frac{\pi n}{\ell} x\right)$$

$$EI\left(\frac{\pi n}{\ell}\right)^4 - \frac{P}{2}\left(\frac{\pi n}{\ell}\right)^2 + 2k = 0$$

$$P = P_n = 2\left(\frac{\pi n}{\ell}\right)^2 EI + \frac{4k\ell^2}{(\pi n)^2}$$

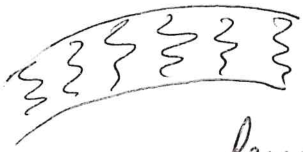
$w = v_1 + v_2$: $\quad P_c = \dfrac{2\pi^2 EI}{\ell^2} = \min\limits_n \dfrac{2\pi^2 EI}{\ell^2} n^2$

$u = v_1 - v_2$: $\quad P_c = \min\limits_n P_n,$

$$P_n = \frac{2\pi^2 EI}{\ell^2} n^2 + \frac{4k\ell^2}{\pi^2 n^2}.$$

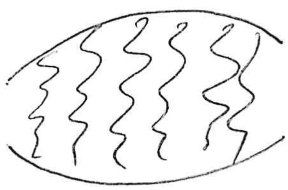
Two modes of instability

1) $w \neq 0,\ u = 0$

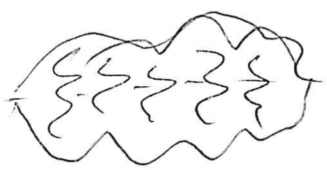


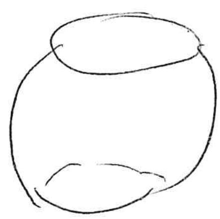
bending

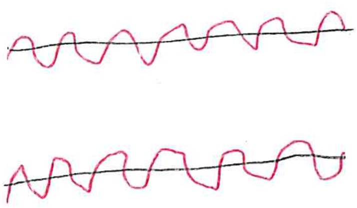
2) $u \neq 0,\ w = 0$



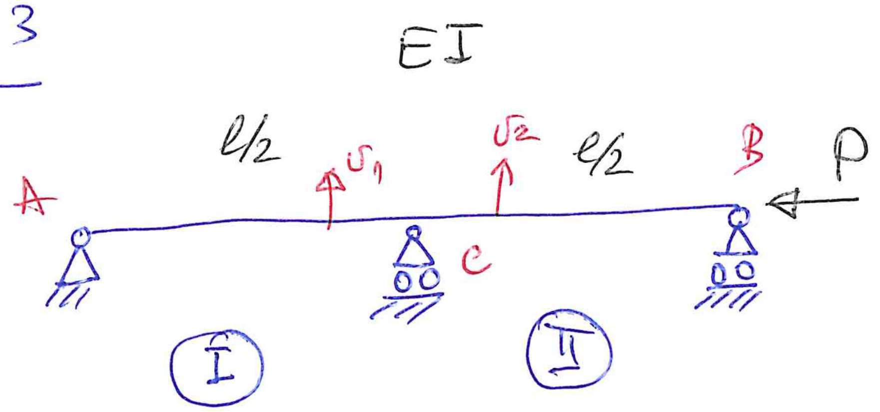
barrelling







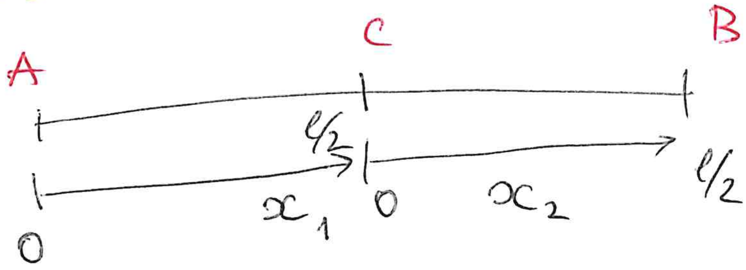
Example 3



$(P_c - ?)$

I. $EI\, v_1^{(4)} + P v_1'' = 0$

II. $EI\, v_2^{(4)} + P v_2'' = 0$



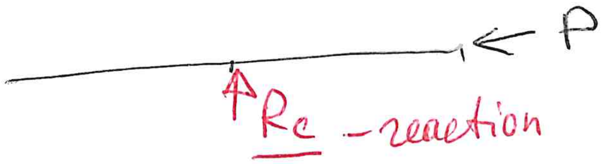
BCs : $v_1(0) = 0, \quad v_1''(0) = 0 \quad \leftarrow$ p. A

$v_2(l/2) = 0, \quad v_2''(l/2) = 0 \quad \leftarrow$ p. B

p. C $\quad v_1(l/2) = v_2(0) = 0$

$v_1'(l/2) = v_2'(0)$

$v_1''(l/2) = v_2''(0)$



$V_1 = V_2$ if no concentrated transverse force !

$V_1 + R_c = V_2$?

$v_1(\frac{l}{2}) = v_2(0)$ and forget about V_1, V_2

Standard way

$$v_1(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 x + C_4$$

$$v_2(x) = C_5 \cos\beta x + C_6 \sin\beta x + C_7 x + C_8$$

$$\beta^2 = \frac{P}{EI}$$

BCs:

$v_1(0)=0 \Rightarrow \underline{C_1 + C_4 = 0} \longrightarrow \underline{C_4 = 0}$

$v_1''(0)=0 \Rightarrow -C_1\beta^2\cos\beta\cdot 0 = 0 \quad \underline{C_1 = 0}$

v_1:

$$v_1(x) = C_2 \sin\beta x + C_3 x$$

$v_1(l/2)=0 \Rightarrow \underline{C_2 \sin\beta\frac{l}{2} + C_3\frac{l}{2} = 0}$

$$v_1'(x) = C_2\beta\cos\beta x + C_3$$

$$v_1''(x) = -C_2\beta^2\sin\beta x$$

v_2:

$v_2(0)=0 : \underline{C_5 + C_8 = 0}$

$\cancel{v_2''(0)=0}$

$v_2(l/2)=0 : \underline{C_5\cos\beta\frac{l}{2} + C_6\sin\beta\frac{l}{2} + C_7\frac{l}{2} + C_8 = 0}$

$v_2''(l/2)=0 : \underline{-C_5\beta^2\cos\beta\frac{l}{2} - C_6\beta^2\sin\frac{\beta l}{2} = 0}$

$$v_2'(0) = C_6 + C_7$$

$$v_2''(0) = -C_5\beta^2$$

$$v_1'(\ell/2) = v_2'(0)$$

$$\Rightarrow \quad \underline{c_2 \beta \cos\frac{\beta \ell}{2} + c_3 = c_6 \beta + c_7}$$

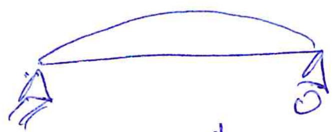
$$v_1''(\ell/2) = v_2''(0)$$

$$\Rightarrow \quad \underline{-c_2 \beta^2 \sin\frac{\beta \ell}{2} = -c_5 \beta^2}$$

Unknowns are $\underline{c_3,\ c_2,\ c_6,\ c_7,\ c_8,\ c_5}$

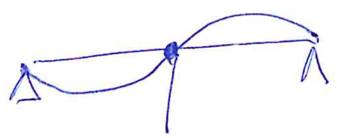
<u>6×6 matrix</u> $\underline{\det A = 0}$

1st mode



Solution coinsides with the 2nd mode of simply supported beam

2nd mode



$v''=0$

$$\boxed{P_c = \frac{4\pi^2 EI}{\ell^2}}$$