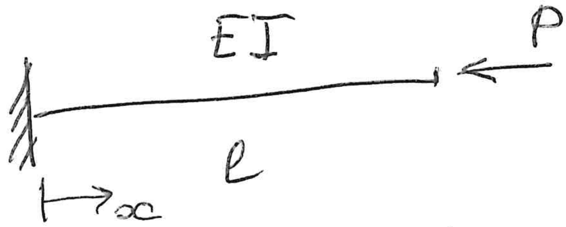
Example 2

Cantilever beam



$\boxed{P_c - ?}$

and shape

$$\underline{EI\, v^{(4)} + P v'' = 0}$$

BCs: $v(0)=0, \quad v'(0)=0$

$EI\, \underline{v''(l)=0}$

$-EI\, v'''(l) - P v'(l) = 0$

$\beta \sim P$

$$v^{(4)} + \underline{\beta}^2 v'' = 0$$

$$v(0)=0, \quad v'(0)=0, \quad v''(l)=0$$

$$v'''(l) + \underline{\beta}^2 v'(l) = 0$$

$$v(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 x + C_4$$

$$v(0) = \underline{C_1 + C_4 = 0}$$

$$v'(0) = \underline{\beta C_2 + C_3 = 0}$$

$$v''(l) = \underline{-C_1 \beta^2 \cos\beta l - C_2 \beta^2 \sin\beta l}$$

$$v'''(x) = C_1 \beta^3 \sin\beta x - C_2 \beta^3 \cos\beta l$$

$$v'(x) = -C_1 \beta \sin\beta x + C_2 \beta \cos\beta x + C_3$$

$$v(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 x + C_4$$

$$v'(x) = -C_1\beta \sin\beta x + C_2 \beta \cos\beta x + C_3$$

$$v''(x) = -C_1\beta^2 \cos\beta x - C_2\beta^2 \sin\beta x$$

$$v'''(x) = C_1\beta^3 \sin\beta x - C_2\beta^3 \cos\beta x$$

$$v(0) = \boxed{C_1 + C_4 = 0} \qquad (1)$$

$$v'(0) = \boxed{C_2\beta + C_3 = 0} \qquad (2)$$

$$v''(l) = \boxed{-C_1\beta^2\cos\beta l - C_2\beta^2 \sin\beta l = 0} \qquad (3)$$

$$v'''(l) = C_1\beta^3 \sin\beta l - C_2\beta^3\cos\beta l$$

$$(4)$$

$$v'''(l) + \beta^2 v'(l) = \boxed{C_1\beta^3\sin\beta l - C_2\beta^3\cos\beta l + \beta^2\left[-C_1\beta\sin\beta l + C_2\beta\cos\beta l + C_3\right] = 0}$$

$$\text{Eq. (4)}: \quad C_1\left[\cancel{\beta^3 \sin\beta l - \beta^3\sin\beta l}\right] + C_2\left[\cancel{-\beta^3\cos\beta l + \beta^3\cos\beta l}\right] + C_3 = 0$$

$$\Rightarrow \boxed{C_3 = 0}$$

$$\text{Eq. (2)}: \quad C_2\beta + C_3 = 0 \quad \Rightarrow \boxed{C_2 = 0}$$

Eq.(1): $C_4 = -C_1$

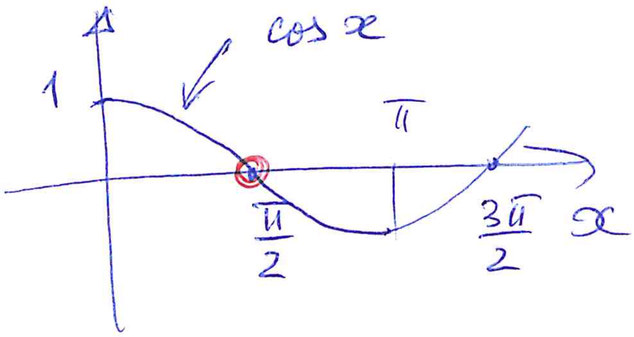
Eq.(3): $-C_1\beta^2 \cos\beta l - \underset{=0}{C_2}\beta^2 \sin\beta l = 0$

$$\boxed{C_1\beta^2 \cos\beta l = 0}$$

$\Rightarrow$ $C_1 \neq 0$ (cannot be zero otherwise $v(x) \equiv 0$!)

$\beta^2 \neq 0$

$$\boxed{\cos\beta l = 0}$$



$\beta l = \frac{\pi}{2}$

$\beta = \frac{\pi}{2l}$

$$\boxed{\beta = \sqrt{\frac{P}{EI}}}$$

$$\boxed{P = P_c = \frac{\pi^2 EI}{4l^2}}$$

In general, we have $\beta l = \frac{\pi}{2} + \underline{\pi k}$, $k = 1\ldots$

P_k - critical forces

$$\boxed{P_c = \min_k P_k}$$

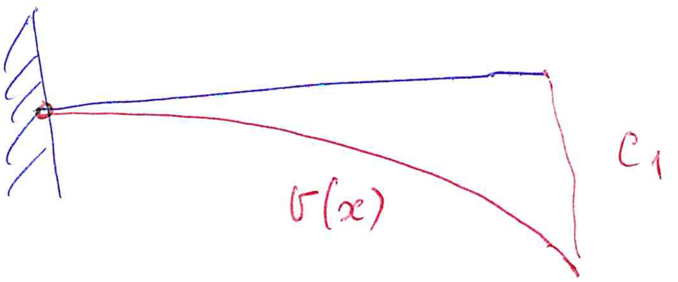
Shape ? $P_c = \dfrac{\pi^2 EI}{4\ell^2}$

$$v(x) = C_1 \cos\beta x + \cancel{C_2} \sin\beta x + \cancel{C_3} x + \underbrace{C_4}_{= -C_1} =$$

$$= C_1 \cos\beta x - C_1 =$$

$$= C_1 [\cos\beta x - 1]$$

Eigen-mode



C1 is unknown

General idea (linear theory)

$v(x) = \underbrace{c_1, c_2, c_3, c_4}_{\text{unknown}}$

Substitute this form into BCs:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \qquad (*)$$

non-zero solution is possible if

$$\det A = 0$$

we find critical parameter β

Then from $(*)$ we find $(c_1 \dots c_4)$.

For example,

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ -\beta^2 \cos\beta l & -\beta^2 \sin\beta l & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$