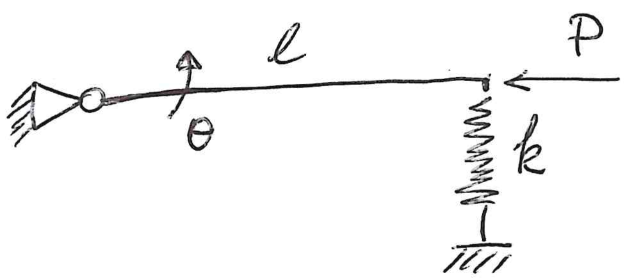
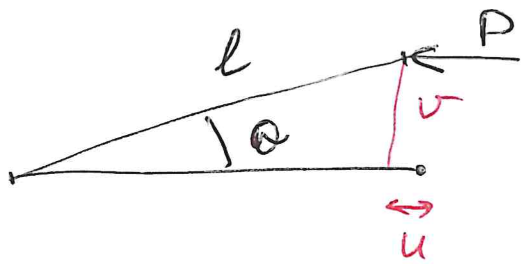
Example 1



P_c - ?

1 Energy : $\theta \approx 0$

$$\Pi = U + V, \quad U = \frac{1}{2} k v^2, \quad V = -Pu$$



$$v = \ell\theta$$

$$u = \frac{\ell}{2}\theta^2$$

$$\begin{cases} v = \ell \sin\theta \cong \ell\theta \\ u = \ell - \ell\cos\theta = \ell - \ell\left(1 - \frac{\theta^2}{2}\right) = \ell\frac{\theta^2}{2} \end{cases}$$

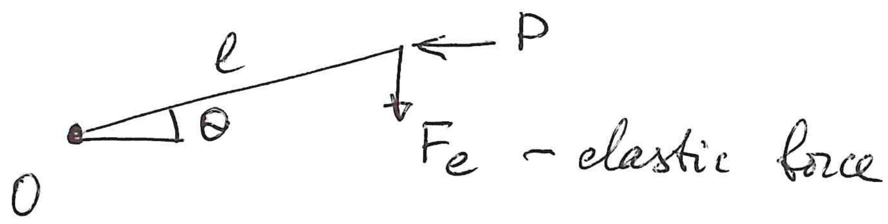
$$\Pi = \frac{1}{2} k \ell^2 \theta^2 - P\frac{\ell}{2}\theta^2$$

$$\frac{d\Pi}{d\theta} = k\ell^2\theta - P\ell\theta = 0$$

1) $\theta = 0$

2) $P = k\ell$ $\qquad$ $P_c = k\ell$

2. Static criterium



Total Moment is zero:

$$P l \sin\theta = F_e\, l \cos\theta \quad , \quad F_e = k v = k l \sin\theta$$

Linearization leads to

$$\sin\theta \to \theta$$
$$\cos\theta \to 1$$
$$F_e = k l \theta$$

Equilibrium equation

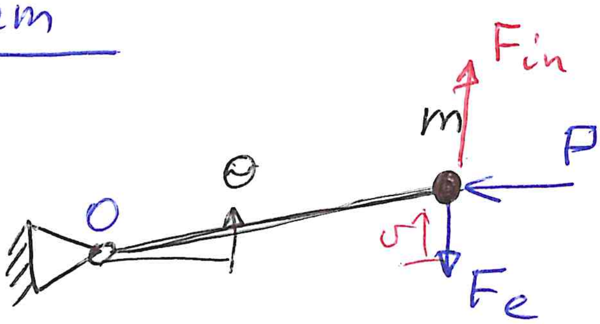
$$P l \theta = k l^2 \theta$$

$\Rightarrow$ 1) $\theta = 0$

2) $P = k l$

3. Dynamic criterium

mass !



Equation of motion ?

$\boxed{\text{d'Alembert principle}}$

$$F_{in} = -m\ddot{v}$$

$l\cos\theta$

M_O : $\quad P l \sin\theta - F_e l\cos\theta - m\ddot{v} = 0$

$v = l\sin\theta$

Linearization at $\theta = 0$:

$$Pl\theta - kl^2\theta - ml^2\ddot{\theta} = 0$$

$$ml\ddot{\theta} = (P - kl)\theta$$

$$\theta = \theta_0 e^{\lambda t}$$

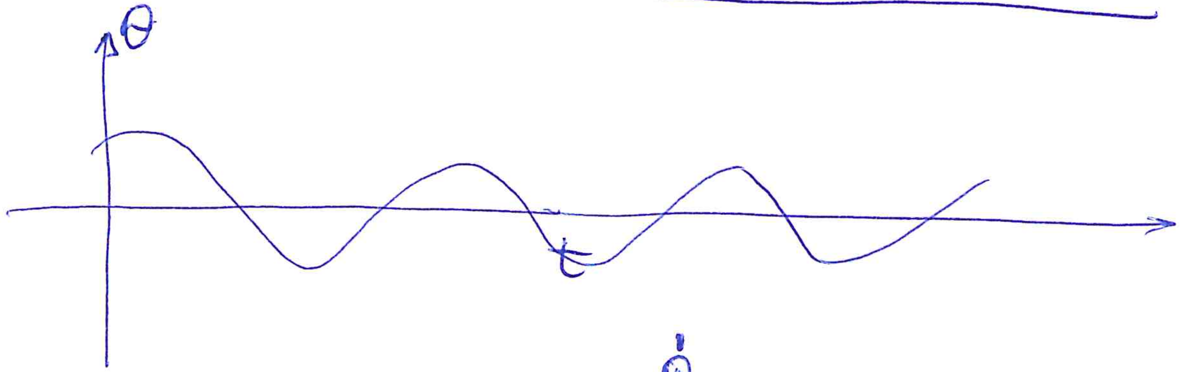
$$ml\lambda^2 = (P - kl)$$ – characteristic equation

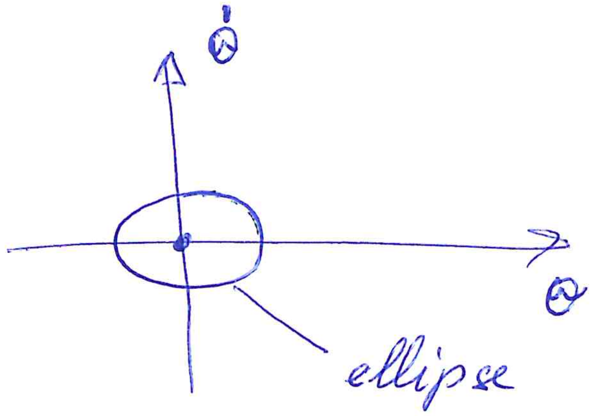
$$\lambda_{1,2} = \sqrt{\frac{P-kl}{ml}}$$

1) $P < kl \equiv P_c$

$$\lambda_{1,2} = \pm i \sqrt{\frac{P_c - P}{ml}}$$

$$\underline{\theta = A \cos\sqrt{\frac{P_c - P}{ml}}\, t + B \sin\sqrt{\frac{P_c - P}{ml}}\, t}$$





$\underline{P < kl}$ => stable state

2) $P > kl = P_c$

$$\lambda_{1,2} = \pm\sqrt{\frac{P-P_c}{ml}}$$

$$\theta = \underline{A\, e^{\sqrt{\frac{P-P_c}{ml}}\,t}} + B\, e^{\sqrt{\frac{P-P_c}{ml}}\,t}$$

$$\theta(t) \nearrow \infty \quad \text{at } t \to \infty$$

Unstable !

$$\Longrightarrow \boxed{P_c = kl}$$

---

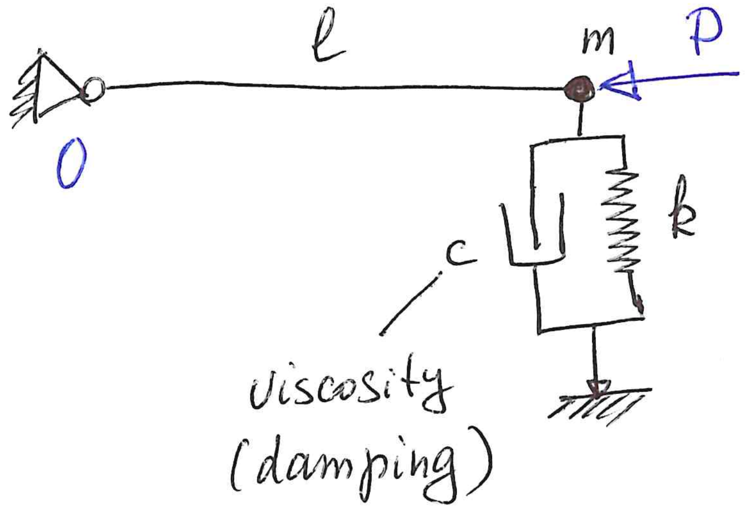
3) $P = P_c$: $\quad m\ddot{\theta} = (\underbrace{P - kl}_{=\,0})\,\theta$

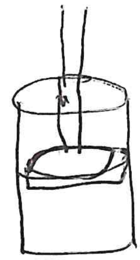
$$m\ddot{\theta} = 0$$

$$\theta = \theta_0 + \theta_1 t$$

Viscosity

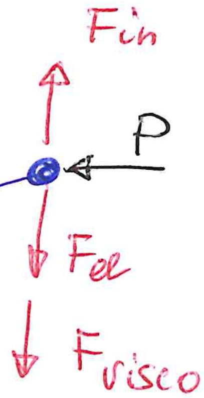
P_c - ?





$\boxed{\theta \approx 0}$

no energy (total)



M_O:

$$0 = M_P + M_{F_{in}} + M_{F_{el}} + \underline{M_{F_{visc}}}$$

new

$$F_{visco} = c\dot{v} = c l \dot{\theta}$$

$$\underbrace{m l^2 \ddot{\theta}}_{M_{F_{in}}} = \underbrace{P l \theta}_{M_P} - \underbrace{k l^2 \theta}_{M_{F_{el}}} - \underbrace{c l^2 \dot{\theta}}_{M_{F_{visco}}}.$$

static: $P l \theta - k l^2 \theta = 0$ no viscosity no inertia

quasistatic: $P l \theta - k l^2 \theta - c l \dot{\theta} = 0$

no inertia

$$m\ell\ddot{\theta} = (P - k\ell)\theta - c\ell\dot{\theta}$$

$$\theta = \theta_0 e^{\lambda t} \quad \to \quad \lambda - ? \qquad / \, \underline{\mathrm{Re}\,\lambda -} ?$$

$$m\ell\lambda^2 - (P - k\ell) + c\ell\lambda = 0$$

$$\overbrace{m\ell}^{a}\lambda^2 + \overbrace{c\ell}^{b}\lambda \overbrace{- (P - k\ell)}^{c} = 0$$

$$\boxed{\begin{gathered} ax^2 + bx + c = 0 \\ x_{\pm} = \frac{-b \pm \sqrt{D}}{2a} \\ D = b^2 - 4ac \end{gathered}}$$

$$\lambda_{1,2} = \frac{-c\ell \pm \sqrt{(c\ell)^2 + 4m\ell(P - k\ell)}}{2m\ell}$$

$$\lambda_{1,2} = \frac{-c\ell}{2m\ell} \pm \sqrt{\frac{c^2}{4m^2} - \frac{P_c - P}{m\ell}}$$

$$\underline{\underline{\mathrm{Re}\,\lambda}} - ? \qquad \gtrless 0$$

1. $P < P_c$

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\underbrace{\frac{c^2}{4m^2} - \frac{P_c - P}{m\ell}}_{< \frac{c^2}{4m^2}}} \quad \frac{P_c-P}{m\ell} > 0$$

a) c is small

$$\frac{c^2}{4m^2} - \frac{P_c}{m\ell} < 0$$

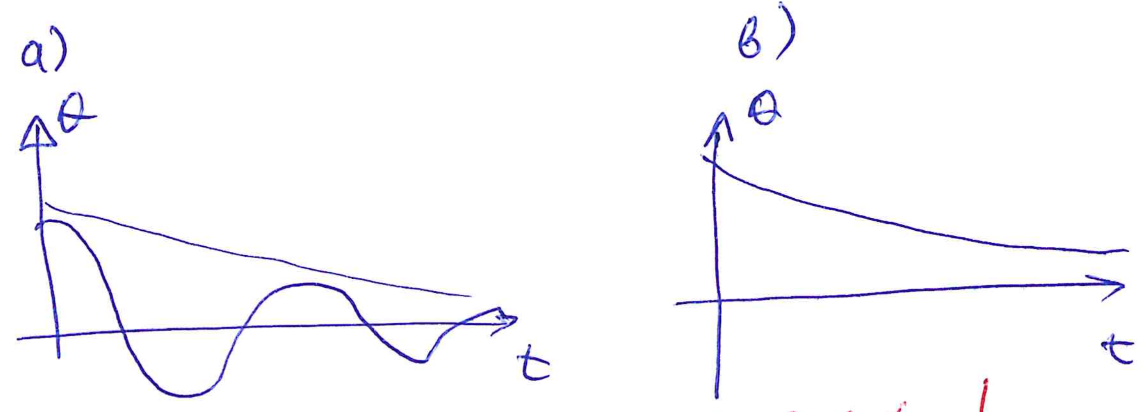
$$\lambda_{1,2} = -\frac{c}{2m} \pm i\sqrt{\frac{P_c - P}{m\ell} - \frac{c^2}{4m^2}}$$

$$\operatorname{Re} \lambda = -\frac{c}{2m} < 0$$

b) c is not small

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{P_c - P}{m\ell}}$$

$$\operatorname{Re} \lambda < 0$$



Conclusion: asymptotic stability!

2. $P > P_c$

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\underbrace{\frac{c^2}{4m^2} + \underbrace{\frac{P-P_c}{m\ell}}_{>0}}_{>\frac{c^2}{4m^2}}}$$

all roots are real.

$$\lambda_1 = -\frac{c}{2m} - \sqrt{\frac{c^2}{4m^2} + \frac{P-P_c}{m\ell}} < 0$$

$$\lambda_2 = -\frac{c}{2m} + \sqrt{\frac{c^2}{4m^2} + \frac{P-P_c}{m\ell}} > 0$$

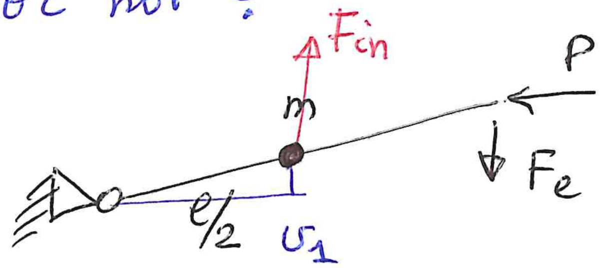
$\Rightarrow$ asymptotic instability

$P = P_c \equiv k\ell$

Conclusion: viscosity or damping stabilizes the system.

Distribution of mass

important or not ?



$$F_{in} = -m\ddot{v}_1 = -m\frac{\ell}{2}\ddot{\theta}$$

$$M_{in} = F_{in}\frac{\ell}{2}$$

Equation of motion became

$$P\ell\theta - k\ell^2\theta - m\frac{\ell^2}{4}\ddot{\theta} = 0$$

$$\ddot{\theta} = \frac{P - k\ell}{m\ell/4}\theta$$

$$P_c = k\ell$$

true mass distribution : I - moment of inertia

$$P\ell\theta - k\ell^2\theta - mI\ddot{\theta}$$

$$\ddot{\theta} = \frac{P - P_c}{mI/\ell}\theta \quad \Rightarrow P_c = k\ell$$

For conservative systems distribution of mass does not influence critical loads