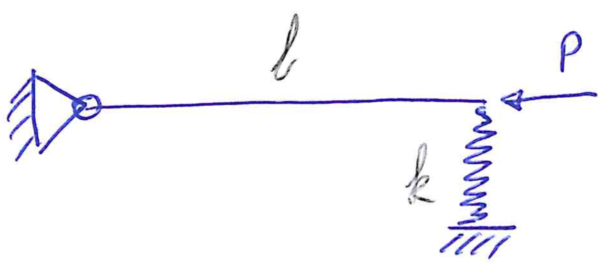
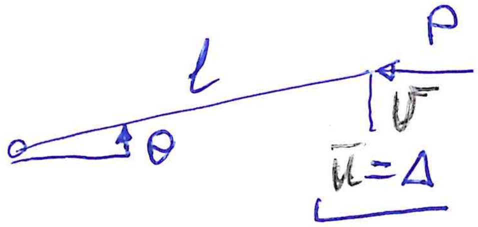
Example 1



1. P_c - ?
2. Energy analysis

$$\Pi = U + V$$

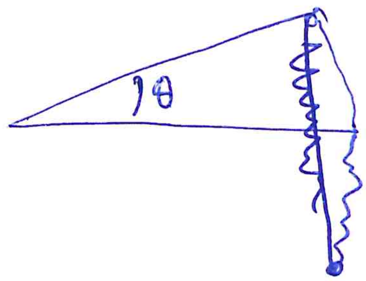


$$U = \frac{1}{2} k v$$

θ is small,

$$v \sim l \sin\theta = l\theta$$

$$\Rightarrow \boxed{U = \frac{1}{2} k l^2 \theta^2}$$



$$V = -P\Delta, \quad \Delta = l - l\cos\theta$$

$$\cos\theta = 1 - \frac{1}{2}\theta^2$$

$$V = -Pl(1 - \cos\theta) = -Pl\left(1 - 1 + \frac{1}{2}\theta^2\right) =$$

$$= -\frac{Pl}{2}\theta^2$$

$$\Rightarrow \Pi = \frac{1}{2} k l^2 \theta^2 - \frac{Pl}{2}\theta^2 = \frac{1}{2} l (kl - P)\theta^2$$

quadratic form $\Pi \sim \theta^2$

$$\frac{d\Pi}{d\theta} = l(kl - P)\theta$$

$$\frac{d\Pi}{d\theta} = 0 \quad : \quad \theta = 0 \text{ – trivial solution}$$

$$\theta \neq 0$$

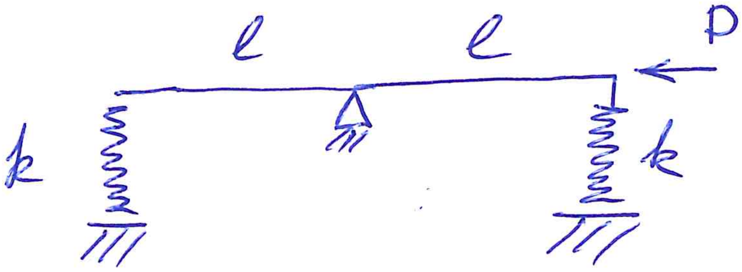
$$\boxed{P = kl = P_c} \; !$$

$$\frac{d^2\Pi}{d\theta^2} = l(kl - P)$$

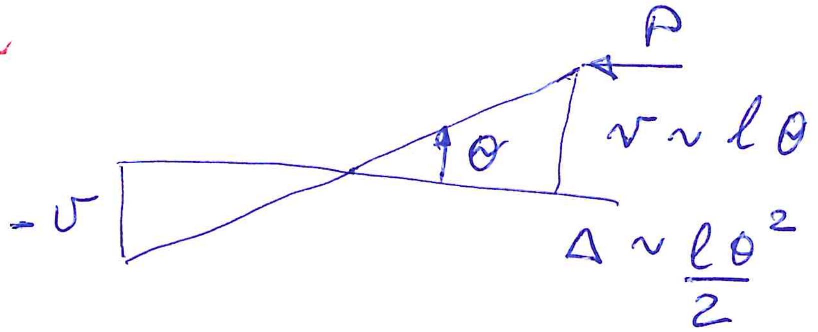
$P < P_c \equiv kl$ stable

$P > P_c$ unstable

Example 2



P_c – ?



$$\Pi = U + V \,, \quad U = \frac{1}{2} k v^2 + \frac{1}{2} k v^2$$

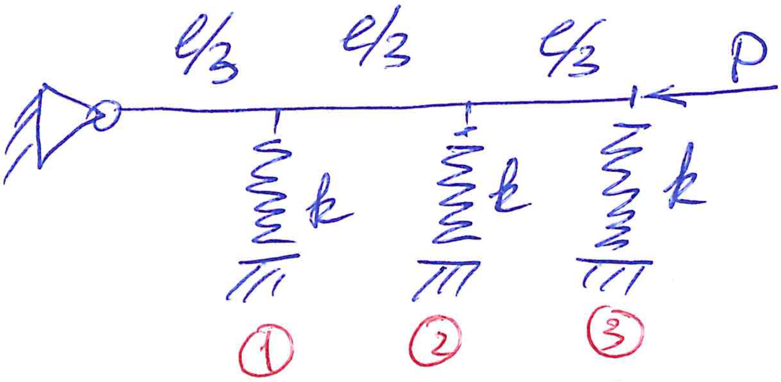
$$V = -\frac{Pl}{2}\theta^2$$

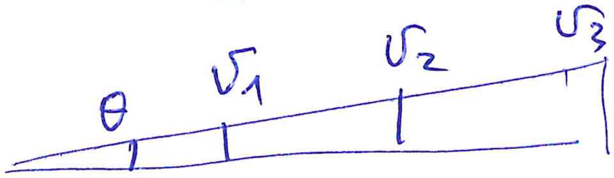
$$\Pi = k l^2 \theta^2 - \frac{Pl}{2}\theta^2$$

$$\frac{d\Pi}{d\theta} = l\,(2kl - P)\,\theta = 0 \Rightarrow \boxed{P_c = 2kl}$$

$$\frac{d^2\Pi}{d\theta^2} = l\,(2kl - P) \lessgtr 0$$

Example 3





$$v_1 = \frac{l}{3}\theta \quad , \quad v_2 = \frac{2}{3} l\theta , \quad v_3 = l\theta$$

$$U = \frac{1}{2} k v_1^2 + \frac{1}{2} k v_2^2 + \frac{1}{2} k v_3^2 =$$

$$= \frac{1}{2} k \left[\left(\frac{l}{3}\right)^2 + \left(\frac{2}{3} l\right)^2 + l^2 \right] \theta^2 =$$

$$= \frac{1}{2} k \Big[\underbrace{\frac{1}{9} + \frac{4}{9} + 1}_{14/9} \Big] l^2 \theta^2$$

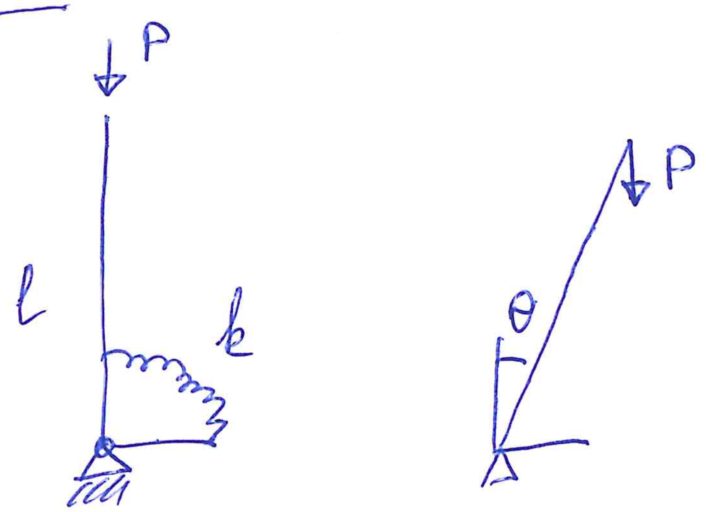
$$V = -\frac{P l \theta^2}{2}$$

$$\Pi = \frac{1}{2} \frac{14}{9} k l^2 \theta^2 - \frac{P l \theta^2}{2}$$

$$\frac{d\Pi}{d\theta} = \left(\frac{14}{9} k l^2 - P l \right) \theta = 0$$

$$\boxed{P_c = \frac{14}{9} k l}$$

Example 4

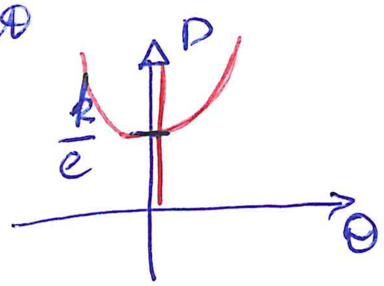


$$\Pi = \frac{1}{2} k \theta^2 - P\Delta, \qquad \Delta = l - l\cos\theta$$

$$\Pi = \frac{1}{2} k \theta^2 - Pl(1-\cos\theta)$$

$$\frac{d\Pi}{d\theta} = k\theta - Pl\sin\theta = 0$$

exact



$$\begin{cases} \theta = 0 & \text{trivial} \\ P = \frac{k}{l}\frac{\theta}{\sin\theta} \end{cases}$$

Linearized theory

$$\Delta \sim l - l\left(1 - \frac{\theta^2}{2}\right) = \frac{l\theta^2}{2}$$

$$\Pi = \frac{1}{2} k\theta^2 - \frac{Pl\theta^2}{2}$$

$$\frac{d\Pi}{d\theta} = k\theta - Pl\theta = 0 \Rightarrow \theta = 0 \text{ or } P = k/l \equiv P_c$$

"Postbuckling or next approximation"

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$$

$$y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \\ + \frac{1}{2\cdot 3}y'''(0)x^3 + \frac{1}{2\cdot 3\cdot 4}y''''(0)x^4 + \dots$$

$$V = -Pl\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4\right)$$

$$\frac{d\Pi}{d\theta} = k\theta - Pl\theta + \frac{Pl}{6}\theta^3 = 0$$

$$\frac{d\Pi}{d\theta} = \theta\left[k - Pl + \frac{Pl}{6}\theta^2\right] = 0$$

$\theta = 0$ – trivial solution

non-trivial solution:

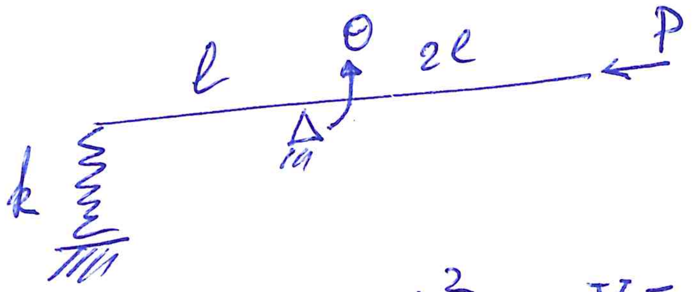
$$\frac{Pl}{6}\theta^2 - Pl + k = 0$$

$$\theta^2 = \frac{k - Pl}{Pl/6}$$

$$\theta = \pm\sqrt{\frac{k - Pl}{Pl/6}}$$

Example 5

P_c -?



$$U = \frac{k}{2}(l\theta)^2 , \quad V = -P\left[2l - 2l\cos\theta\right]$$

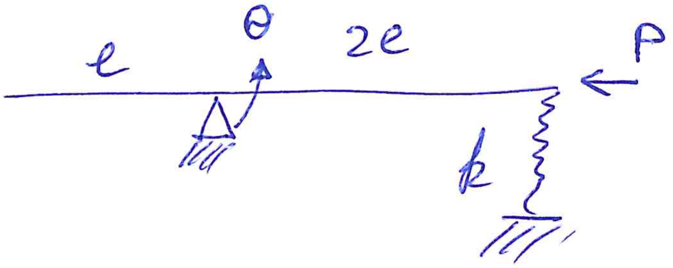
$$= -Pl\theta^2$$

$$\Pi = \frac{k}{2} l^2 \theta^2 - Pl\theta^2$$

$$\frac{d\Pi}{d\theta} = kl^2\theta - 2Pl\theta = 0$$

$$\boxed{P_c = \frac{kl}{2}} \qquad \forall \theta \neq 0$$

Example 6

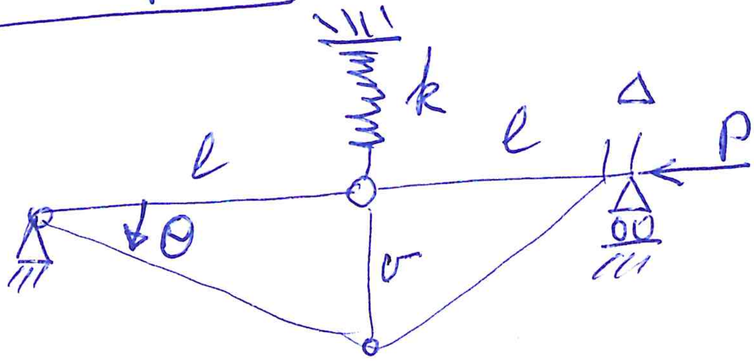


$$\Pi = \frac{1}{2} k (2\ell\theta)^2 - P\ell\theta^2$$

$$\frac{d\Pi}{d\theta} = \left[k4\ell^2 - P\ell \right] \theta = 0$$

$$\boxed{P_c = 4k\ell}$$

Example 7



P_c - ?

$\theta \approx 0$

$\cos\theta = 1 - \frac{1}{2}\theta^2 + \dots$

$$U = \frac{1}{2} k v^2 = \frac{1}{2} k (\ell\theta)^2$$

$$\Delta = 2\ell - 2\ell\cos\theta = \ell\theta^2$$

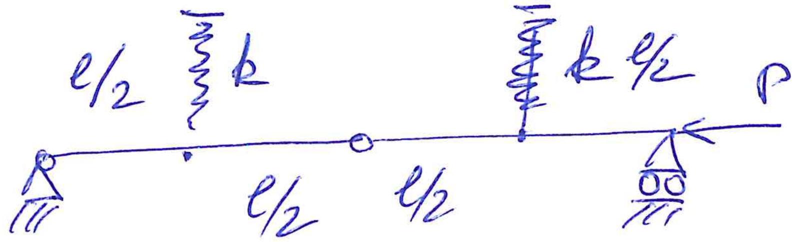
$$V = -P\Delta = -P\ell\theta^2$$

$$\Pi = \frac{1}{2} k\ell^2\theta^2 - P\ell\theta^2$$

$$\frac{d\Pi}{d\theta} = \left[k\ell^2 - 2P\ell\right]\theta = 0$$

$$\boxed{P_c = \frac{k\ell}{2}}$$

Example 8



P_c - ?

$$U = \frac{k}{2} v_1^2 + \frac{k}{2} v_2^2 \quad \Big| \quad v_1 = v_2$$

$$U = k\left(\frac{l}{2}\theta\right)^2, \quad V = -Pl\theta^2$$

$$\Pi = \left(\frac{kl^2}{4} - Pl\right)\theta^2$$

$$\frac{d\Pi}{d\theta} = 0 \Rightarrow \boxed{P_c = \frac{kl}{4}}$$