



Università degli Studi di Cagliari

UNICA  
UNIVERSITÀ  
DEGLI STUDI  
DI CAGLIARI

Facoltà di Ingegneria e Architettura

Dipartimento di Ingegneria Civile, Ambientale e Architettura

**Magistrale Ingegneria Civile 2 anno a.a. 2021/22**

# Instabilità delle strutture e calcolo a rottura

> **Lezione 17**

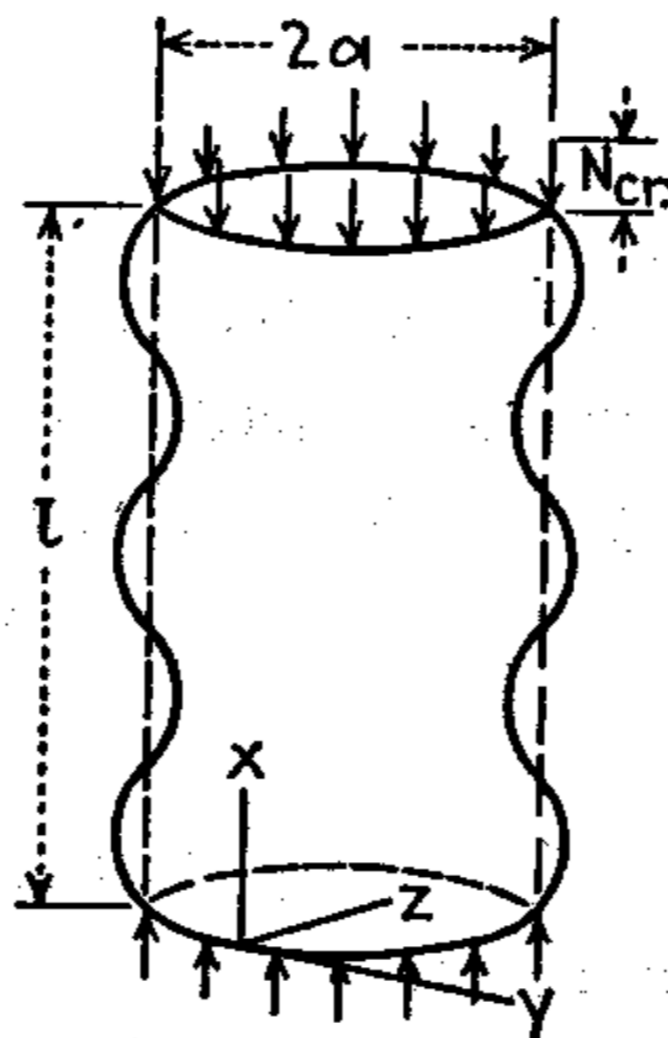
Instabilità: le lastre curve  
(tubi a cupole)

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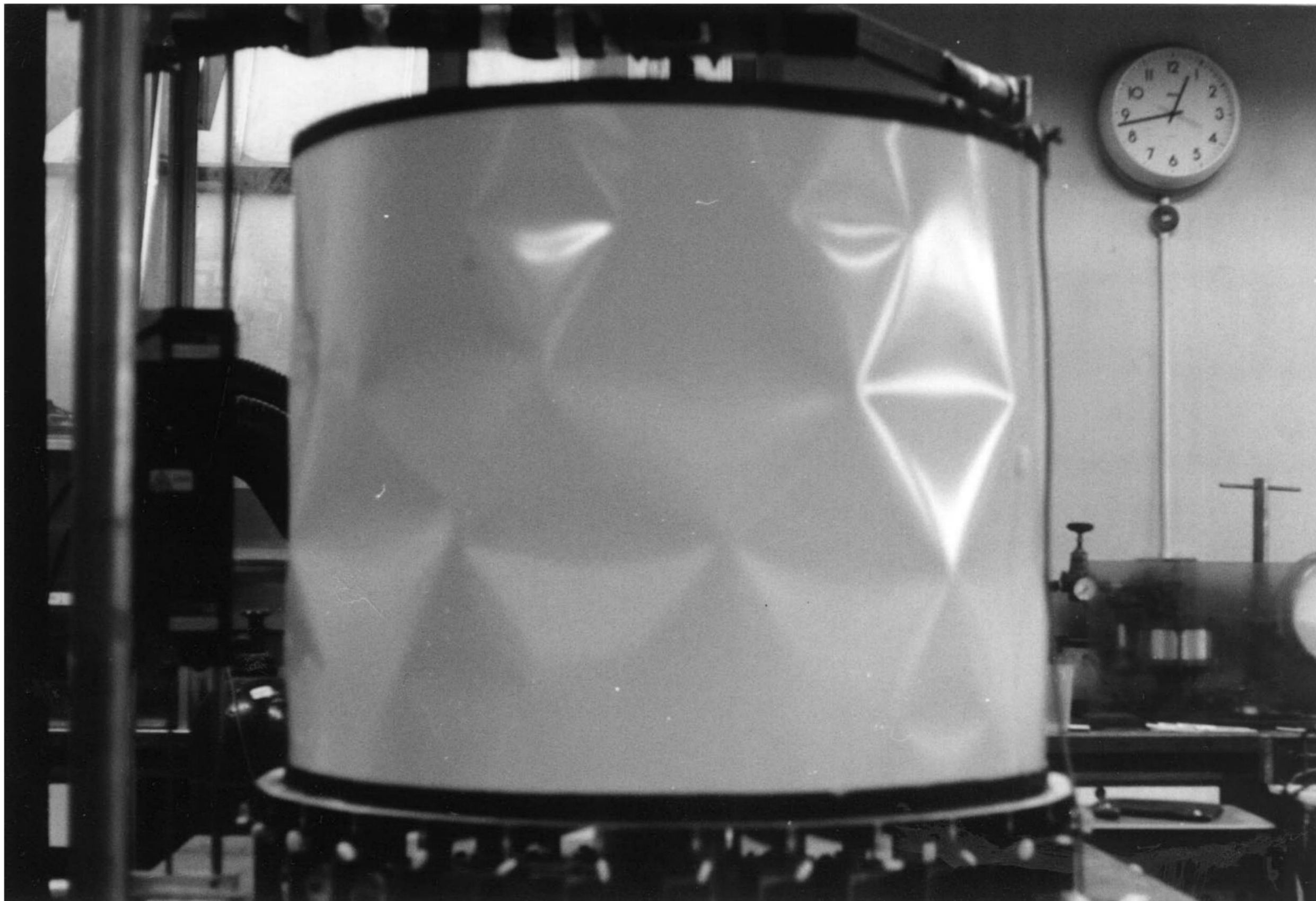
# LASTRE CILINDRICHE (TUBI)



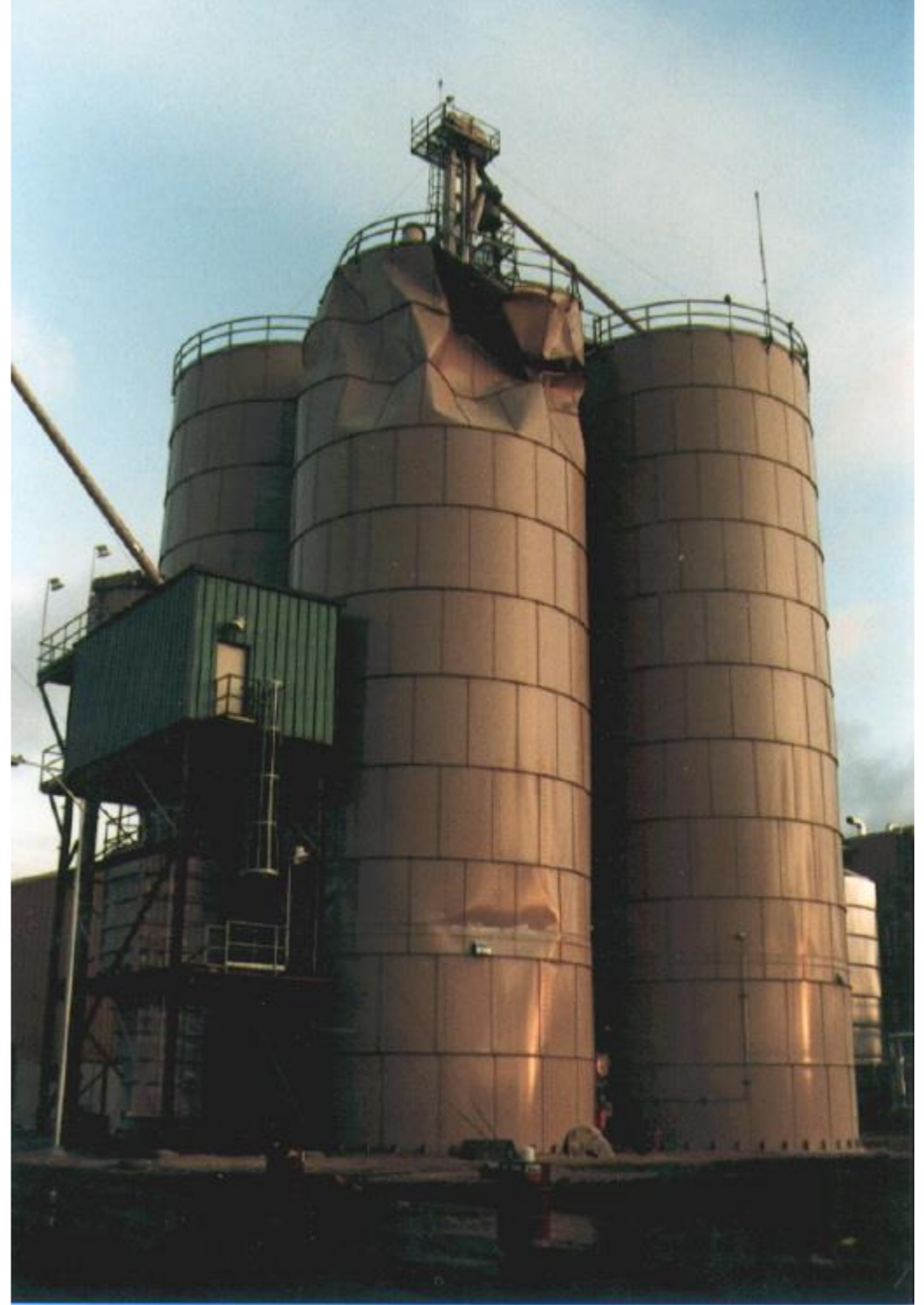




<https://www.youtube.com/watch?v=XVJrcizcdwE>



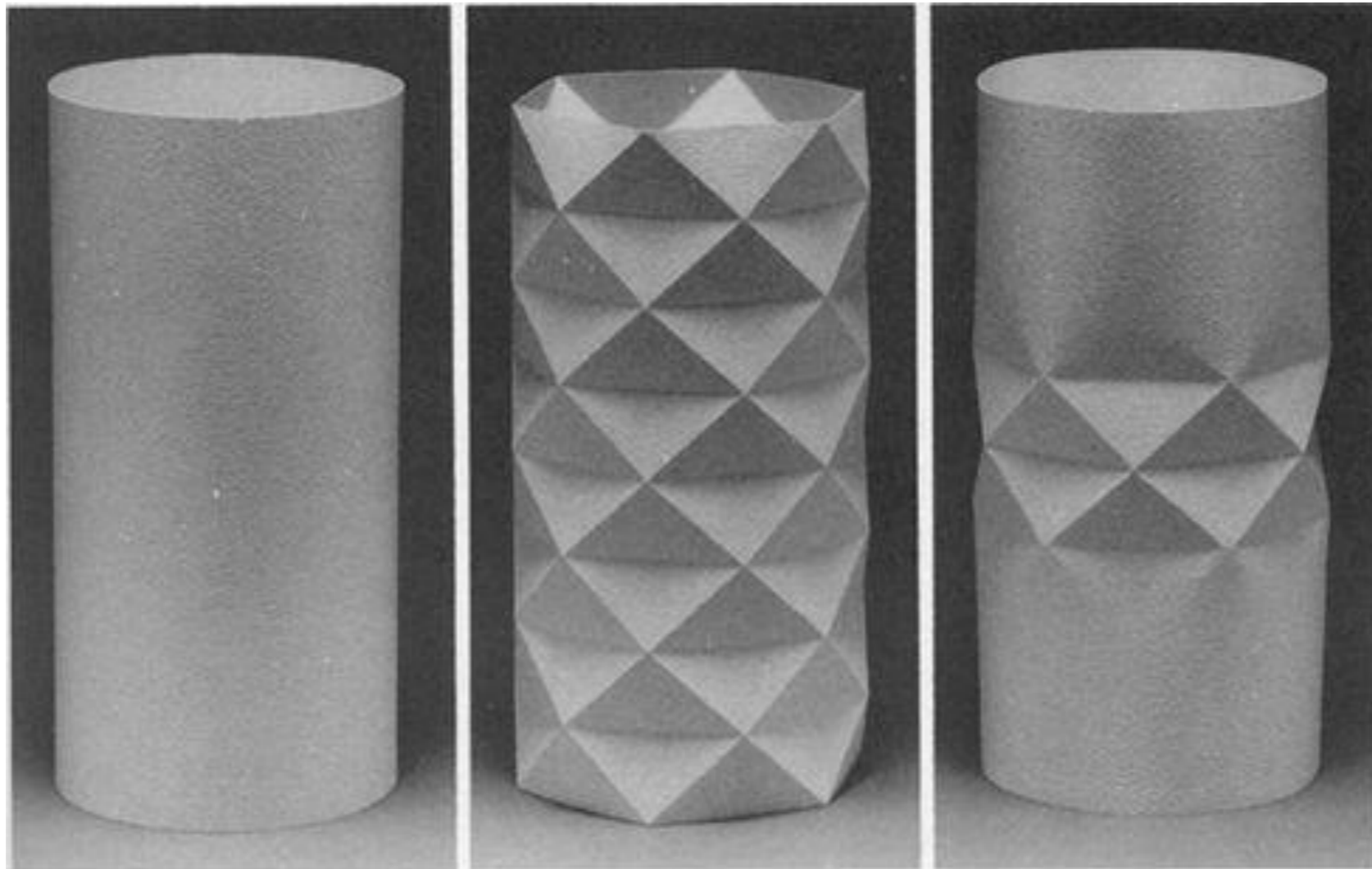


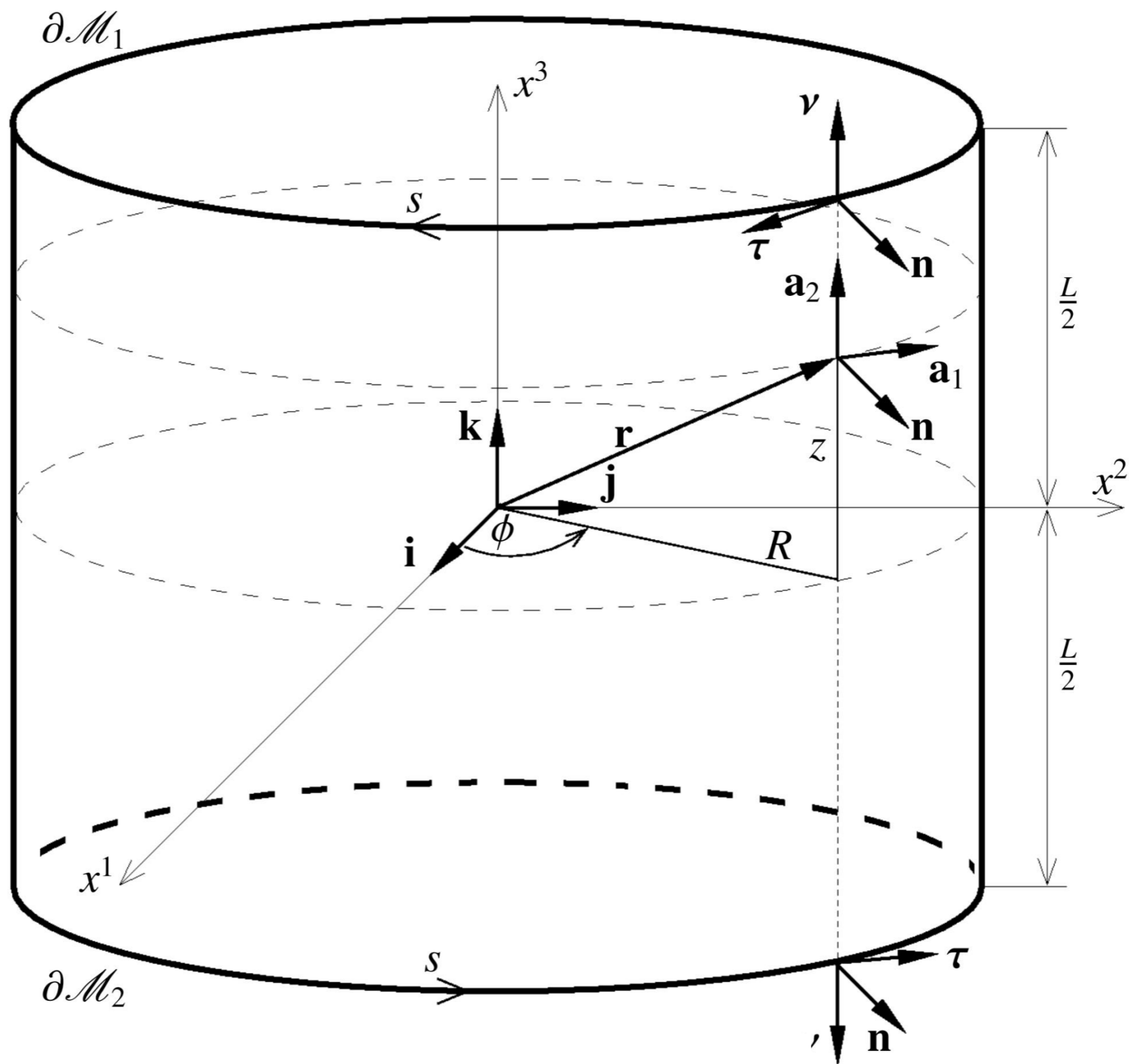


<https://shellbuckling.com/>

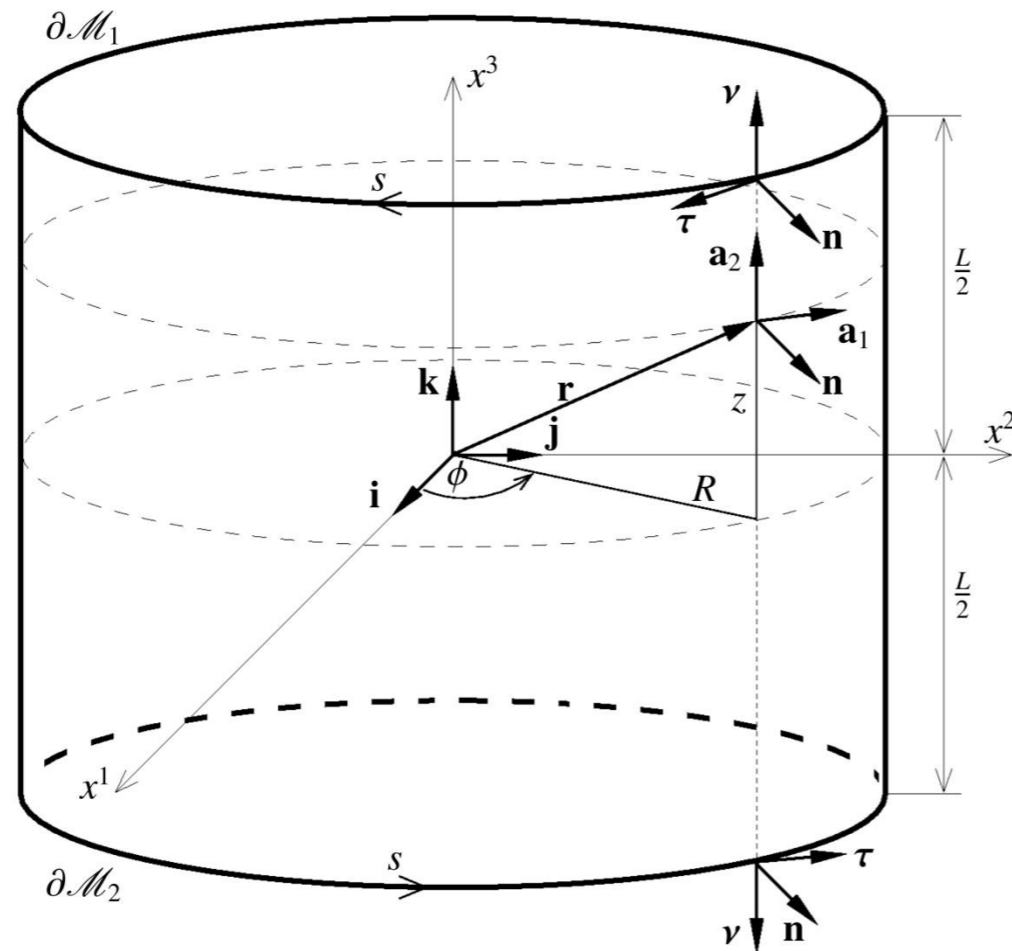
<https://www.youtube.com/watch?v=y5VRww1Ypwk>

<https://orilab.art/natural/yoshimura>





$$\mathbf{r} = R \cos \phi \mathbf{i} + R \sin \phi \mathbf{j} + z \mathbf{k} ,$$



$$\mathbf{r} = R \cos \phi \mathbf{i} + R \sin \phi \mathbf{j} + z \mathbf{k} ,$$

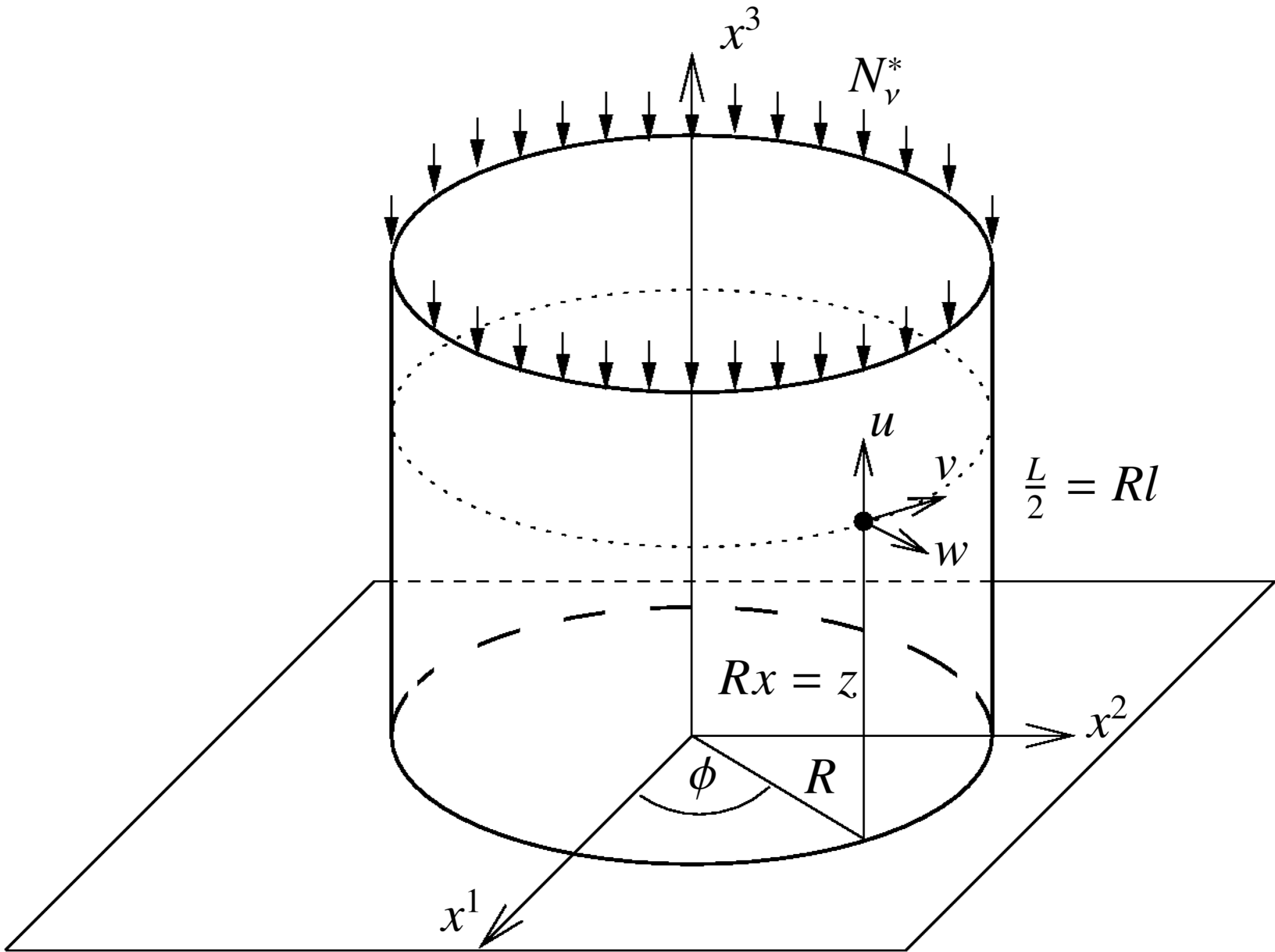
$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial \phi} = -R \sin \phi \mathbf{i} + R \cos \phi \mathbf{j} ,$$

$$\mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k} ,$$

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{n}}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{n})} = -\frac{1}{R} (\sin \phi \mathbf{i} - \cos \phi \mathbf{j}) ,$$

$$\mathbf{a}^2 = \frac{\mathbf{n} \times \mathbf{a}_1}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{n})} = \mathbf{k} ,$$

$$\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} = \cos \phi \mathbf{i} + \sin \phi \mathbf{j} .$$



## Selected theoretical models used in stability analysis:

- Flügge (1932)
- Donnell (1933)
- Mushtari-Galimov (1957)
- Sanders (1963)
- Koiter (1967)
- Budiansky (1978)
- Stumpf (1984)
- Pietraszkiewicz (1984)

# Membrane prebuckling state

Axial compressive force component  $N_\nu^* = -\frac{2\rho}{\epsilon}$  generates membrane prebuckling state

$$u_0(\phi, x) = -2\epsilon\rho(1 + 3\epsilon\rho)x$$

$$v_0(\phi, x) = 0$$

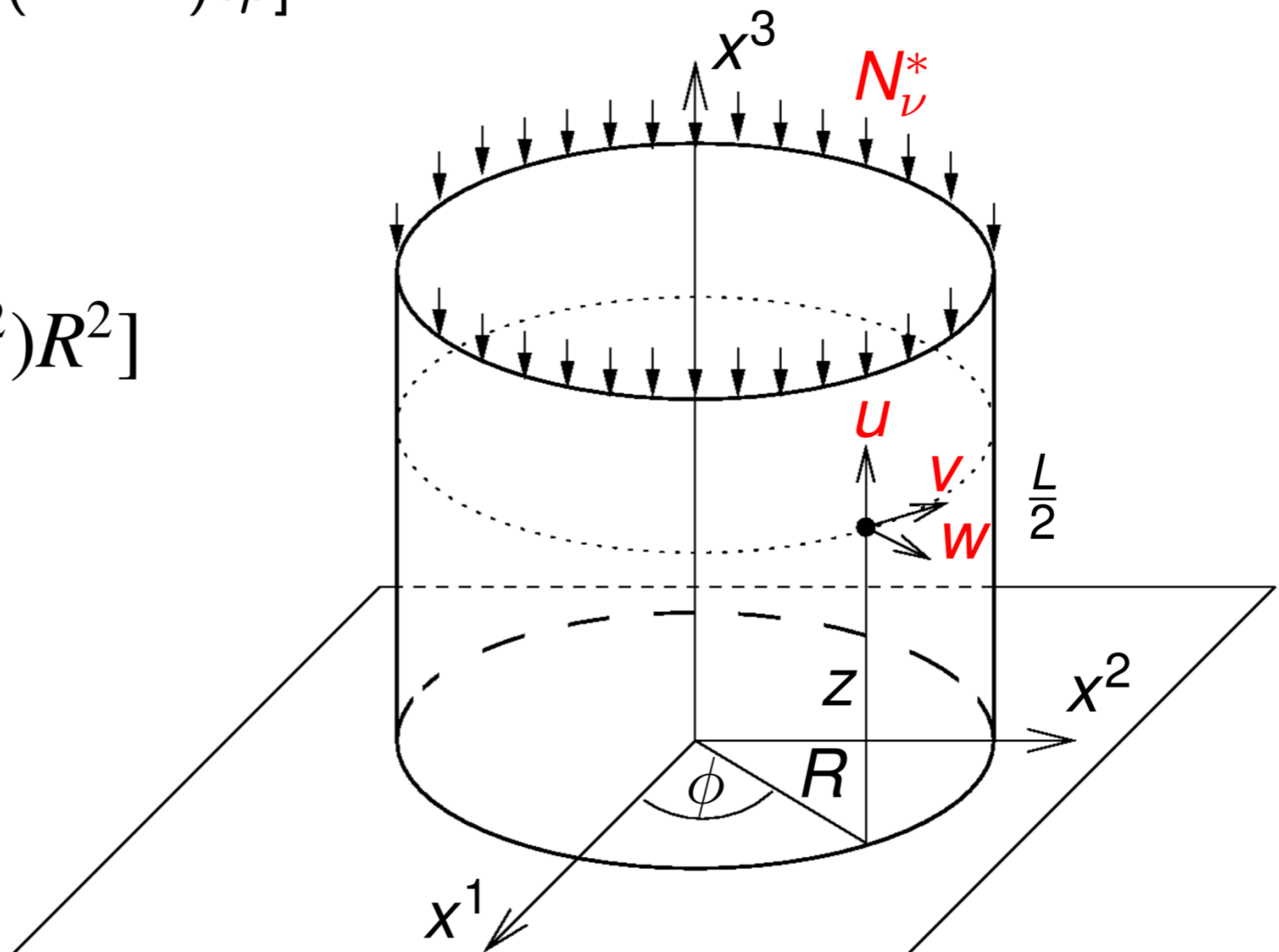
$$w_0(\phi, x) = 2\epsilon\nu\rho[1 + (2 - \nu)\epsilon\rho]$$

$$N_\nu^* = -\frac{2\rho}{\epsilon}$$

$$\epsilon^2 = h^2/[12(1 - \nu^2)R^2]$$

$$\rho = 1$$

$$\sigma_{cl} = 2\epsilon Eh.$$



$$N_1^1 = \frac{D}{R^2} N_\phi(\phi, x), \quad N_2^2 = \frac{D}{R^2} N_x(\phi, x), \quad N_1^2 = R^2 N_2^1 = \frac{D}{R} N_{x\phi}(\phi, x),$$
$$M_1^1 = \frac{D}{R} M_\phi(\phi, x), \quad M_2^2 = \frac{D}{R} M_x(\phi, x), \quad M_1^2 = R^2 M_2^1 = DM_{x\phi}(\phi, x),$$

$$N_\phi = O(1), \quad N_{x\phi} = O(1), \quad N_x = -\frac{2\rho}{\epsilon} + O(1),$$
$$M_\phi = O(\epsilon), \quad M_{x\phi} = O(\epsilon), \quad M_x = O(\epsilon),$$

$$u_1 = u_0 + \mu u, \quad v_1 = v_0 + \mu v, \quad w_1 = w_0 + \mu w$$

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Opoka, S. and Pietraszkiewicz, W., 2009. On refined analysis of bifurcation buckling for the axially compressed circular cylinder. *International Journal of Solids and Structures*, 46(17), pp.3111-3123.

# Bifurcation buckling: Linear stability equations and homogeneous work-conjugate boundary conditions

$$A_1 w'''' + A_2 w'''' + A_3 u'' + A_4 v'' + A_5 v'' + A_6 w'' = 0$$

$$B_1 (w'' + \nu w''') + B_2 u'' + B_3 u'' + B_4 [(1 + \nu)v'' + 2\nu w'] = 0$$

$$C_1 (w'''' + 2w'''' + w''''') + C_2 (u'' + \nu u''') + C_3 v'' + C_4 v'' + C_5 w'' + C_6 w'' + C_7 u' + C_8 v' + C_9 w = 0$$

$$d_1 \equiv D_1 w'' + D_2 u' + D_3 (v' + w) = 0 \quad \text{or} \quad u = 0$$

$$d_2 \equiv E_1 w' + E_2 u + E_3 v' = 0 \quad \text{or} \quad v = 0$$

$$d_3 \equiv F_1 [w''' + (2 - \nu)w''] + F_2 (u'' + \nu u''') + F_3 v'' + F_4 w' = 0 \quad \text{or} \quad w = 0$$

$$d_4 \equiv G_1 (w'' + \nu w''') + G_2 v' + G_3 u' + G_4 w = 0 \quad \text{or} \quad w' = 0$$

at  $x = \pm l = \pm \frac{L}{2R}$ , where  $\frac{\partial(\cdot)}{\partial x} = (\cdot)'$ ,  $\frac{\partial(\cdot)}{\partial \phi} = (\cdot)''$

$$\begin{aligned}
A_1 &= 4\epsilon^2(1 - \nu^2)[1 - \epsilon(2 - 5\nu)\rho - \epsilon^2(7 + 2\nu)\rho^2] + O(\epsilon^5), \\
A_2 &= 4\epsilon^2(1 - \nu^2)[1 - \epsilon(2 - 4\nu - \nu^2)\rho - \epsilon^2(6 + \nu + 2\nu^2)\rho^2] + O(\epsilon^5), \\
A_3 &= -(1 + \nu)[1 - 2\epsilon(2 - 3\nu)\rho - 2\epsilon^2(4 + 6\nu - 3\nu^2)\rho^2] + O(\epsilon^3), \\
A_4 &= -(1 - \nu)\{1 - \epsilon(6 - 4\nu)\rho + 2\epsilon^2[2(1 - \nu^2) - (3 + 8\nu)\rho^2]\} + O(\epsilon^3), \\
A_5 &= -2\{1 - 2\epsilon(1 - 4\nu)\rho + 2\epsilon^2[2(1 - \nu^2) - (3 - 8\nu^2)\rho^2]\} + O(\epsilon^3), \\
A_6 &= -2\{1 - 2\epsilon(1 - 4\nu)\rho + 2\epsilon^2[1 - \nu^2 - (3 - 8\nu^2)\rho^2]\} + O(\epsilon^3), \\
B_1 &= 4\epsilon^3(1 - \nu^2)\rho[\nu - \epsilon(1 + 2\nu)\rho] + O(\epsilon^5), \\
B_2 &= -2[1 - 2\epsilon(5 - \nu - \nu^2)\rho + 2\epsilon^2(2 - 8\nu - 3\nu^2 + 2\nu^3)\rho^2] + O(\epsilon^3), \\
B_3 &= -(1 - \nu)[1 - 2\epsilon(4 - \nu)\rho - 2\epsilon^2\nu(6 + \nu)\rho^2] + O(\epsilon^3), \\
B_4 &= -1 + \epsilon(6 - 4\nu)\rho + 2\epsilon^2(3 + 8\nu)\rho^2 + O(\epsilon^3), \\
C_1 &= -2\epsilon^2(1 - \nu^2)[1 - \epsilon(6 - 4\nu)\rho - 2\epsilon^2(3 + 8\nu)\rho^2] + O(\epsilon^5), \\
C_2 &= 4\epsilon^3(1 - \nu^2)\rho[\nu - \epsilon(1 + 4\nu - 2\nu^2)\rho] + O(\epsilon^5), \\
C_3 &= 4\epsilon^2(1 - \nu^2)[1 - \epsilon[6 - 4\nu - \nu^2]\rho - \epsilon^2(6 + 17\nu + 6\nu^2)\rho^2] + O(\epsilon^5), \\
C_4 &= 4\epsilon^2(1 - \nu^2)[1 - \epsilon(6 - 5\nu)\rho - \epsilon^2(7 + 22\nu)\rho^2] + O(\epsilon^5), \\
C_5 &= -4\epsilon(1 - \nu^2)[\rho - \epsilon(\nu + 4(1 - \nu)\rho^2)] + O(\epsilon^3), \\
C_6 &= 4\epsilon^2(1 - \nu^2) + O(\epsilon^3), \\
C_7 &= -2\nu[1 - 2\epsilon(4 - 3\nu)\rho - 6\epsilon^2(6 - \nu)\nu\rho^2] + O(\epsilon^3), \\
C_8 &= -2[1 - 2\epsilon(3 - 4\nu)\rho + 2\epsilon^2(1 - \nu^2 - (3 + 16\nu - 8\nu^2)\rho^2)] + O(\epsilon^3), \\
C_9 &= -2[1 - 2\epsilon(3 - 4\nu)\rho + \epsilon^2(1 - \nu^2 - 2(3 + 16\nu - 8\nu^2)\rho^2)] + O(\epsilon^3),
\end{aligned}$$

$$D_1 = -4\epsilon^3 \nu(1 - \nu^2)\rho[\nu - \epsilon(1 + 2\nu)\rho] + O(\epsilon^5),$$

$$D_2 = 2[1 - 2\epsilon(5 - \nu - \nu^2)\rho + 2\epsilon^2(2 - 8\nu - 3\nu^2 + 2\nu^3)\rho^2] + O(\epsilon^3),$$

$$D_3 = 2\nu[1 - \epsilon(6 - 4\nu)\rho - 2\epsilon^2(3 + 8\nu)\rho^2] + O(\epsilon^3),$$

$$E_1 = 4\epsilon^2(1 - \nu^2)[1 - \nu - \epsilon(2 - 6\nu + 3\nu^2)\rho - \epsilon^2(6 - 5\nu + 2\nu^2)\rho^2] + O(\epsilon^5),$$

$$E_2 = -(1 - \nu)[1 - 2\epsilon(2 - 3\nu)\rho - 2\epsilon^2(4 + 6\nu - 3\nu^2)\rho^2] + O(\epsilon^3),$$

$$E_3 = -(1 - \nu)[1 - \epsilon(6 - 4\nu)\rho + \epsilon^2(4 - 4\nu^2 - 6\rho^2 - 16\nu\rho^2)] + O(\epsilon^3),$$

$$F_1 = -2\epsilon^2(1 - \nu^2)[1 - \epsilon(6 - 4\nu)\rho - 2\epsilon^2(3 + 8\nu)\rho^2] + O(\epsilon^5),$$

$$F_2 = 4\epsilon^3(1 - \nu^2)\rho[\nu - \epsilon(1 + 4\nu - 2\nu^2)\rho] + O(\epsilon^5),$$

$$F_3 = 4\epsilon^2(1 - \nu^2)[1 - \epsilon(6 - 4\nu - \nu^2)\rho - \epsilon^2(6 + 17\nu + 6\nu^2)\rho^2] + O(\epsilon^5),$$

$$F_4 = -2\epsilon(1 - \nu^2)[2\rho - \epsilon(\nu + 8(1 - \nu)\rho^2)] + O(\epsilon^3),$$

$$G_1 = -1 + 2\epsilon(1 - \nu)\rho + 2\epsilon^2(3 + \nu^2)\rho^2 + O(\epsilon^3),$$

$$G_2 = 2\nu[1 - \epsilon(3 - 2\nu)\rho - \epsilon^2(9 + 2\nu + 2\nu^2)\rho^2] + O(\epsilon^3),$$

$$G_3 = 2\epsilon\nu\rho[1 + 2\nu + \epsilon(3 + 4\nu)\rho] + O(\epsilon^3),$$

$$G_4 = \nu[1 - 2\epsilon(2 - \nu)\rho - 2\epsilon^2(6 + 2\nu + \nu^2)\rho^2] + O(\epsilon^3).$$

# Solution method

## 1 Solutions of the stability equations

$$\left. \begin{aligned} u(\phi, x) &= Ue^{\rho x} \cos(n\phi) \\ v(\phi, x) &= Ve^{\rho x} \sin(n\phi) \\ w(\phi, x) &= We^{\rho x} \cos(n\phi) \end{aligned} \right\} \Rightarrow \begin{aligned} V &= V(\nu, \epsilon, n, \rho; p_j, U) \\ W &= W(\nu, \epsilon, n, \rho; p_j, U) \\ p_j &\text{ - roots of characteristic eq.} \end{aligned}$$

## 2 Substitution of solutions into boundary conditions

Solutions exist if

$$\text{Det}[\dots] = 0 \quad \Rightarrow \quad f(\nu, \epsilon, n, \rho, p_j, l) = 0 \quad \Rightarrow \quad \rho = \rho_{crit}$$

$$\begin{aligned}
 u(\phi, x) &= \sum_j U_j e^{p_j x} \cos(n\phi) - \frac{1 - \epsilon(6 - 8\nu)\rho + \epsilon^2[1 - \nu^2 - 2(3 + 16\nu - 8\nu^2)\rho^2]}{\nu[1 - 2\epsilon\rho(4 - 3\nu + 3\epsilon(6 - \nu)\nu\rho)]} S x + Z, \\
 v(\phi, x) &= \sum_j V(\nu, \epsilon, n, \rho; p_j, U_j) e^{p_j x} \sin(n\phi), \\
 w(\phi, x) &= \sum_j W(\nu, \epsilon, n, \rho; p_j, U_j) e^{p_j x} \cos(n\phi) + S,
 \end{aligned}$$

where non-zero  $A_k$ 's are named  $S$  and  $Z$ , respectively.

axially compressed Euler column with simply-supported (clamped) boundaries

$$\rho = \frac{\pi^2}{16l^2\epsilon} \quad \left( \rho = \frac{\pi^2}{4l^2\epsilon} \right), \quad \blacktriangle \text{ clamped column}$$

■ simply-supported column



# Nomenclature for boundary conditions

C–family and S–family				
C1(S1):	$u = 0$	$v = 0$	$w = 0$	$w' = 0$ ( $d_4 = 0$ )
C2(S2):	$d_1 = 0$	$v = 0$	$w = 0$	$w' = 0$ ( $d_4 = 0$ )
C3(S3):	$u = 0$	$d_2 = 0$	$w = 0$	$w' = 0$ ( $d_4 = 0$ )
C4(S4):	$d_1 = 0$	$d_2 = 0$	$w = 0$	$w' = 0$ ( $d_4 = 0$ )
C5(S5):	$u = 0$	$v = 0$	$d_3 = 0$	$w' = 0$ ( $d_4 = 0$ )
C6(S6):	$d_1 = 0$	$v = 0$	$d_3 = 0$	$w' = 0$ ( $d_4 = 0$ )
C7(S7):	$u = 0$	$d_2 = 0$	$d_3 = 0$	$w' = 0$ ( $d_4 = 0$ )
C8(S8):	$d_1 = 0$	$d_2 = 0$	$d_3 = 0$	$w' = 0$ ( $d_4 = 0$ )

Classical cases:

C1: clamped BC

S2: simply-supported BC

# SIMPLIFIED VERSION (Timoshenko)

$$D \frac{d^4 w}{dx^4} + N_z \frac{d^2 w}{dx^2} + Eh \frac{w}{a^2} = 0$$

$$w = -A \sin \frac{m\pi x}{l}$$



$$l = \frac{L}{2R}$$

if  $l \leq 0.1$  then the cylinder is regarded as short

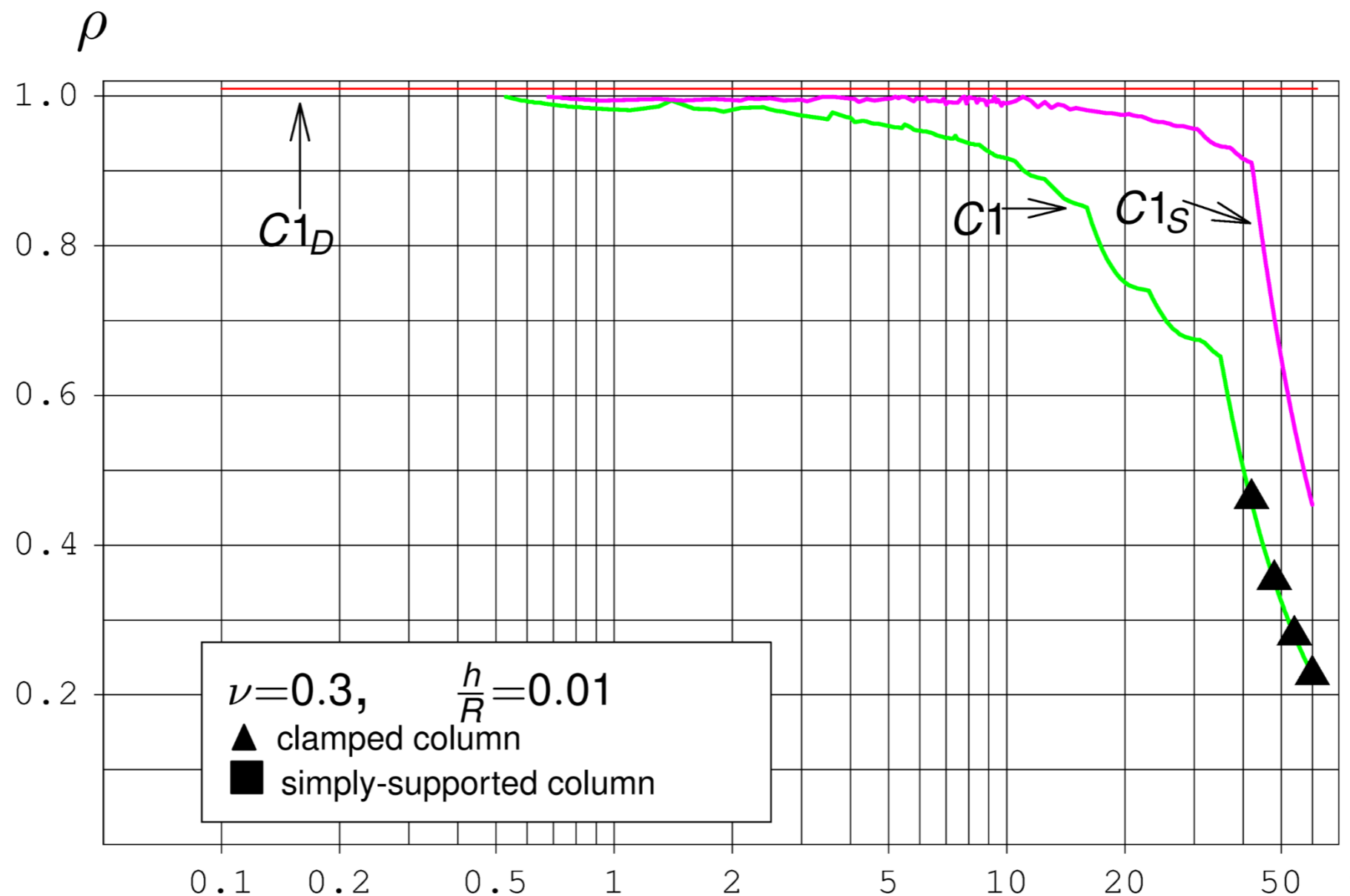
the practical cylinder lengths (PCL) cover the interval  $l \in (0.1, 20)$

long cylinders  $l \geq 20$ .

experimental cylinder lengths (ECL) when  $l \in (0.2, 5)$ .

# Influence of different approximations in the derivation of the stability problem

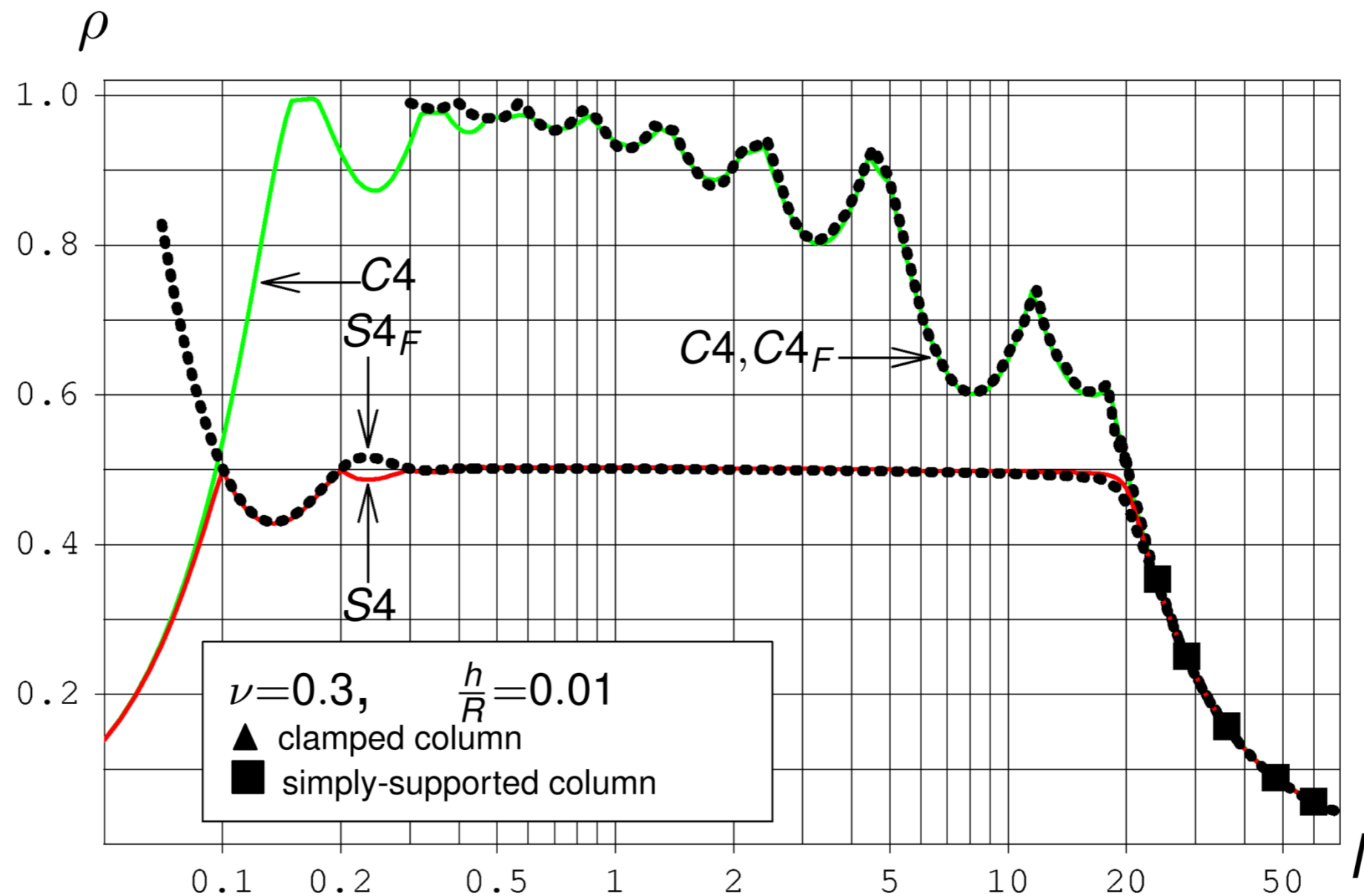
Even (energetically) small terms influence  $\rho_{crit}$



# Comparison with results obtained from Flügge equations

For all curves we noted good overall agreement but short cylinders with bc S4 or C4 behave differently

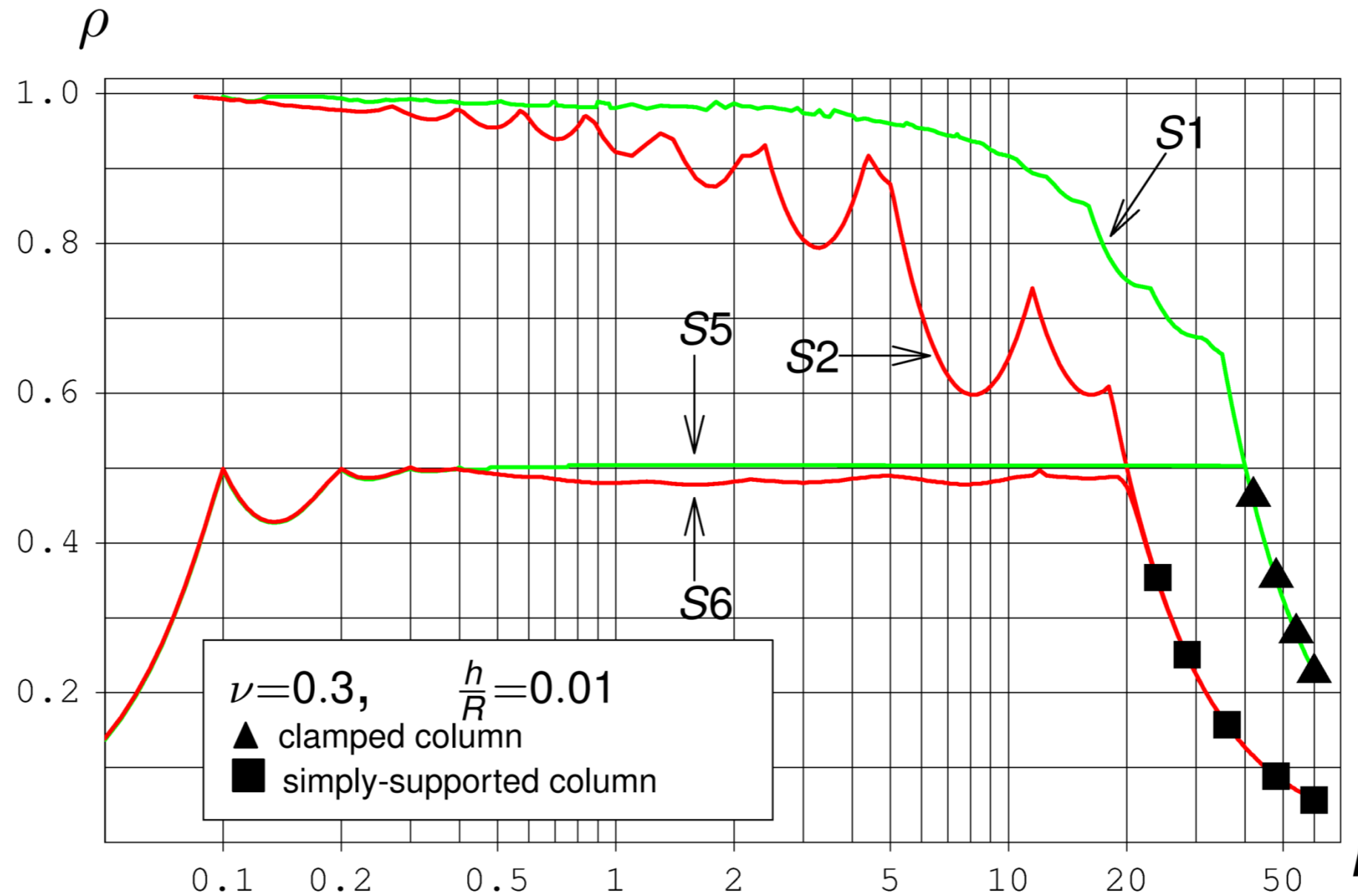
$$l = \frac{L}{2R}$$



Similar result for S4 case: [Simmonds and Danielson \(1970\)](#)

# Relaxation of geometric boundary conditions as a factor for decreasing the buckling load

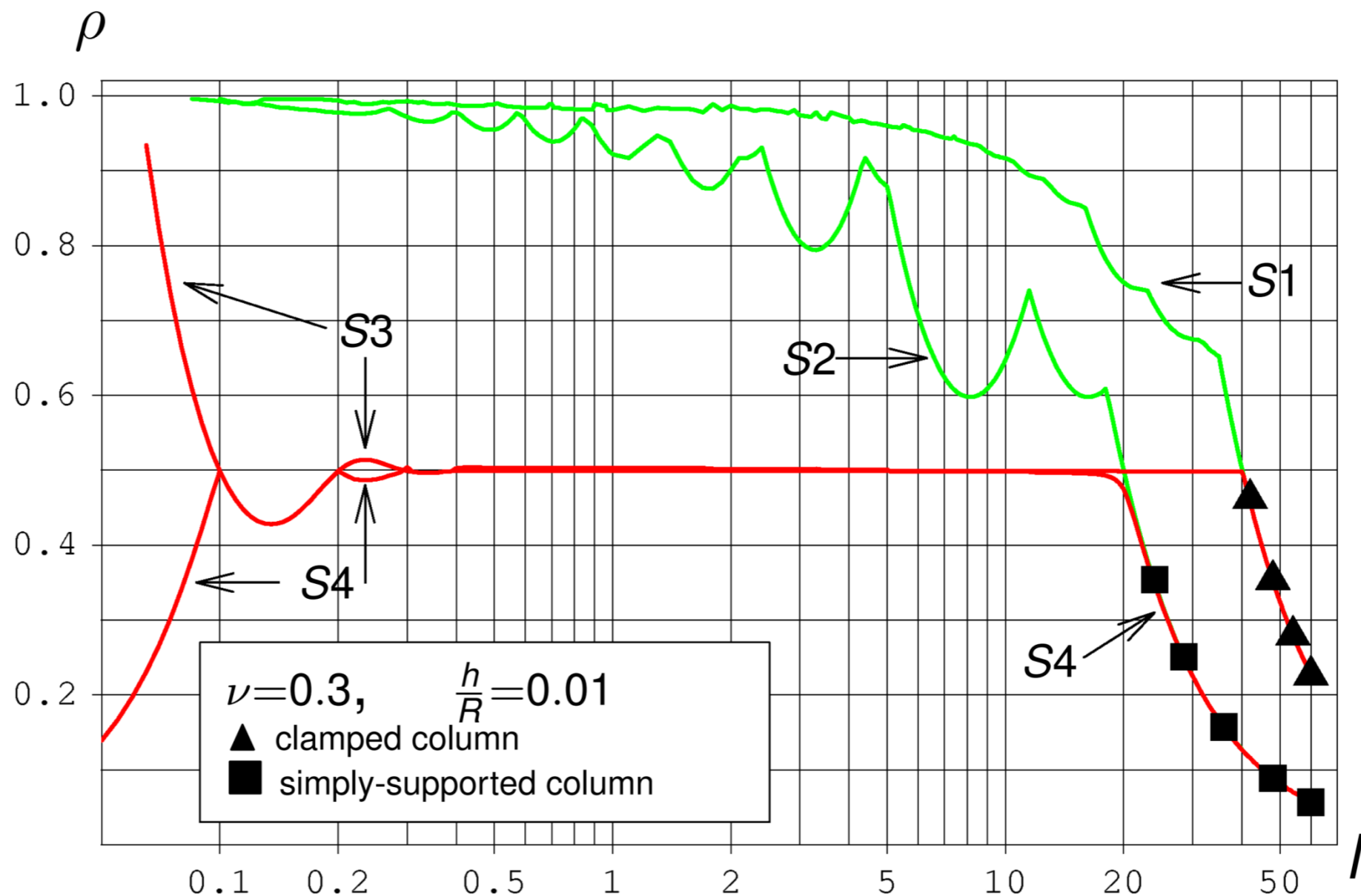
The exchange of  $u = 0$  for  $d_1 = 0$ :  $S1 \rightarrow S2$ ,  $S5 \rightarrow S6$



Small influence in range of ECL

# Relaxation of geometric boundary conditions as a factor for decreasing the buckling load

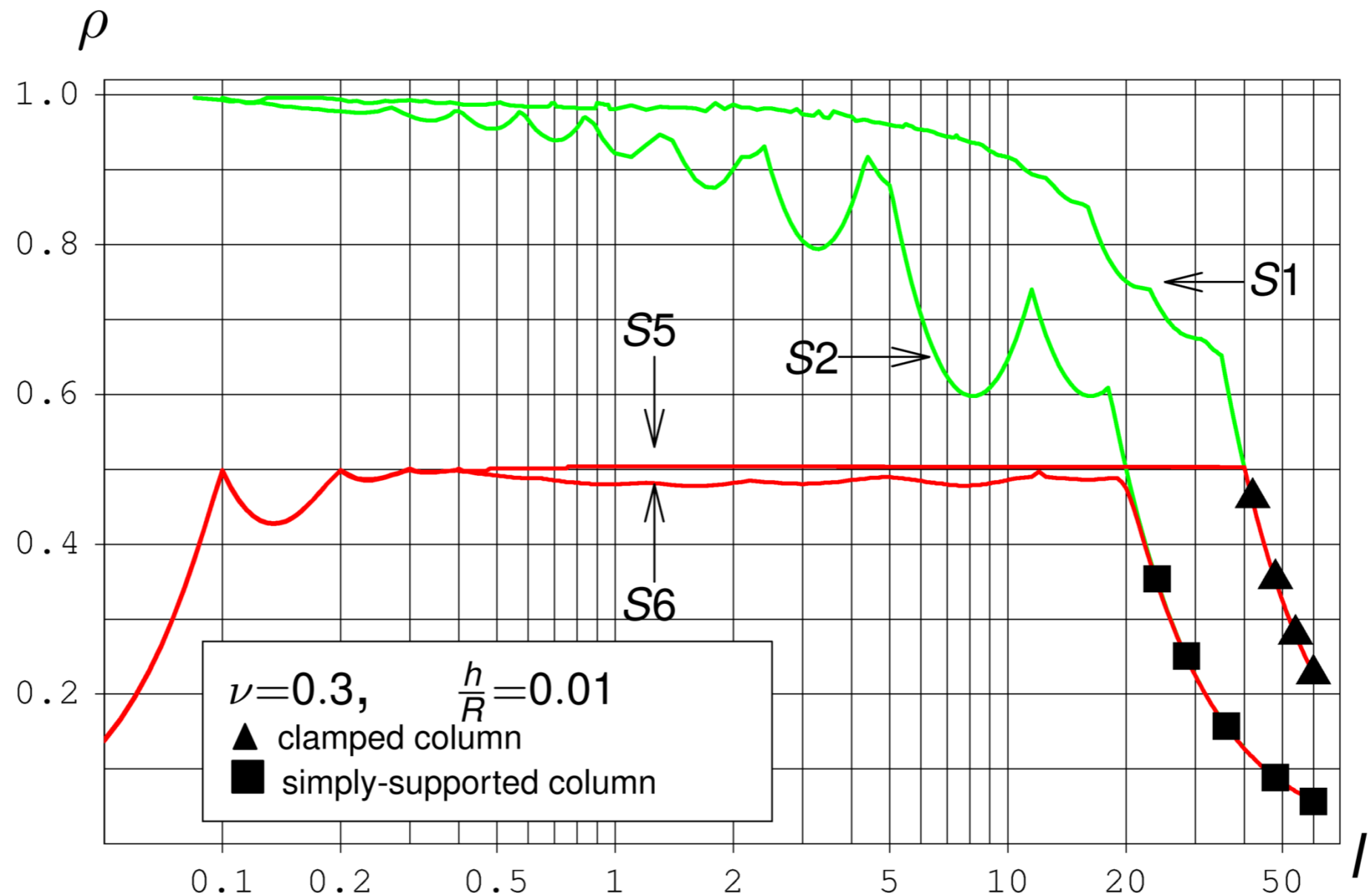
The exchange of  $\nu = 0$  for  $d_2 = 0$ :  $S1 \rightarrow S3$ ,  $S2 \rightarrow S4$



Fall down to one half of the classical value Ohira (1961)

# Relaxation of geometric boundary conditions as a factor for decreasing the buckling load

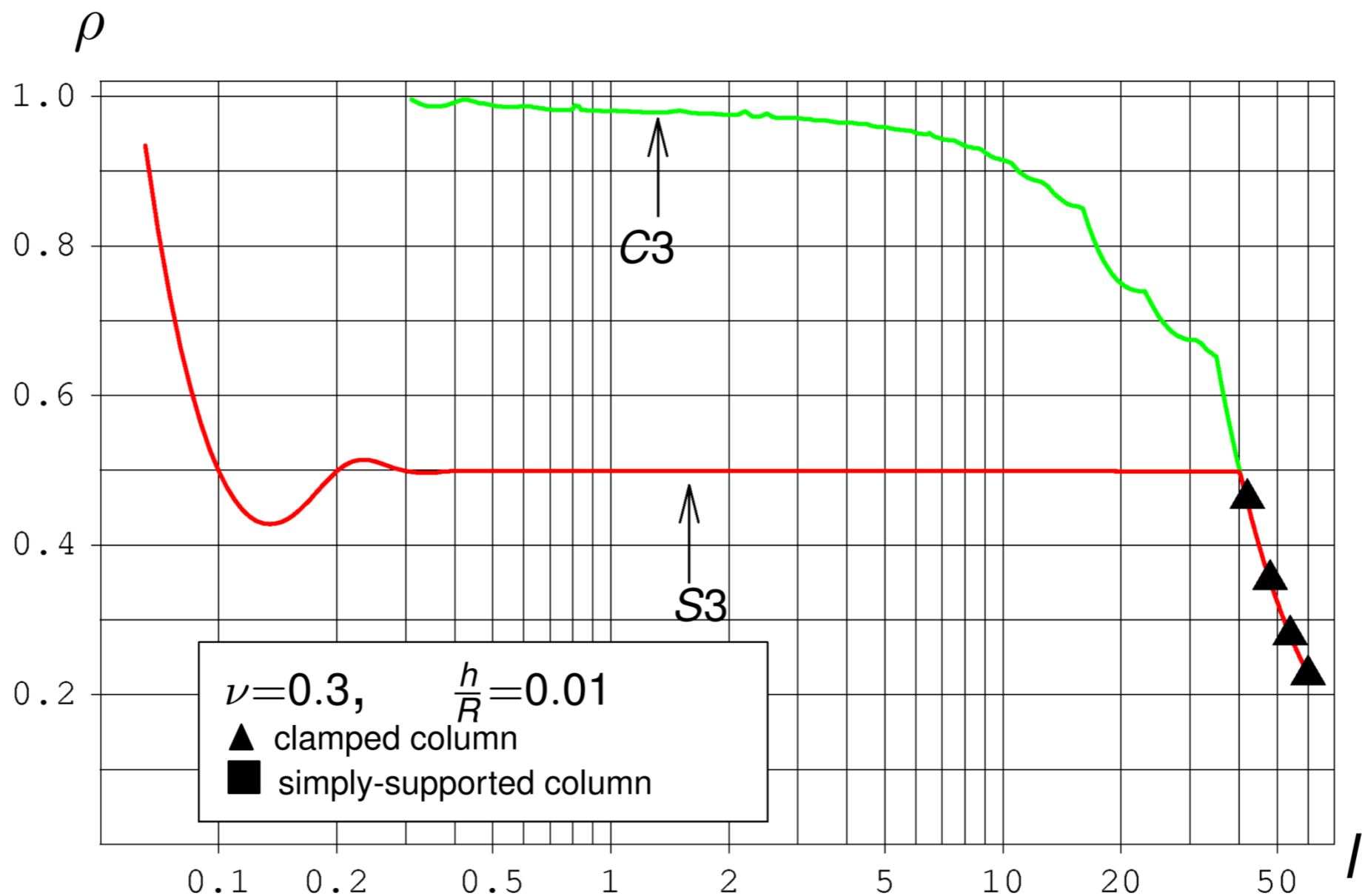
The exchange of  $w = 0$  for  $d_3 = 0$ :  $S1 \rightarrow S5$ ,  $S2 \rightarrow S6$



Fall down to one half of the classical value in range of ECL

# Relaxation of geometric boundary conditions as a factor for decreasing the buckling load

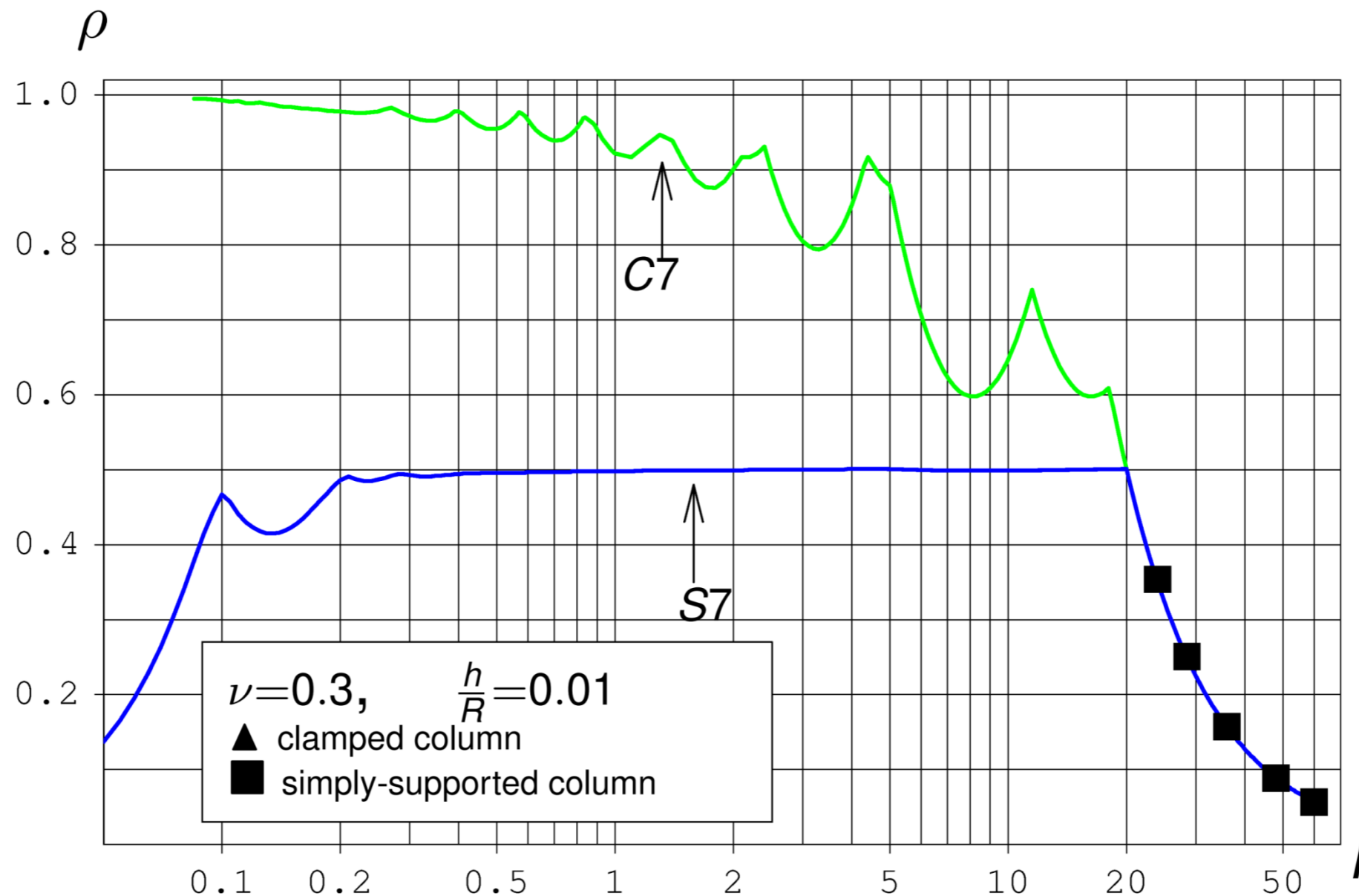
The exchange of  $w' = 0$  for  $d_4 = 0$ : C3  $\rightarrow$  S3



Fall down to one half of the classical value in range of ECL

# Behaviour of long cylinder as Euler column

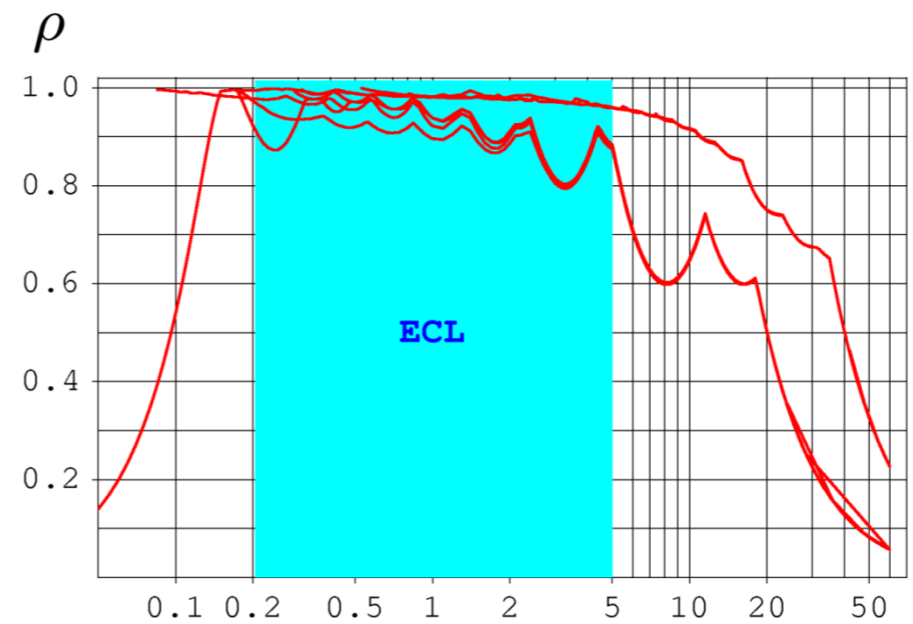
Long cylinder behaves as clamped Euler column  
iff  $u = v = 0$  or  $u = w = 0$  (the constraint  $u = 0$  is not sufficient)



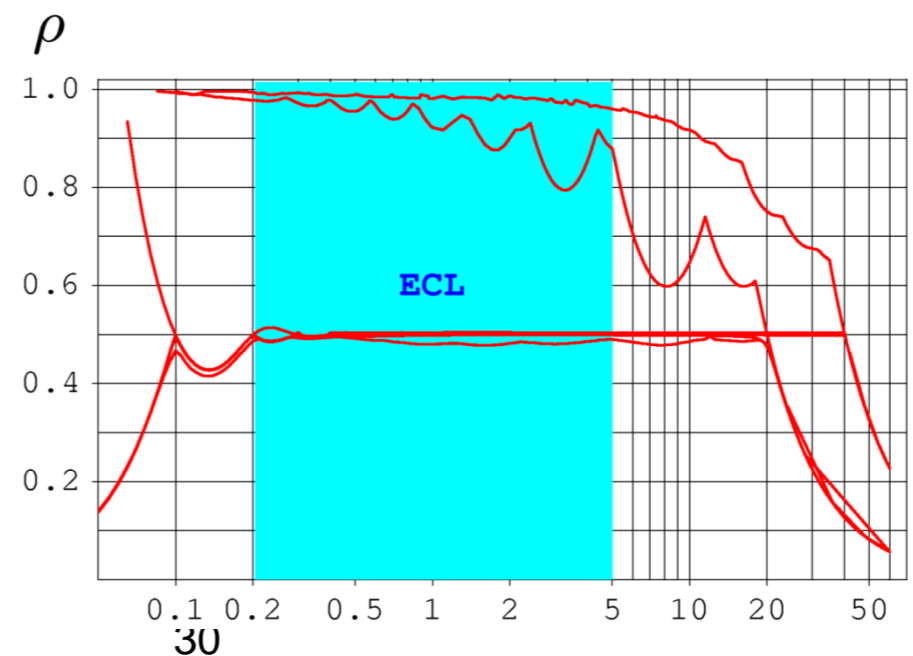
# The influence of suppression of rotations at the boundary

- Cylinders with clamped BCs

experimental cylinder lengths  
(ECL)



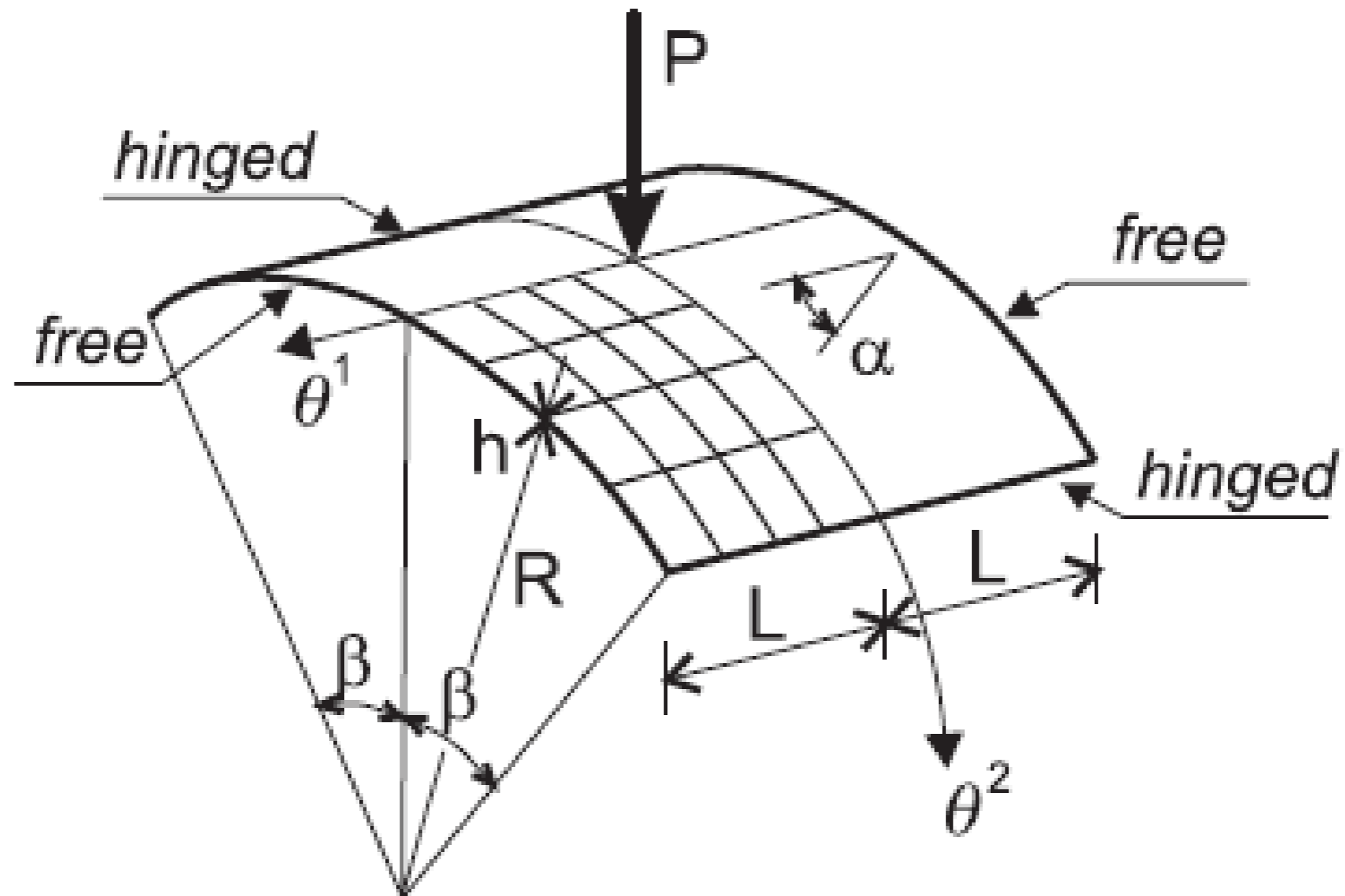
- Cylinders with simply-supported BCs



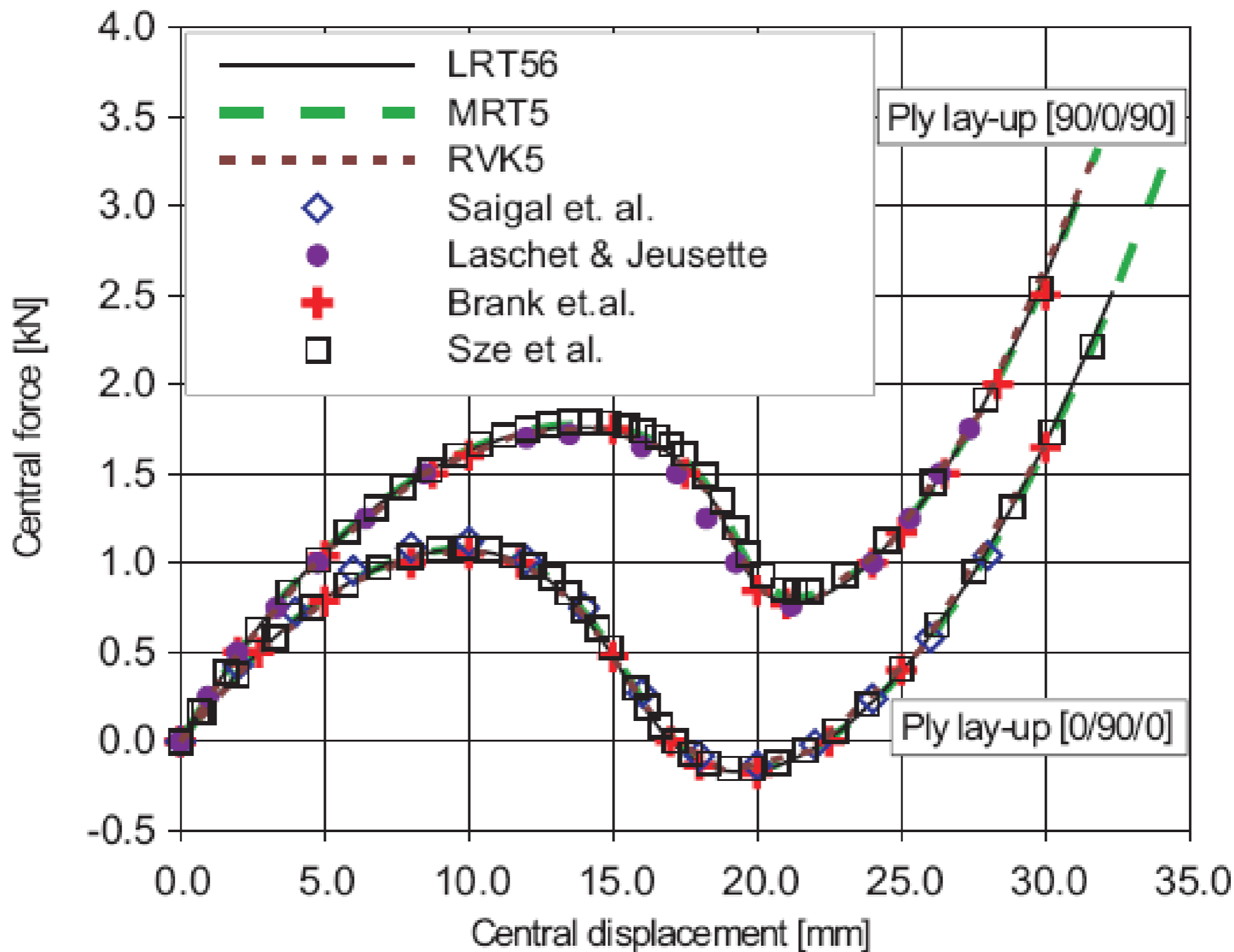
# Elementi finiti

## Buckling e postbuckling

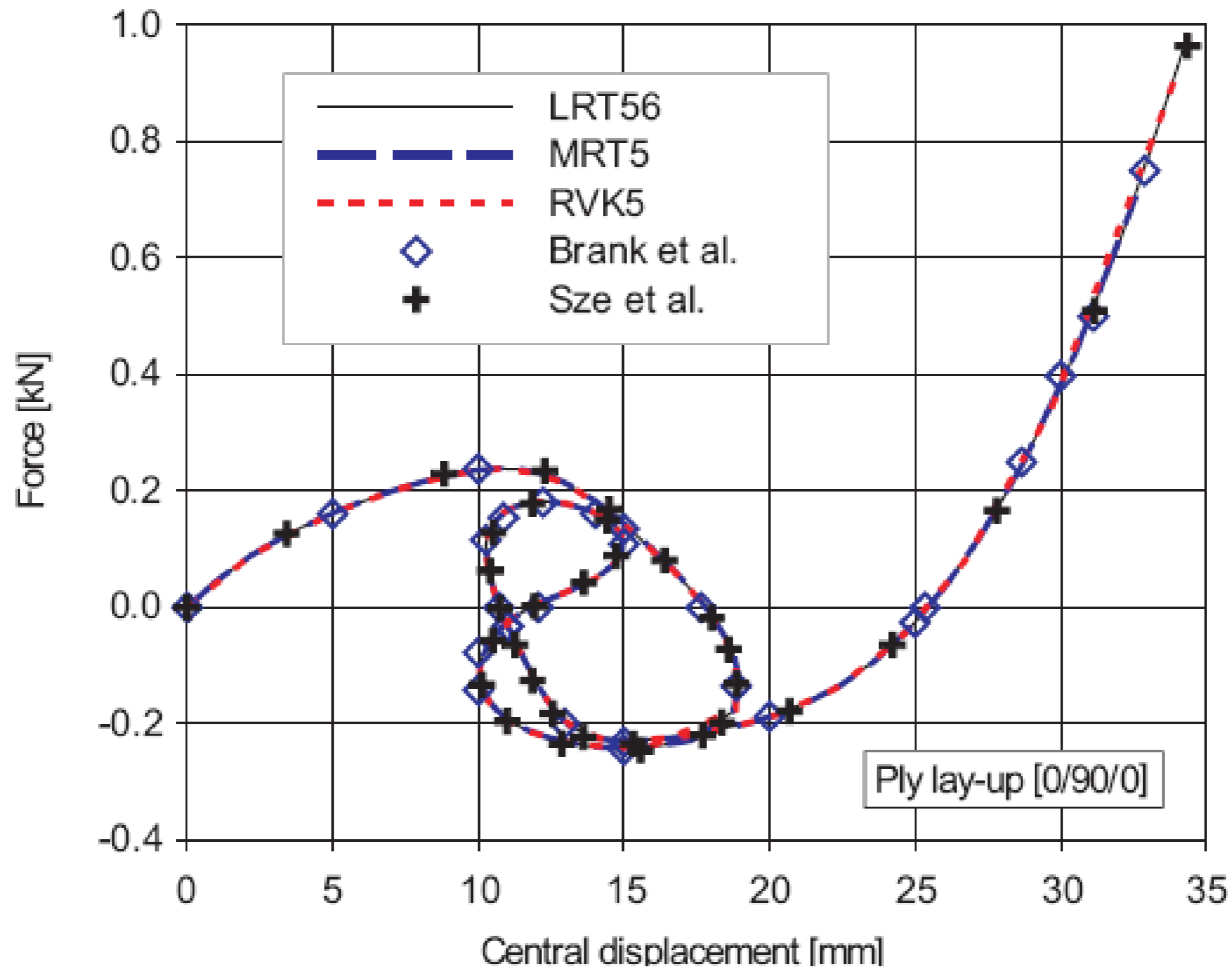
# Hinged laminated cylindrical panels under point load



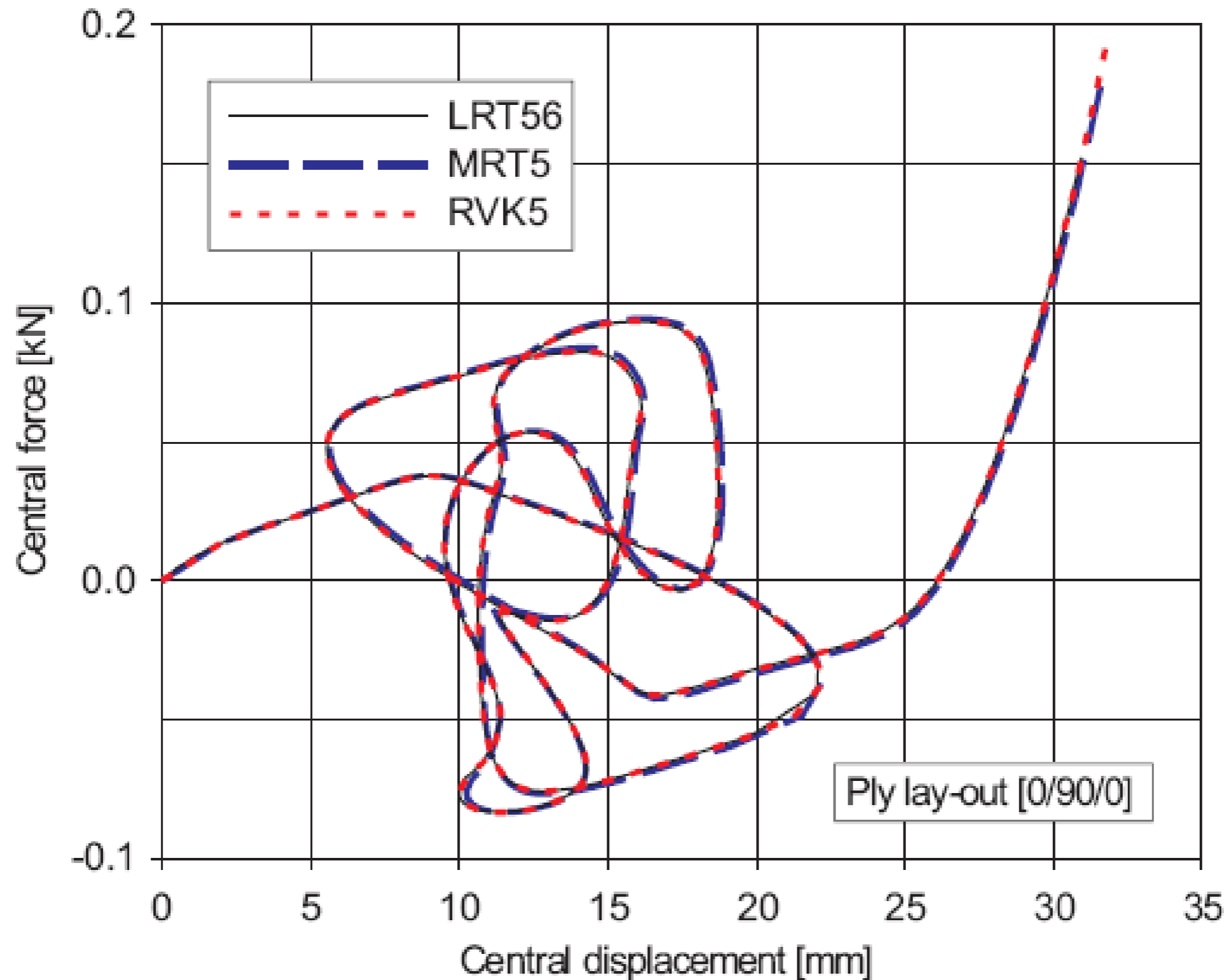
# Central deflection for the cross ply laminate panel A ( $h=12.6$ mm)



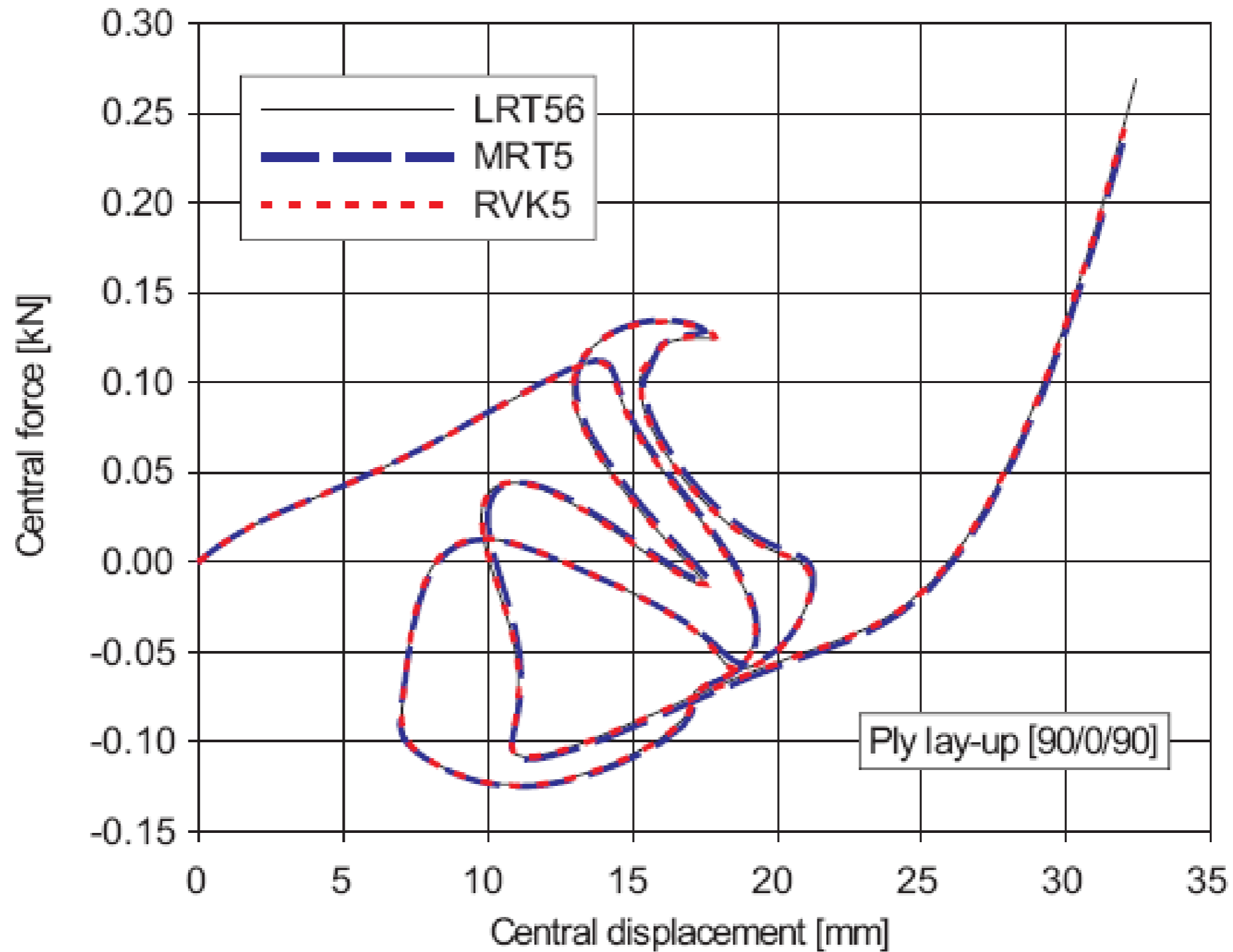
# Central deflection for the cross ply [0/90/0] laminate panel B (h=6.3 mm)



# Central deflection for the laminate panel C ( $h=3.15$ mm)

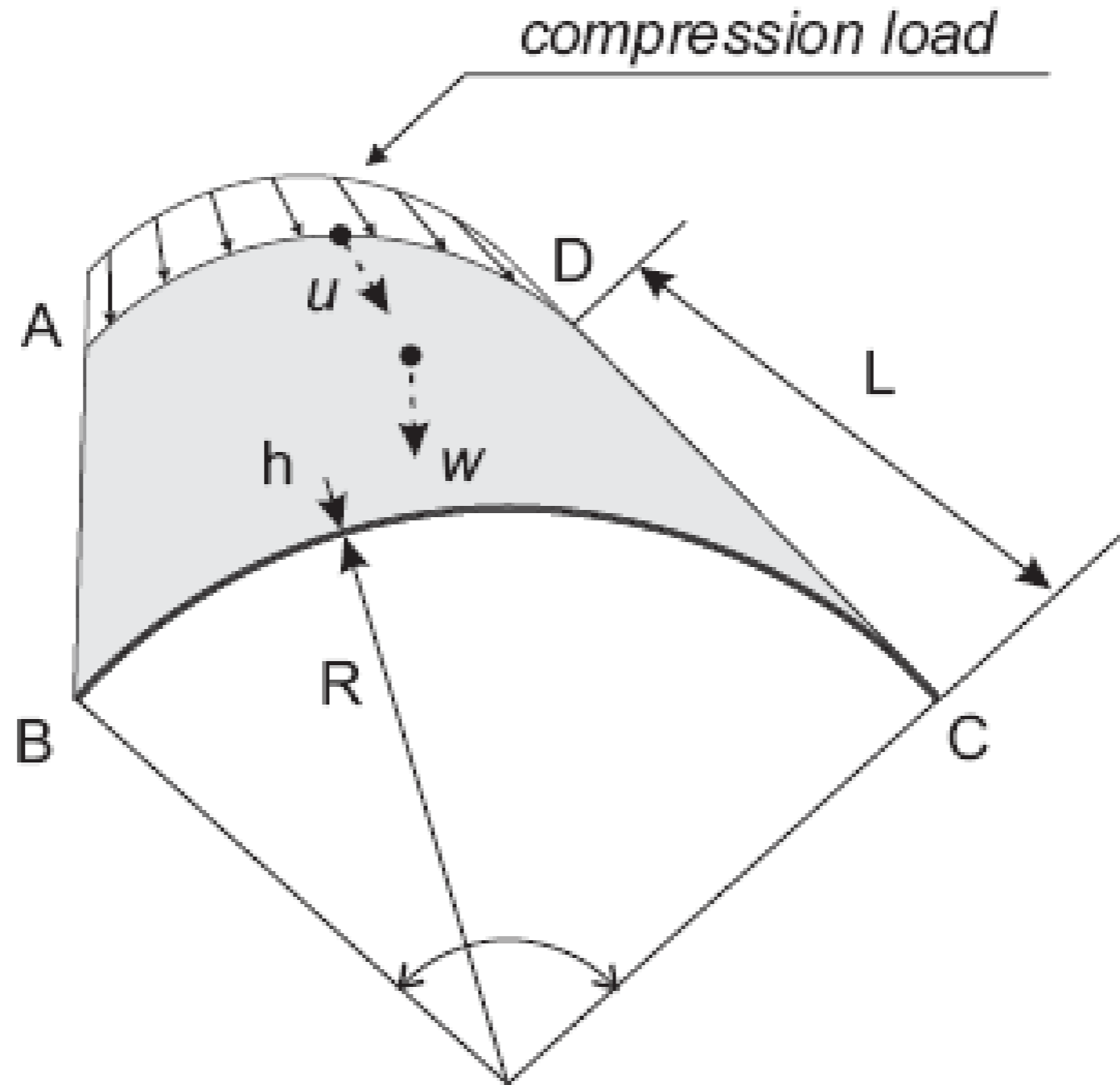


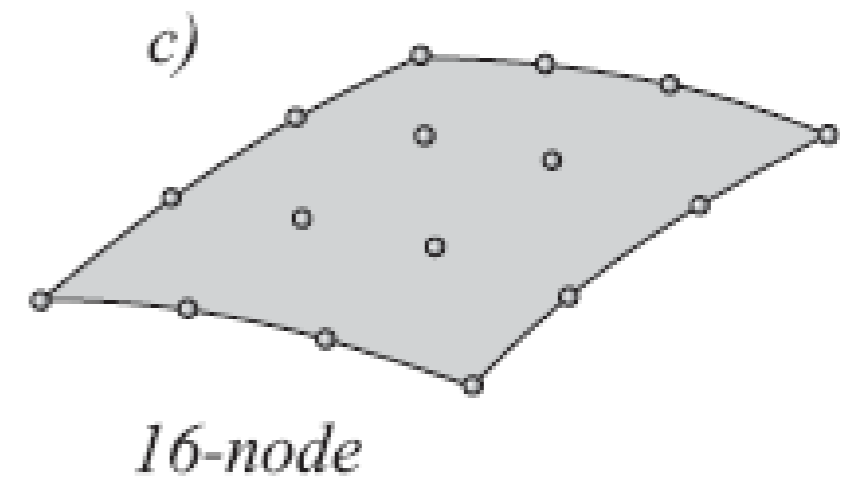
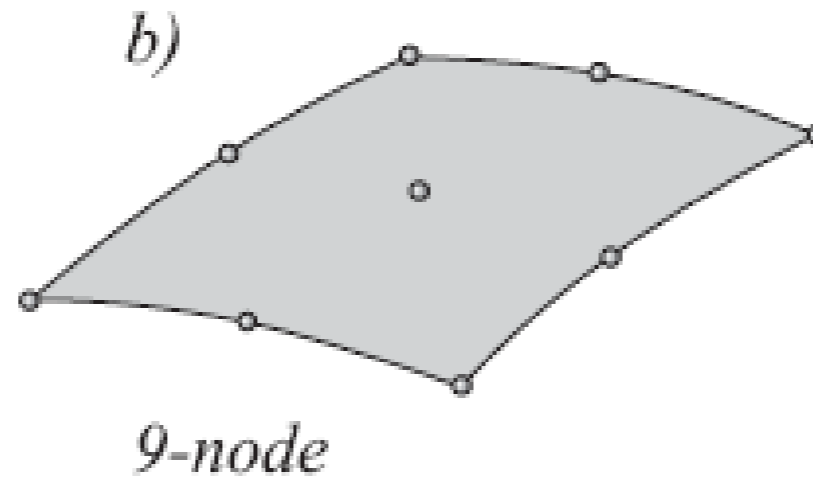
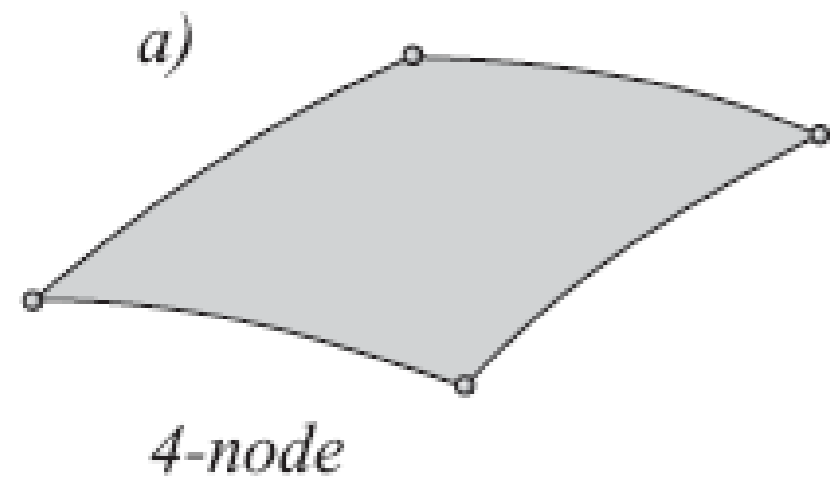
# Central deflection for the laminate panel C ( $h=3.15$ mm)



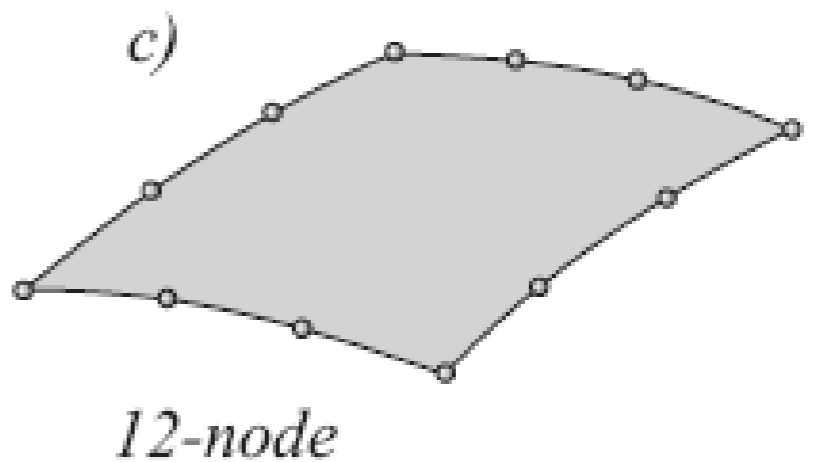
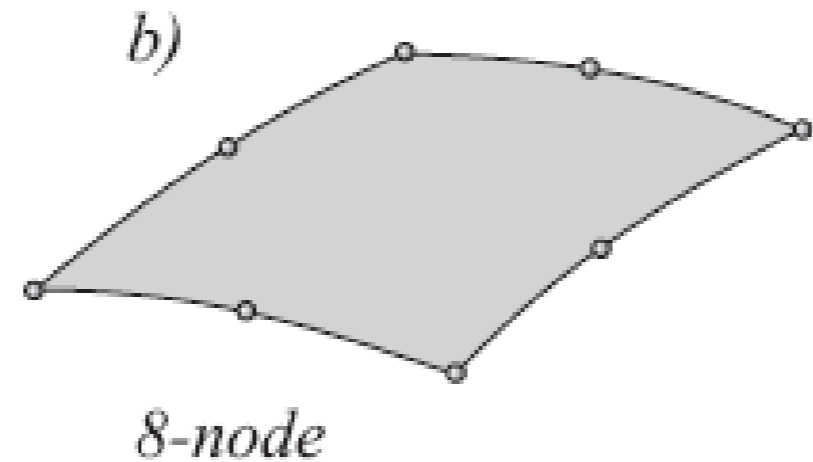
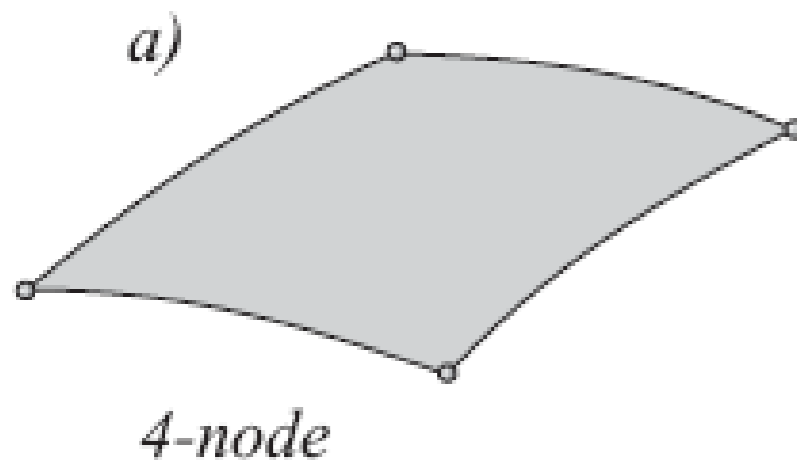


# Axial compression of composite cylindrical panel





Lagrange family of shell elements: a) linear; b) quadratic; c) cubic interpolation

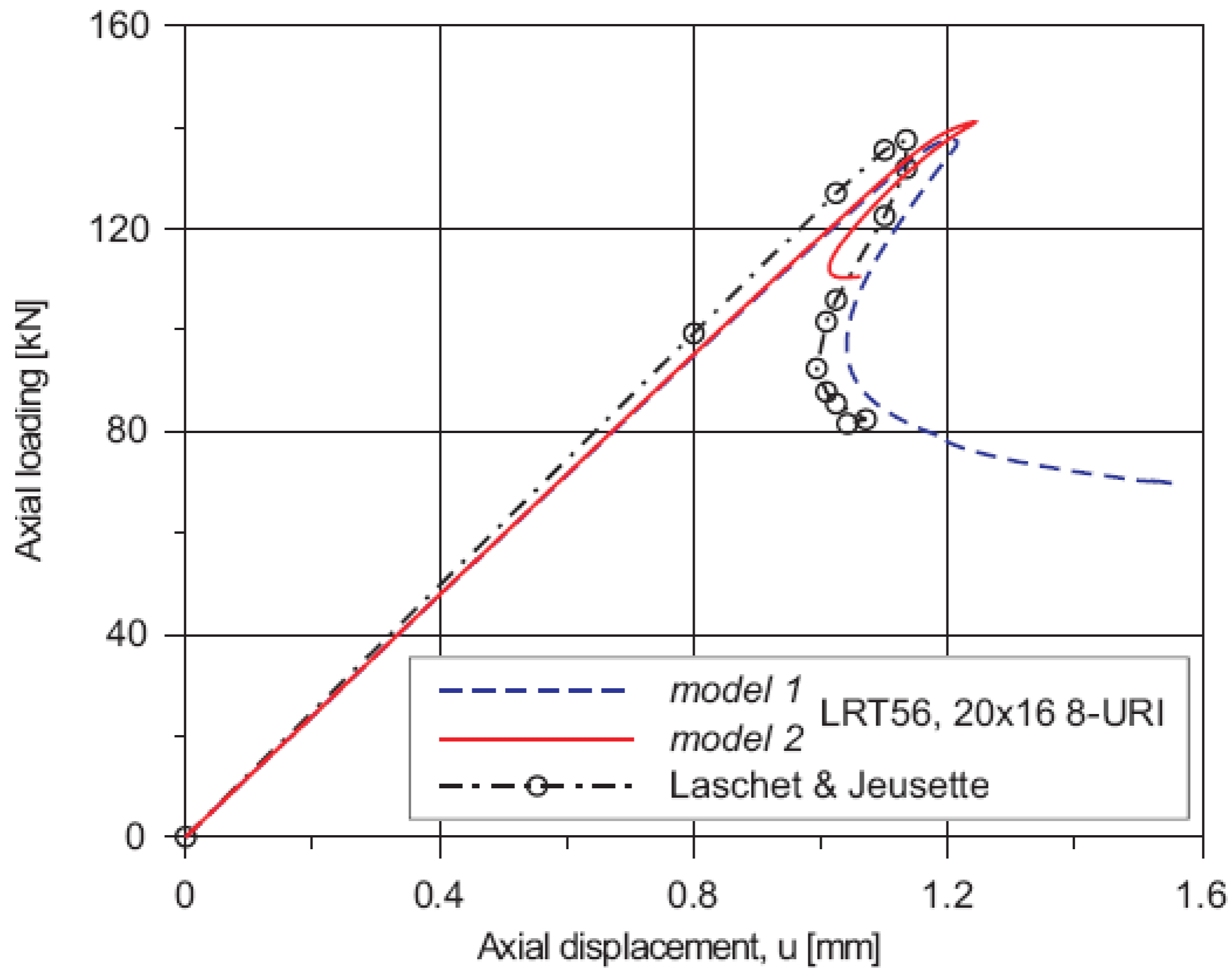


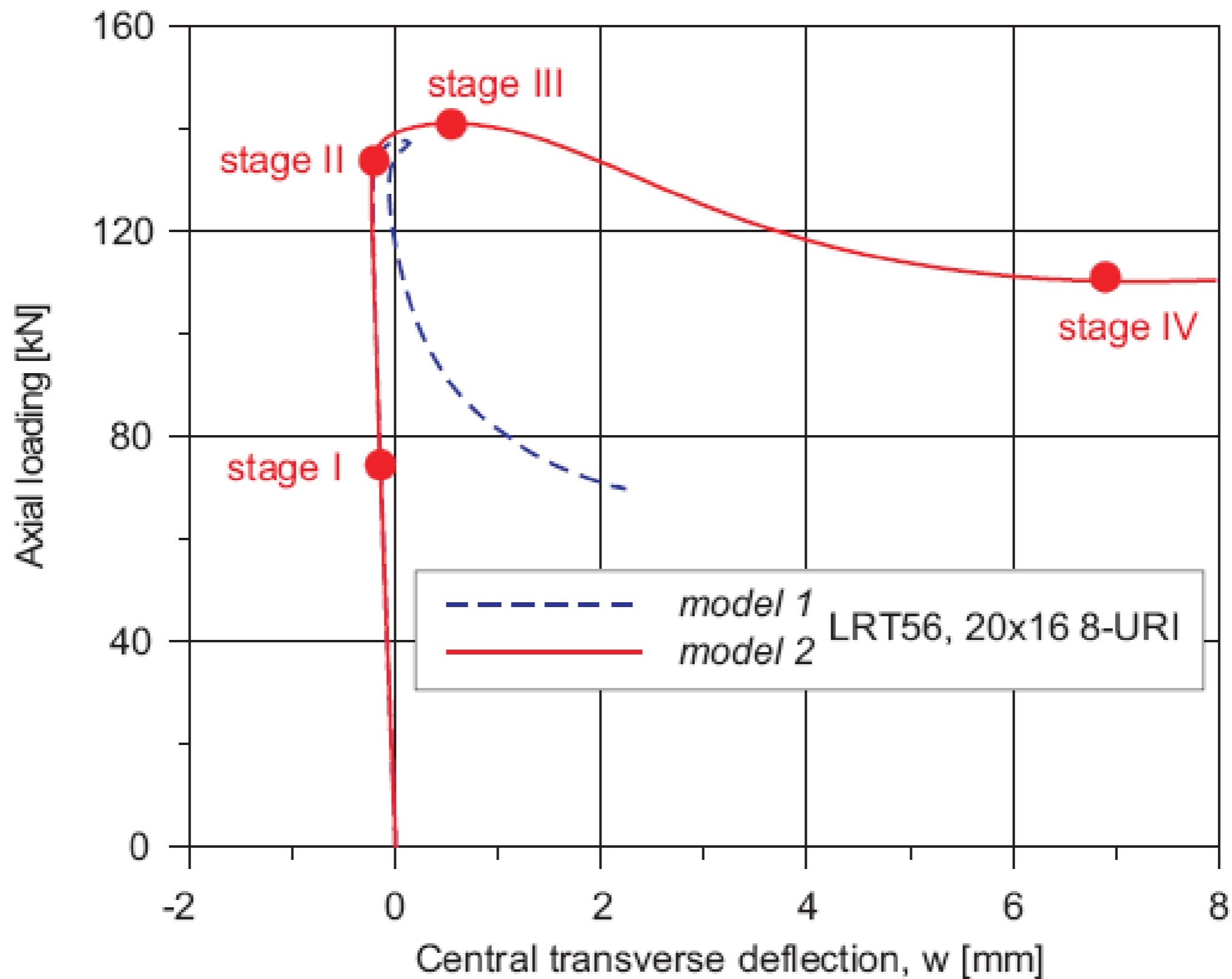
Serendipity family of shell elements: a) linear; b) quadratic; c) cubic interpolation

Model	Mesh	Critical load [kN]	
		Linear buckling	Incremental analysis
<i>model 1</i> LRT56 8-URI elements	10×8	-	138.9
	16×8	-	137.8
	20×16	-	137.7
<i>model 2</i> LRT56 8-URI elements	10×8	-	143.5
	16×8	-	143.3
	20×16	-	140.9
8-node elements Jun & Hong	8×10	-	143.2
16-node elements Laschet & Jeusette	8×10	143.9	137.8
	12×18	140.3	-

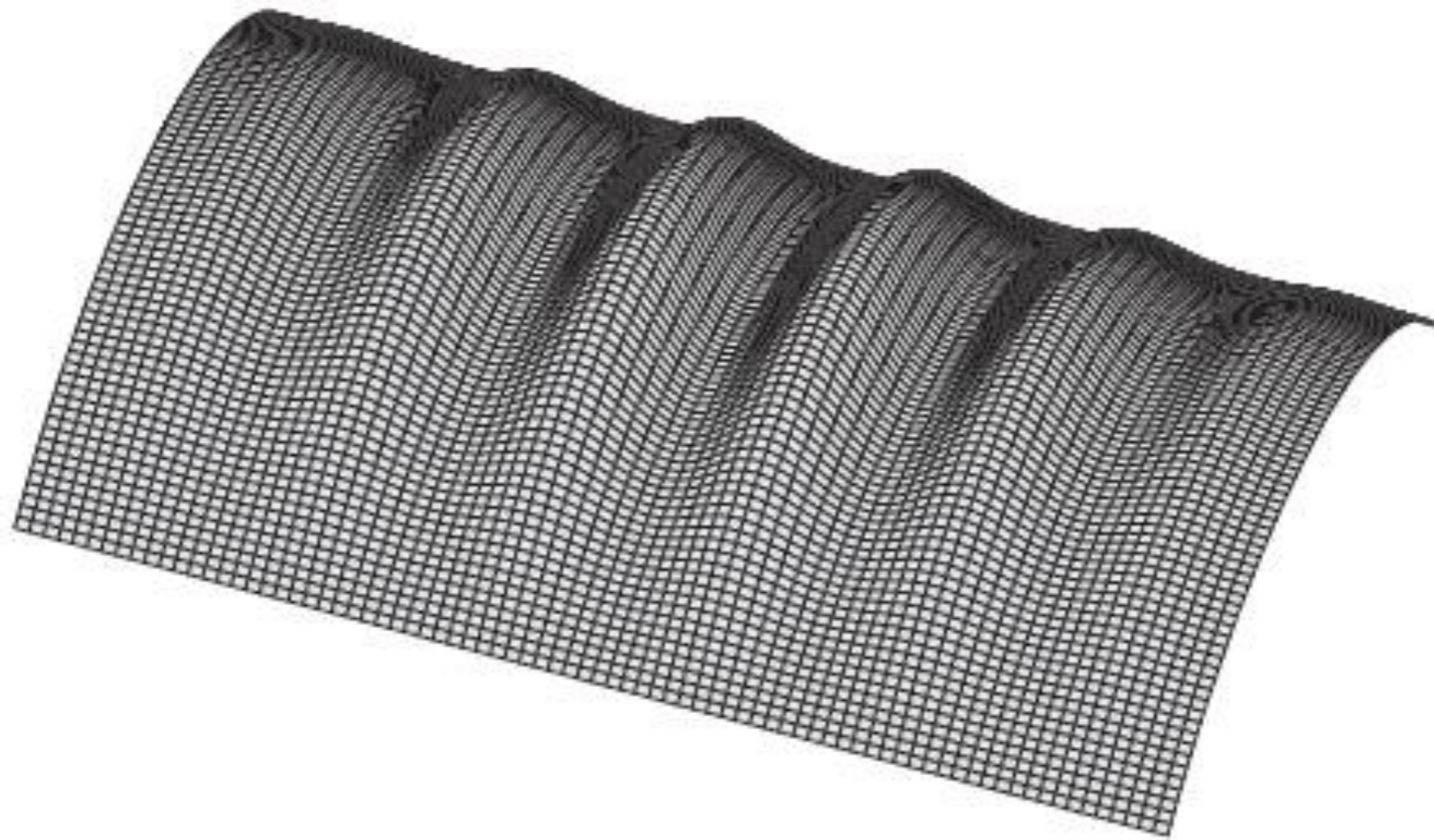


4-node elements Wagner	4×12	145.6	-
	4×16	142.2	-
	4×20	140.8	-
	4×40	140.0	-
	4×80	139.6	-
4-node elements Brank & Carrera	32×32	-	150
MSC Nastran QUAD4 elements Krejčí [1]	20×20	144.56	144.35
	40×40	141.56	142.34
	80×80	140.34	140.38
Snell & Morley	Experiment	134	



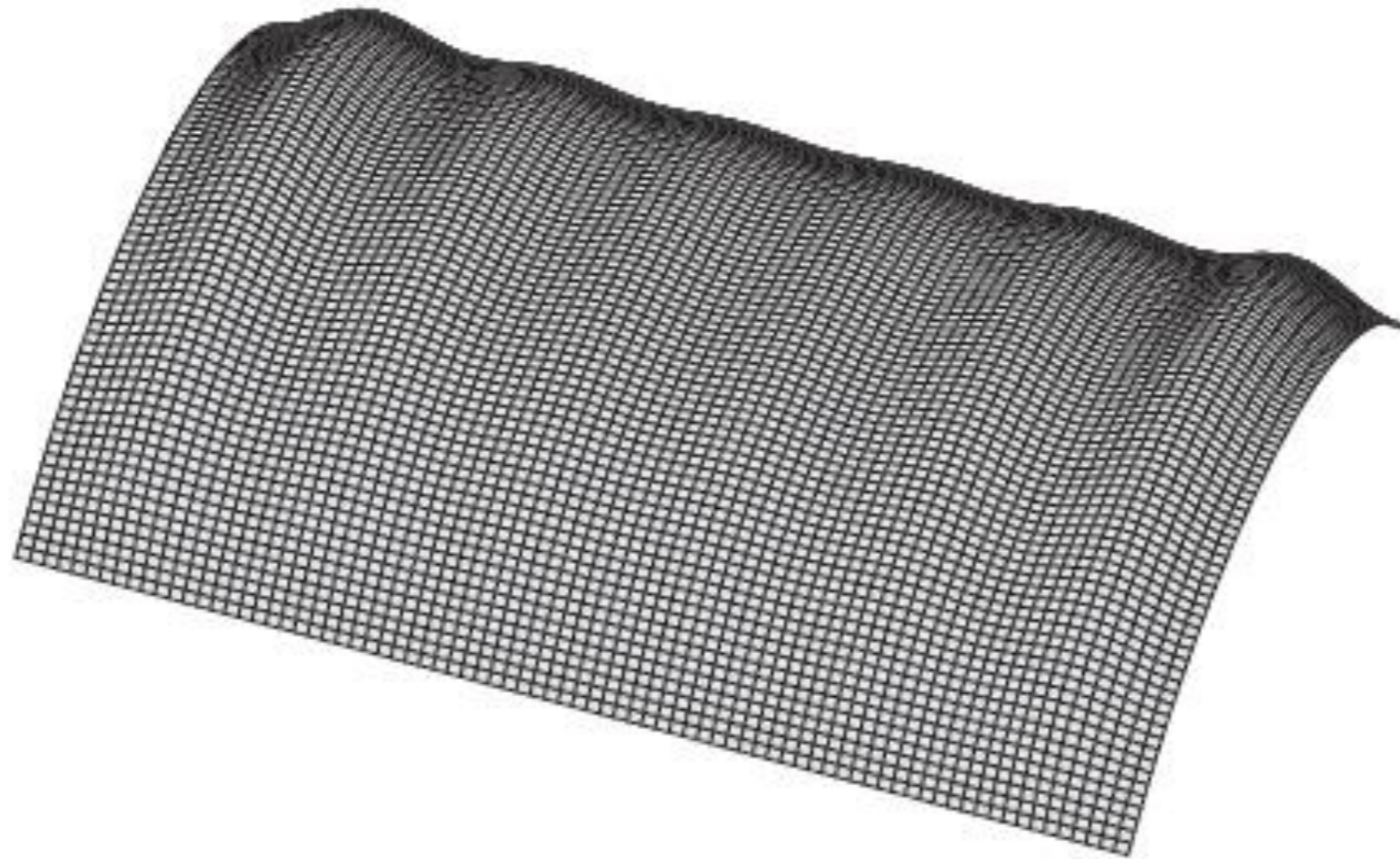


The first buckling mode (MSC Nastran 80X80 QUAD4 elements,  
 $P_{crit} = 140.34 \text{ kN}$ )



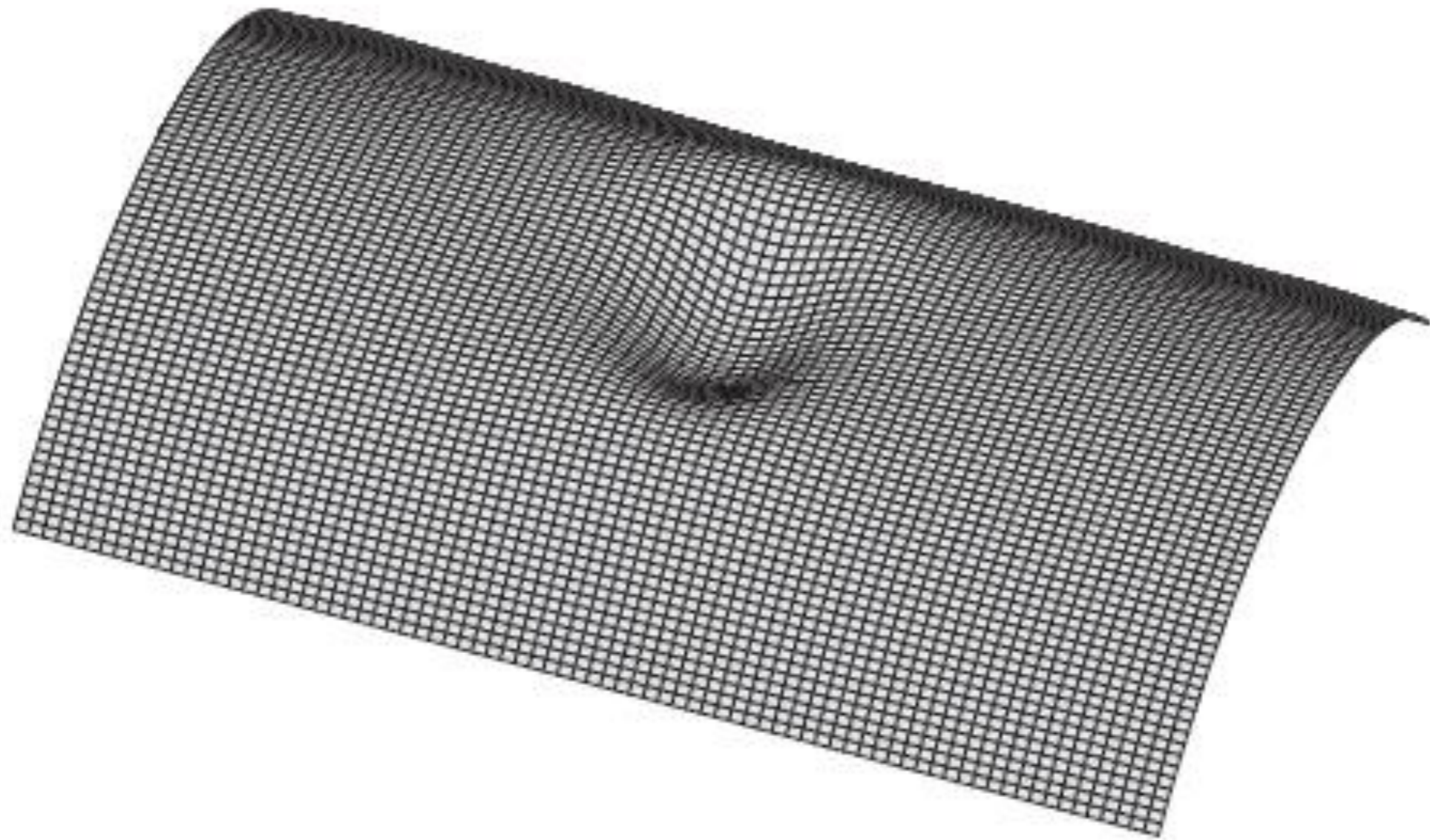


# Pre-buckling deformation at $P=136.60$ kN (MSC Nastran 80x80 QUAD4 elements)

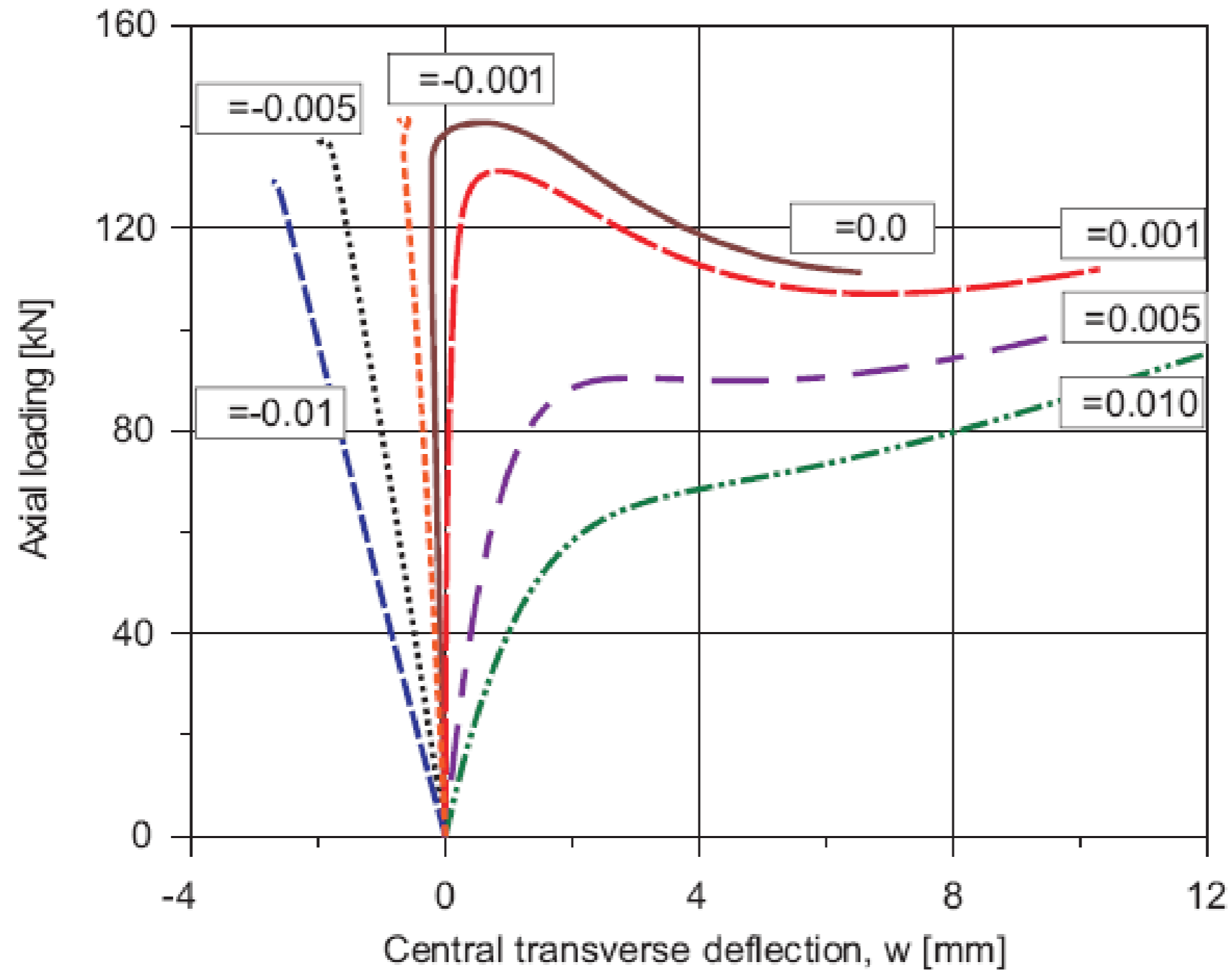




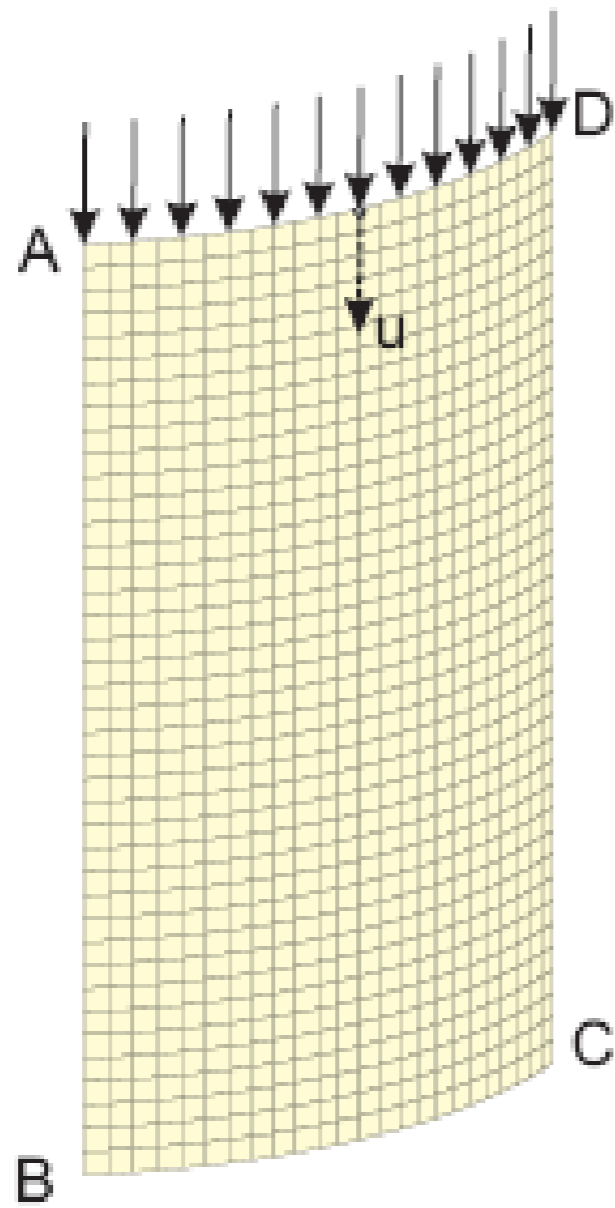
## Post-buckling deformation at $P=110.26$ kN (MSC Nastran 80x80 QUAD4 elements)



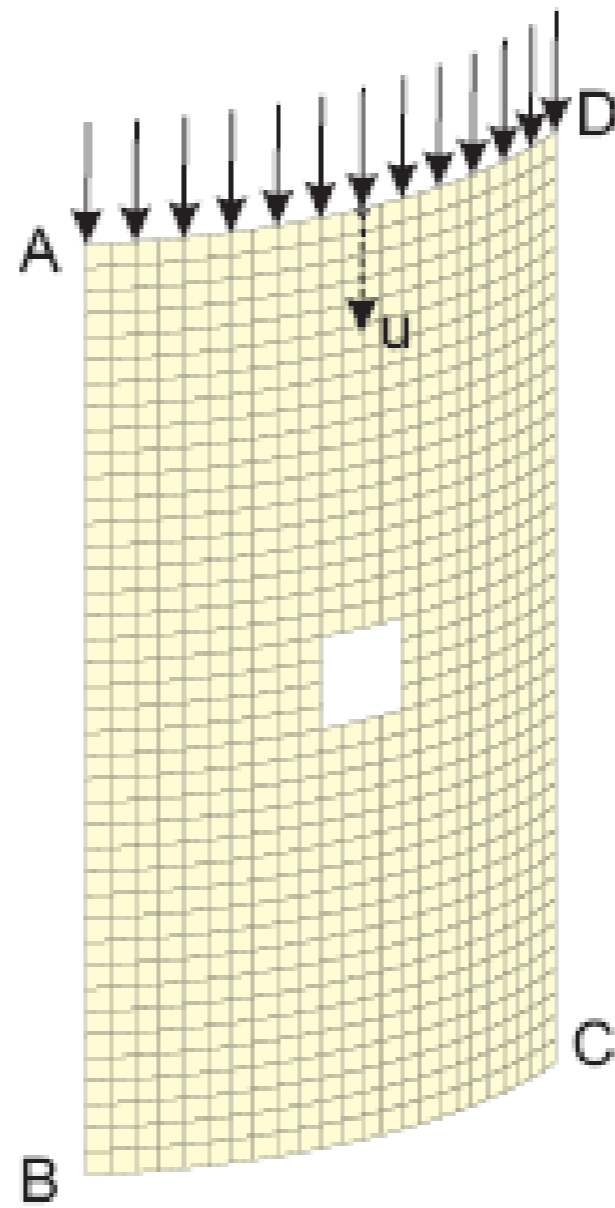
# Imperfection sensitivity of the layered panel buckling



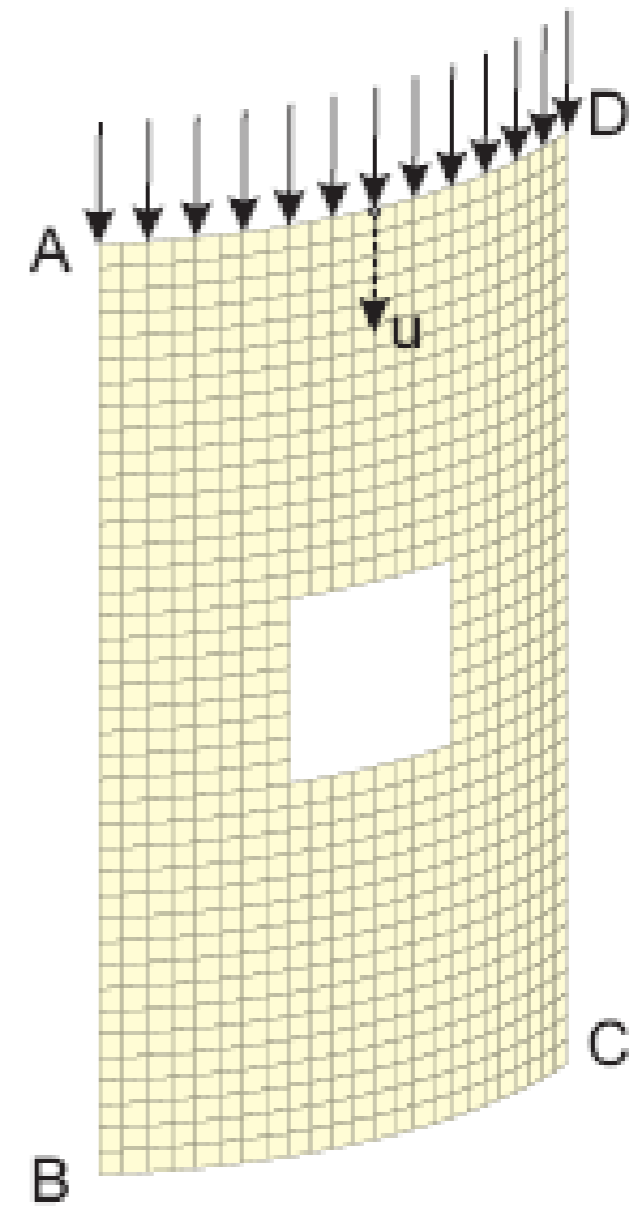
# Buckling of composite cylindrical panels with square cut-outs



Case I

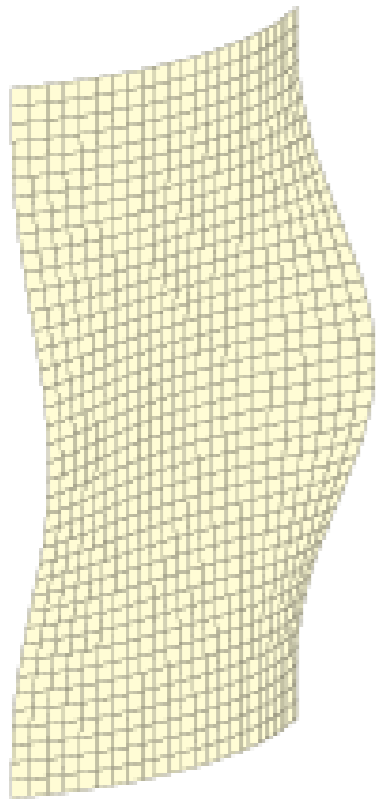


Case II



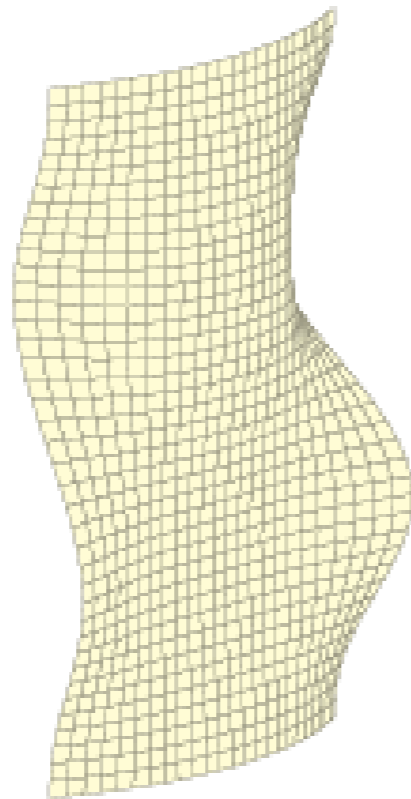
Case III

# Linear buckling modes obtained for Case I with model 24X40 QUAD4 elements



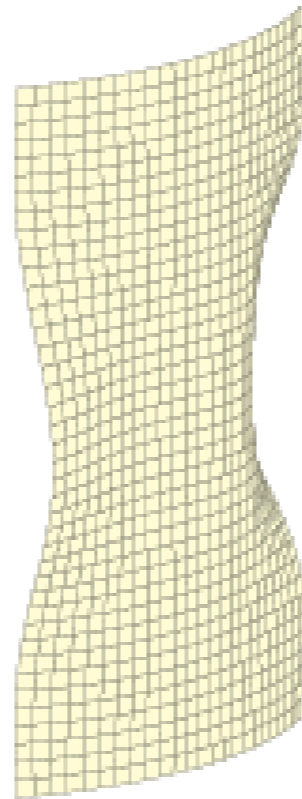
Mode 1

$P_{crit} = 24.43 \text{ kN}$



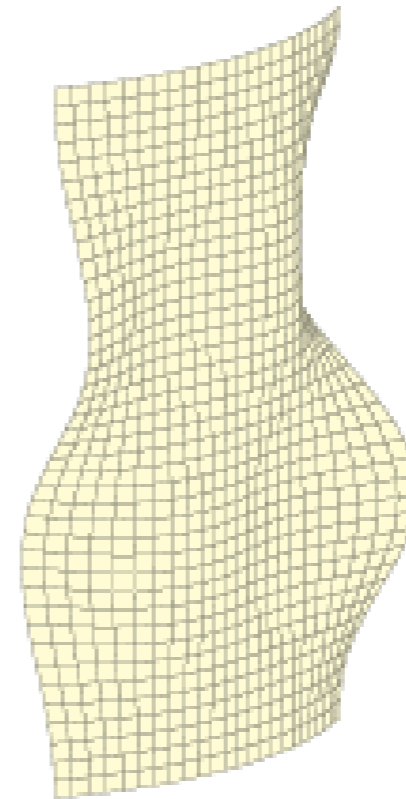
Mode 2

$P_{crit} = 27.55 \text{ kN}$



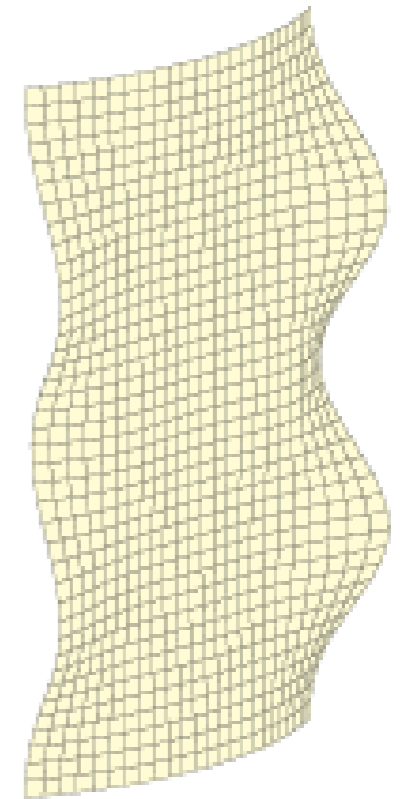
Mode 3

$P_{crit} = 28.47 \text{ kN}$



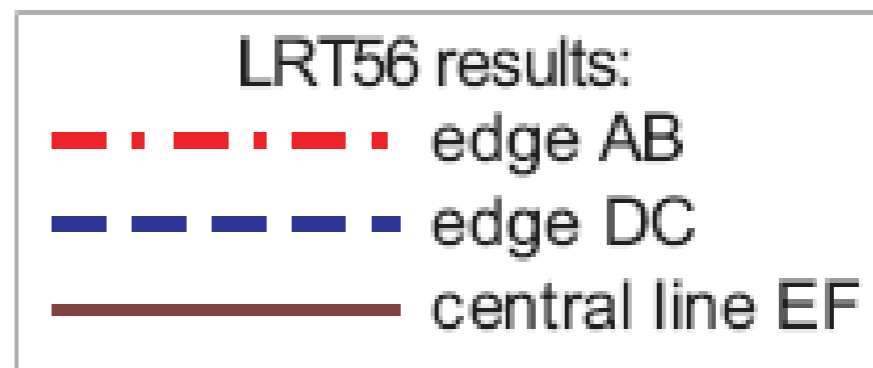
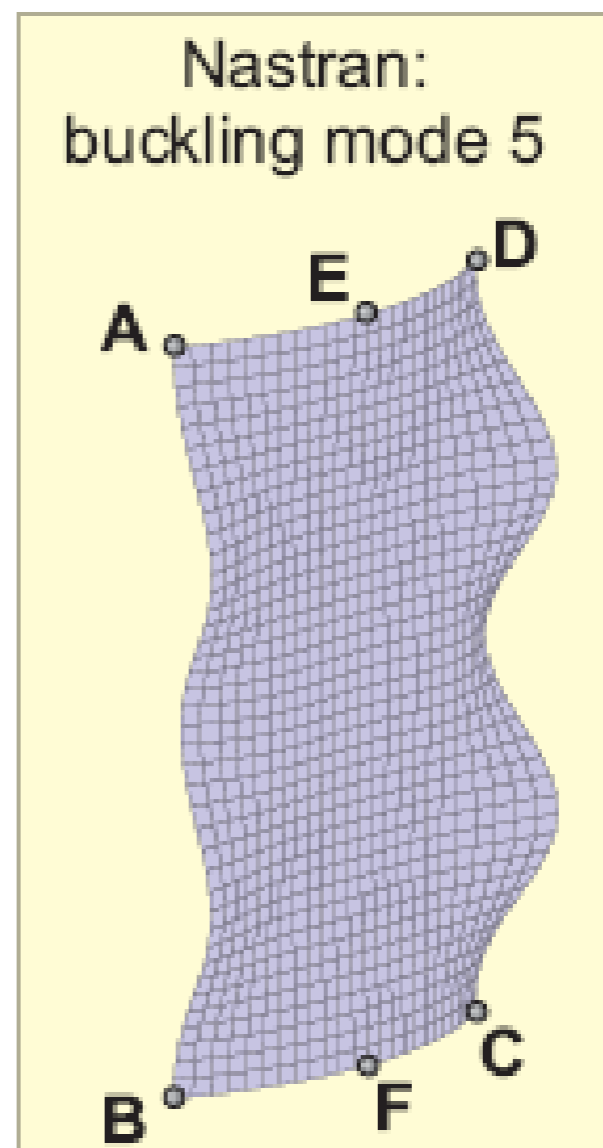
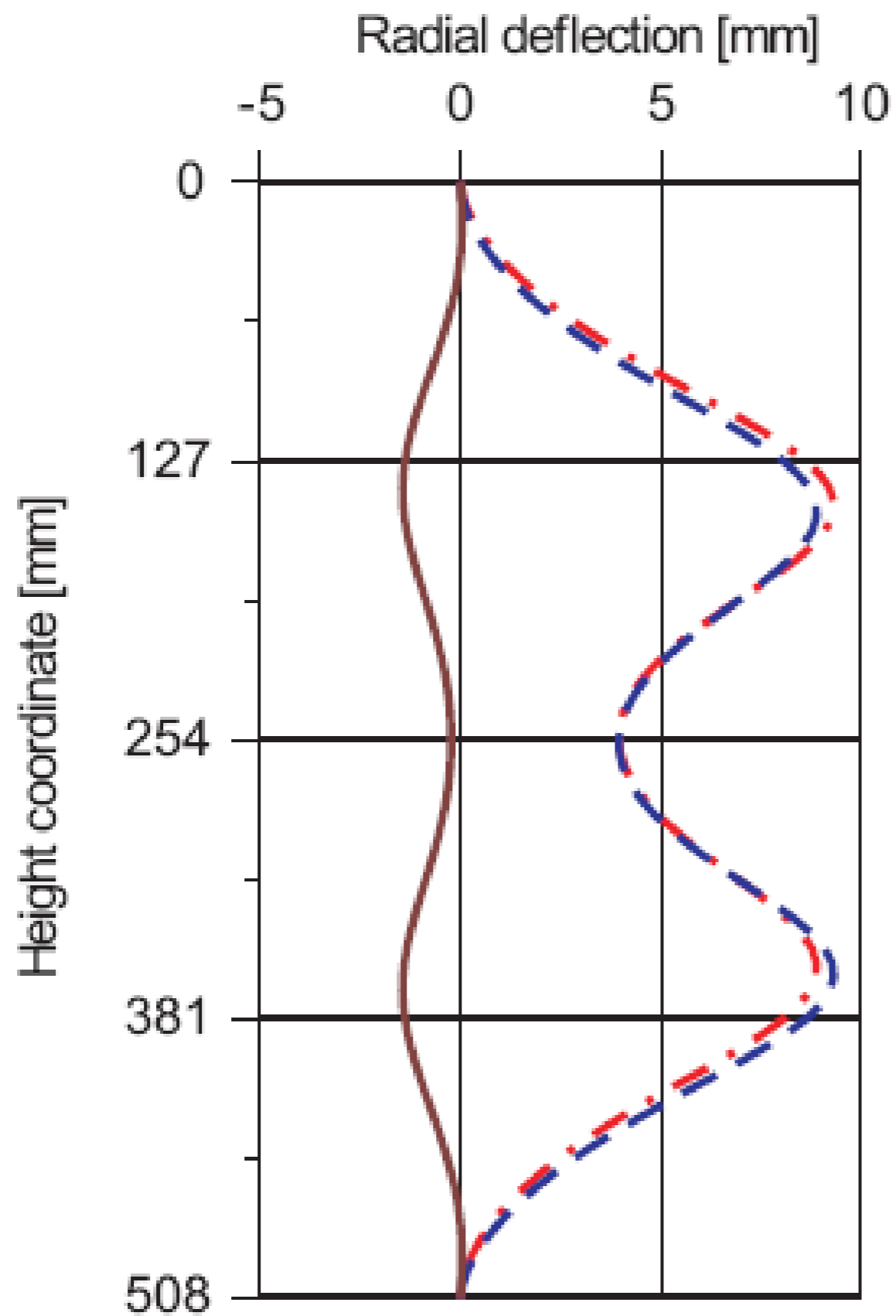
Mode 4

$P_{crit} = 29.31 \text{ kN}$



Mode 5

$P_{crit} = 36.91 \text{ kN}$



# Linear buckling modes obtained for Case II with 37X76

## QUAD4 elements



Mode 1

$P_{crit} = 21.95 \text{ kN}$



Mode 2

$P_{crit} = 26.10 \text{ kN}$



Mode 3

$P_{crit} = 26.53 \text{ kN}$



Mode 4

$P_{crit} = 27.69 \text{ kN}$



Mode 5

$P_{crit} = 34.50 \text{ kN}$

